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Topological quasi varieties

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A *quasi variety* is a class of *algebras* defined by a set of *quasi equations*

$$\bigwedge_{i<n} \Phi_i \Rightarrow \Psi,$$

where each Φ_i and Ψ are *equations*. This well known notion can be generalized in many ways to accommodate the need for more powerful means of expression. The notion studied in this paper encompasses two such generalizations going into different directions. In one direction we follow GRÄTZER and LAKSER [11] who introduce *structures* with *operations* and *relations*. They then generalize a result of MAL'CEV [17] to this setting: A class of structures is a quasi variety if and only if it is closed under the formation of isomorphic images, substructures and reduced products. Continuing in this direction ANDRÉKA, BURMEISTER and NÉMETI [1] consider *partial algebras* and prove the corresponding result. In another direction we follow TAYLOR [22] who considers *topological algebras* and introduces a new type of (infinitary) *topological atomic formula* to express *net convergence*. He then generalizes a result of BIRKHOFF [3] to this setting: A class of topological algebras is definable by topological atomic formulas if and only if it is closed under the formation of continuous homomorphic images, subalgebras and direct products. In this paper we shall consider *topological structures* (with operations, partial operations, relations and a topology) and introduce topological atomic formulas to talk about the topology. Since these new atomic formulas are infinitary, we have to allow for infinite conjunctions in *topological quasi atomic formulas*

$$\bigwedge\{\Phi_i | i\in I\} \Rightarrow \Psi,$$

where each Φ_i and Ψ are atomic formulas and I is a *set*. A *topological quasi variety* is a class of topological structures defined by a class of topological quasi atomic

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formulas. In this setting we show that a class of topological structures is a topological quasi variety if and only if it is closed under the formation of topological isomorphic images, substructures and direct products.

The definition of this notion was motivated by the observation that topological quasi varieties naturally arise in the numerous topological duality theorems in the literature. In each case a category anti equivalence is established between the quasi variety generated by a finite algebra $\mathfrak{P}$ and the class of *compact* members of the topological quasi variety generated by a finite topological structure $\boldsymbol{P}$ having the same universe as $\mathfrak{P}$. Accordingly we call the class of compact members of a topological quasi variety a *compact topological quasi variety.* It easily follows that a class of compact topological structures is a compact topological quasi variety if and only if it is closed under the formation of topological isomorphic images, compact substructures and direct products.

Duality theory is a rather recent topic in universal algebra and the primary sources are DAVEY [6] and DAVEY and WERNER [7]. We are mostly interested in their notion of *full duality* and we review the conceptual framework in which this notion is introduced. We then give a new characterization of this notion in terms of *hull-kernel closed* sets which not only tends to clarify the situation but also leads to new duality results. We show that there is a full duality for the (quasi) variety generated by an arbitrary finite algebra having a *near unanimity term* and only *simple non-trivial subalgebras*. This includes all *dual discriminator algebras* and unifies previous results for quasi primal algebras, distributive lattices, weakly associative lattices, median algebras, Kleene algebras and DeMorgan algebras. The full duality results for quasi primal algebras, weakly associative lattices and median algebras were claimed by WERNER [23] and DAVEY and WERNER [7], however their proofs are not correct. So we not only vindicate their claims but also establish them in a much broader context.

Topological quasi atomical theories appear to be an interesting topic for the logician and — as far as we have been able to ascertain — some of the most obvious problems in this area are still open. We have addressed ourselves to the question of *axiomatizability* and have come up with axiomatizations of several topological quasi atomical theories, many of which arise in the context of duality theory.

Altogether, we have attempted to put some rather diverse but extremely interesting recent developments in universal algebra and model theory into a unifying perspective.

1. Topological quasi varieties

In this section we define topological quasi varieties and investigate their basic properties. Special features are a suitable treatment of partial operations and the introduction of an infinitary first-order language which permits us to express many relevant topological facts.

Given is a similarity type $\mathbf{t}$ determined by a set $\mathbf{Op}$ of *operation symbols*, a set $\mathbf{POp}$ of *partial operation symbols* and a set $\mathbf{Rl}$ of *relation symbols*. A *topological structure* $\mathbf{X}$ of similarity type $\mathbf{t}$ has a (non-empty) topological space X for a universe, for each n-ary operation symbol $f\in\mathbf{Op}$ has a continuous n-ary operation $f^{\mathbf{X}}\colon X^n\to X$, for each n-ary partial operation symbol $g\in\mathbf{POp}$ has a continuous n-ary partial operation $g^{\mathbf{X}}\colon D\to X$, where $D=\operatorname{dom}(g^{\mathbf{X}})\subseteq X^n$ is the (possibly empty) domain of $g^{\mathbf{X}}$, and for each n-ary relation symbol $r\in\mathbf{Rl}$ has a closed n-ary relation $r^{\mathbf{X}}\subseteq X^n$, where X^n is endowed with the product topology. In all constructions involving topological structures operations and relations behave as usual so that we shall only mention them in case something extraordinary or unexpected is happening. Although all topological constructions are standard as well, we shall be a little more explicit in this area because there are some subtleties which are easily overlooked. The situation is quite different with partial operations. There are several options available here which have been pursued in the literature (see, e.g., GRÄTZER [10]). Thus we have to make our choices quite explicit in this area. $\mathbf{X}$ is called an *algebra* in case it has neither partial operations nor relations, i.e. $\mathbf{POp}\cup\mathbf{Rl}=\emptyset$.

To begin with, for $\mathbf{Y}$ to be a *substructure* of $\mathbf{X}$ (in symbols $\mathbf{Y}\subseteq\mathbf{X}$) we require that Y is a subspace of X and for all $g\in\mathbf{POp}$, $g^{\mathbf{Y}}\subseteq g^{\mathbf{X}}$ and $\operatorname{dom}(g^{\mathbf{Y}})=\operatorname{dom}(g^{\mathbf{X}})\cap Y^n$. For φ to be a *continuous homomorphism* from $\mathbf{X}$ to $\mathbf{Y}$ (in symbols $\varphi\colon\mathbf{X}\to\mathbf{Y}$) we require that φ is continuous and for all $g\in\mathbf{POp}$, if $x\in\operatorname{dom}(g^{\mathbf{X}})$, then $\varphi x\in\operatorname{dom}(g^{\mathbf{Y}})$ and $\varphi g^{\mathbf{X}}(x)=g^{\mathbf{Y}}(\varphi x)$. In the presence of partial operations and/or relations homomorphisms are afflicted with some peculiarities which cause much trouble and confusion in this area. To be more specific, if $\varphi\colon\mathbf{X}\to\mathbf{Y}$, then the following conditions are not necessarily satisfied:

(1) If $g\in\mathbf{POp}$ and $\varphi x\in\operatorname{dom}(g^{\mathbf{Y}})$, then there exists $y\in\operatorname{dom}(g^{\mathbf{X}})$ such that $\varphi x=\varphi y$.

(2) If $r\in\mathbf{Rl}$ and $\varphi x\in r^{\mathbf{Y}}$, then there exists $y\in r^{\mathbf{X}}$ such that $\varphi x=\varphi y$.

Notice that (1) implies that $\varphi(X)\subseteq Y$ is closed under $g^{\mathbf{Y}}$, i.e. $\varphi(X)$ determines a substructure of $\mathbf{Y}$, but not vice versa. The fact that $\varphi(X)$ may fail to determine a substructure of $\mathbf{Y}$ will be an important issue later. This situation seriously affects the definition of *surjective homomorphism* and *homomorphic image*, and there are several options available for these notions. The notion of *injective homomorphism* is not entirely clear either. Fortunately we do not have to get involved in these troublesome decisions because we only have to consider *embeddings* where the requirements are quite clear. Consequently we shall use the notions "injective" and "surjective" *only for mappings*. φ is called a *(topological) embedding* (in symbols $\varphi\colon \mathbf{X} \Vert\!\!\rightarrow \mathbf{Y}$) in case φ is a continuous homomorphism and an injective mapping whose inverse is continuous, where

(i) for all $g\in \mathbf{POp}$, $x\in \mathrm{dom}\,(g^{\mathbf{X}})$ iff $\varphi x\in \mathrm{dom}\,(g^{\mathbf{Y}})$;

(ii) for all $r\in \mathbf{Rl}$, $x\in r^{\mathbf{X}}$ iff $\varphi x\in r^{\mathbf{Y}}$.

φ is called a *topological isomorphism* (in symbols $\varphi\colon \mathbf{X} \Vert\!\!\twoheadrightarrow \mathbf{Y}$) if, in addition, φ is a surjective mapping (equivalently, if φ has a two-sided inverse). Now it is clear that a topological embedding is a topological isomorphism with a substructure. For the *direct product* $\mathbf{Y}=\prod\langle \mathbf{X}_i \mid i\in I\rangle$ of the set of structures $\{\mathbf{X}_i \mid i\in I\}$ we require that Y is endowed with the product topology and that for all $g\in \mathbf{POp}$ and all $y\in Y^n$,

$$y\in \mathrm{dom}\,(g^{\mathbf{Y}}) \quad \text{iff} \quad y(i)\in \mathrm{dom}\,(g^{\mathbf{X}_i}) \quad \text{for all} \quad i\in I, \quad \text{and}$$

$$g^{\mathbf{Y}}(y_0, \dots, y_{n-1})(i) = g^{\mathbf{X}_i}(y_0(i), \dots, y_{n-1}(i)).$$

By a *trivial structure* we mean a one element structure with all partial operations and relations nonempty. For example, the direct product $\prod\langle \mathbf{X}_i \mid i\in\emptyset\rangle$ of the empty set of structures is trivial.

If $J\subseteq I$, then the projection

$$\pi_J\colon \mathbf{Y} \to \prod\langle \mathbf{X}_j | j\in J\rangle$$

is afflicted with both peculiarities mentioned above, i.e. (1) and (2) both may fail although π_J clearly is a *surjective mapping*.

If $\mathscr{K}$ is a class of topological structures then $\mathbf{I}\mathscr{K}$, $\mathbf{S}\mathscr{K}$ and $\mathbf{S}_c\mathscr{K}$ denote the classes of topological isomorphic images, substructures and compact substructures of members of $\mathscr{K}$ respectively, and $\mathbf{P}\mathscr{K}$ denotes the class of direct products of *nonempty* subsets of $\mathscr{K}$. First we show that our definition of direct product is correct for the category we have defined.

Lemma 1.1. (Transfer Principle). *Suppose $I\neq\emptyset$ and for each $i\in I$, $\varphi_i\colon \mathbf{Y}\to\mathbf{X}_i$. Then there exists a unique $\psi\colon \mathbf{Y}\to\prod\langle \mathbf{X}_i \mid i\in I\rangle$ such that for all $i\in I$, $\varphi_i=\pi_i\circ\psi$.*

Proof. Suppose $g\in \mathbf{POp}$ and $y\in \mathrm{dom}\,(g^{\mathbf{Y}})$. By hypothesis, for every $i\in I$, $\varphi_i y\in \mathrm{dom}\,(g^{\mathbf{X}_i})$ and $\varphi_i g^{\mathbf{Y}}(y)=g^{\mathbf{X}_i}(\varphi_i y)$. Thus $\psi y(i)\in \mathrm{dom}\,(g^{\mathbf{X}_i})$ for all $i\in I$ and therefore $\psi y\in \mathrm{dom}\,(g^{\mathbf{Z}})$, where $\mathbf{Z}=\prod\langle \mathbf{X}_i \mid i\in I\rangle$. Finally, $\psi g^{\mathbf{Y}}(y)(i)=\varphi_i g^{\mathbf{Y}}(y)= =g^{\mathbf{X}_i}(\varphi_i y)=g^{\mathbf{X}_i}(\psi y(i))=g^{\mathbf{Z}}(\psi y)(i)$ for all $i\in I$.

Next we show that the usual separation properties can be augmented to obtain subdirect representations in this setting.

Lemma 1.2 (Separation Principle). *For any nontrivial structure $\mathbf{Y}$, $\mathbf{Y}\in \mathbf{ISP}\mathscr{K}$ if and only if*

(i) *If* $x, y \in Y$, *where* $x \neq y$, *then there exist* $\mathbf{X} \in \mathscr{K}$ *and* $\varphi\colon \mathbf{Y} \to \mathbf{X}$ *such that* $\varphi x \neq \varphi y$.

(ii) *If* $g \in \mathbf{POp}$ *and* $y \notin \operatorname{dom}(g^{\mathbf{Y}})$, *then there exist* $\mathbf{X} \in \mathscr{K}$ *and* $\varphi\colon \mathbf{Y} \to \mathbf{X}$ *such that* $\varphi y \notin \operatorname{dom}(g^{\mathbf{X}})$.

(iii) *If* $r \in \mathbf{Rl}$ *and* $y \notin r^{\mathbf{Y}}$, *then there exist* $\mathbf{X} \in \mathscr{K}$ *and* $\varphi\colon \mathbf{Y} \to \mathbf{X}$ *such that* $\varphi y \notin r^{\mathbf{X}}$.

(iv) *If D is a subset of the power set of Y directed by inclusion,* $\delta\colon D \to Y$ *is a net in Y and* $y \in Y$, *where* $\delta d \overset{d \in D}{\not\to} y$, *then there exist* $\mathbf{X} \in \mathscr{K}$ *and* $\varphi\colon \mathbf{Y} \to \mathbf{X}$ *such that* $\varphi \circ \delta d \overset{d \in D}{\not\to} \varphi y$.

Proof. The necessity of (i)—(iv) is obvious. So assume (i)—(iv). We consider all instances mentioned in conditions (i)—(iv), and for each instance we choose $\mathbf{X} \in \mathscr{K}$ and $\varphi\colon \mathbf{Y} \to \mathbf{X}$ "witnessing" its occurrence. So let $\mathbf{X}_i \in \mathscr{K}$, $\varphi_i\colon \mathbf{Y} \to \mathbf{X}_i$, where $i \in I$, be a set of "witnesses". By the Transfer Principle, there exists unique

$$\psi\colon \mathbf{Y} \to \prod \langle \mathbf{X}_i \mid i \in I \rangle$$

such that for all $i \in I$, $\varphi_i = \pi_i \circ \psi$. We shall show that ψ is a topological embedding. As usual, the witnesses for (i) make ψ an injective mapping. To show that ψ^{-1} is continuous it suffices to establish that ψ is a closed mapping. So suppose $W \subseteq Y$, where $\psi(W) \subseteq \psi(Y)$ is not closed. Then there exists $y \in Y$ such that ψy is a limit point of $\psi(W)$ but $\psi(y) \notin \psi(W)$. Let D be the set of $U \subseteq Y$ such that $\psi(U)$ is a neighborhood of ψy. D is directed by inclusion. For each $U \in D$ there exists $\delta U \in Y$ such that $\psi \circ \delta U \in \psi(U) \cap \psi(W)$. Thus $\psi \circ \delta U \overset{U \in D}{\to} \psi y$. If $\delta U \overset{U \in D}{\not\to} Y$, then by construction we obtain from (iv) $\mathbf{X}_i \in \mathscr{K}$ and $\varphi_i\colon \mathbf{Y} \to \mathbf{X}_i$ such that $\varphi_i \circ \delta U \overset{U \in D}{\not\to} \varphi_i y$ and therefore $\pi_i \circ \psi \circ \delta U \overset{U \in D}{\not\to} \pi_i \circ \psi y$, contradicting net convergence in direct products. Thus $\delta U \overset{U \in D}{\to} y$ so that y is a limit point of W, but $y \notin W$. It follows that W is not closed so that ψ^{-1} is continuous.

Next, suppose $g \in \mathbf{POp}$ and $y \notin \operatorname{dom}(g^{\mathbf{Y}})$. Then by construction we obtain from (ii) $\mathbf{X}_i \in \mathscr{K}$ and $\varphi_i\colon \mathbf{Y} \to \mathbf{X}_i$ such that $\varphi_i y \notin \operatorname{dom}(g^{\mathbf{X}_i})$. Thus $\psi y(i) \notin \operatorname{dom}(g^{\mathbf{X}_i})$ and therefore $\psi y \notin \operatorname{dom}(g^{\mathbf{Z}})$, where $\mathbf{Z} = \prod \langle \mathbf{X}_i \mid i \in I \rangle$.

Since the witnesses for (iii) take care of the relations, ψ is indeed a topological embedding.

The Separation Principle considerably simplifies for compact Hausdorff structures.

Corollary 1.3 (Compact Hausdorff Separation Principle). *Let* $\mathscr{K}$ *be a class of Hausdorff structures and let* $\mathbf{Y}$ *be a nontrivial compact structure. Then* $\mathbf{Y} \in \mathbf{IS}_c\mathbf{P}\mathscr{K}$ *if and only if* $\mathbf{Y}$ *satisfies* (i), (ii) *and* (iii) *of Lemma 1.2.*

Proof. By the proof of Lemma 1.2, it clearly suffices to show that under the assumption of (i), (ii) and (iii) of Lemma 1.2 ψ^{-1} is continuous. This however follows from the fact that Y is a compact space, ψ is an injective mapping and $\prod\langle X_i \mid i\in I\rangle$ is a Hausdorff space.

Corollary 1.4 (Compact Hausdorff Separation Principle for Algebras). *Let $\mathcal{K}$ be a class of Hausdorff algebras and let* **Y** *be a nontrivial compact algebra. Then* $\mathbf{Y}\in\mathbf{IS}_c\mathbf{P}\mathcal{K}$ *if and only if distinct members of Y can be separated by continuous homomorphisms into members of* $\mathcal{K}$.

Remark 1.5. Davey and Werner [7] denote conditions (i) and (iii) of Lemma 1.2 by (SEP) and claim that a compact **Y** belongs to $\mathbf{IS}_c\mathbf{P}\mathcal{K}$ just in case it satisfies (SEP). In the presence of partial operations this is not correct and it requires a good deal of technical detail work to correct their arguments in this case. Unfortunately we shall have to get involved with this issue in Section 3.

Now we introduce just enough language to determine $\mathbf{IS}_c\mathbf{P}\mathcal{K}$ as a topological quasi variety. Let Vb be a *proper* class of *variables*. We define the class Tm of finitary *terms* as usual building up terms from variables using both operation and partial operation symbols. We have two types of *atomic formulas*. First the (finitary) *algebraic* and *relational* atomic formulas

$$\tau \approx \sigma, \quad r\tau_0\tau_1\ldots\tau_{n-1},$$

where $\tau, \sigma, \tau_0, \ldots, \tau_{n-1}\in Tm$, $\approx$ is the identity symbol and $r\in \boldsymbol{Rl}$; and secondly the (infinitary) *topological* atomic formulas

$$\tau_d \xrightarrow{d\in D} \sigma,$$

where $\langle D, \leqq\rangle$ is a directed set, $\tau\colon D\to Tm$ is a net in Tm and $\sigma\in Tm$. A *topological quasi atomic formula* is an expression of the form

$$\bigwedge\{\Phi_\xi \mid \xi\in\Delta\} \Rightarrow \Psi \quad \text{or} \quad \bigvee\{\neg\Phi_\xi \mid \xi\in\Delta\},$$

where $\{\Phi_\xi|\xi\in\Delta\}$ is a (possibly empty) *set* of atomic formulas and Ψ is an atomic formula.

An *assignment* of the variables in the topological structure **X** is a mapping $x\colon Vb\to X$. For $\tau\in Tm$ we define by simultaneous recursion the two notions "τ is defined for x (in X)" and "$\tau^{\mathbf{X}}[x]\in X$" in case τ is defined for x.

(1) If $v\in Vb$, then v is defined for x and $v^{\mathbf{X}}[x]=x(v)$.

(2) If $f\in\mathbf{Op}$, then $f\tau_0\ldots\tau_{n-1}$ is defined for x iff $\tau_0, \ldots, \tau_{n-1}$ are defined for x, and then $f\tau_0\ldots\tau_{n-1}{}^{\mathbf{X}}[x]=f^{\mathbf{X}}(\tau_0^{\mathbf{X}}[x], \ldots, \tau_{n-1}^{\mathbf{X}}[x])$.

(3) If $g\in\mathbf{POp}$, then $g\tau_0\ldots\tau_{n-1}$ is defined for x iff $\tau_0, \ldots, \tau_{n-1}$ are defined for x and $\langle\tau_0^{\mathbf{X}}[x], \ldots, \tau_{n-1}^{\mathbf{X}}[x]\rangle\in\operatorname{dom}(g^{\mathbf{X}})$, and then $g\tau_0\ldots\tau_{n-1}{}^{\mathbf{X}}[x]=g^{\mathbf{X}}(\tau_0^{\mathbf{X}}[x], \ldots, \tau_{n-1}^{\mathbf{X}}[x])$.

Next we define the notion of *satisfaction* for quasi atomic formulas.

$$\mathbf{X} \models \sigma \approx \tau[x]$$

if σ and τ are defined for x and $\sigma^{\mathbf{X}}[x]=\tau^{\mathbf{X}}[x]$.

$$\mathbf{X} \models r\tau_0\dots\tau_{n-1}[x]$$

if $\tau_0, \dots, \tau_{n-1}$ are defined for x and $\langle\tau_0^{\mathbf{X}}[x], \dots, \tau_{n-1}^{\mathbf{X}}[x]\rangle \in r^{\mathbf{X}}$.

$$\mathbf{X} \models \tau_d \xrightarrow{d\in D} \sigma[x]$$

if all τ_d, $d\in D$, and σ are defined for x and $\tau_d^{\mathbf{X}}[x] \overset{d\in D}{\to} \sigma^{\mathbf{X}}[x]$.

$$\mathbf{X} \models \wedge\{\Phi_\xi | \xi\in\Delta\} \Rightarrow \Psi[x]$$

if there exists $\xi\in\Delta$ such that $\mathbf{X} \not\models \Phi_\xi[x]$ or $\mathbf{X} \models \Psi[x]$. Finally

$$\mathbf{X} \models \vee\{\neg\Phi_\xi | \xi\in\Delta\}[x]$$

provided $\mathbf{X} \not\models \Phi_\xi[x]$ for each $\xi\in\Delta$. Notice that for each "disjunctive" topological quasi atomic formula Φ there is a finite set of "implicational" topological quasi atomic formulas which are equivalent to Φ in any nontrivial structure.

A topological structure $\mathbf{X}$ is called a *model* of a class Σ of topological quasi atomic formulas (in symbols $\mathbf{X} \models \Sigma$) if for every $\Phi\in\Sigma$ and every $x: Vb \to X$, $\mathbf{X} \models \Phi[x]$. A class $\mathcal{K}$ of (compact) topological structures is called a *(compact) topological quasi variety* if there exists a class Σ of topological quasi atomic formulas such that $\mathcal{K}$ is the class of (compact) models of Σ. The *topological quasi atomical theory* of $\mathcal{K}$ (denoted by $\mathrm{Th}_{\mathrm{tqa}}\mathcal{K}$) is the class of topological quasi atomic formulas Φ such that each member of $\mathcal{K}$ is a model of Φ. The (compact) topological quasi variety *generated* by $\mathcal{K}$ is the class of (compact) models of $\mathrm{Th}_{\mathrm{tqa}}\mathcal{K}$.

Example 1.6. Some facts which hold in all topological structures by their very definition are expressible by topological quasi atomic formulas which therefore become *logically true*. As usual we shall write $\models\Phi$ in case $\mathbf{X}\models\Phi$ for all topological structures. To begin with, notice that for $g\in\mathbf{POp}$ and $x\in X^n$ we have

$$x\in\mathrm{dom}(g^{\mathbf{X}}) \quad \text{iff} \quad \mathbf{X} \models gv_0\dots v_{n-1} \approx gv_0\dots v_{n-1}[x]$$

so that $\not\models gv_0\dots v_{n-1} \approx gv_0\dots v_{n-1}$! On the other hand,

$$\models gv_0\dots v_{n-1} \approx v_n \Rightarrow gv_0\dots v_{n-1} \approx gv_0\dots v_{n-1}$$

so that for $x\in X^{n+1}$ we have

$$x\in\mathrm{graph}(g^{\mathbf{X}}) \quad \text{iff} \quad \mathbf{X} \models gv_0\dots v_{n-1} \approx v_n[x].$$

Now let $\langle D, \leqq\rangle$ be a directed set and $v\colon D\to Vb^n$. Then

$$\models[\bigwedge_{d\in D} g(v(d)_0, \ldots, v(d)_{n-1}) \approx g(v(d)_0, \ldots, v(d)_{n-1}) \wedge \bigwedge_{i<n} v(d)_i \xrightarrow{d\in D} u_i] \Rightarrow$$

$$\Rightarrow [g(v(d)_0, \ldots, v(d)_{n-1}) \xrightarrow{d\in D} g(u_0, \ldots, u_{n-1})]$$

because for each topological structure $\mathbf{X}$, $g^{\mathbf{X}}$ is continuous. Similarly we can express with topological quasi atomic formulas that $f^{\mathbf{X}}$ is continuous and $r^{\mathbf{X}}$ is closed, where $f\in\mathbf{Op}$ and $r\in\mathbf{Rl}$. We can also say "the graph of $g^{\mathbf{X}}$ is closed", where $g\in\mathbf{POp}$:

$$[\bigwedge_{d\in D} g(v(d)_0, \ldots, v(d)_{n-1}) \approx v(d)_n \wedge \bigwedge_{i\leqq n} v(d)_i \xrightarrow{d\in D} u_i] \Rightarrow g(u_0, \ldots, u_{n-1}) \approx u_n.$$

Let this formula be abbreviated by $\mathrm{Cl}(g)$. Now $\not\models\mathrm{Cl}(g)$, but $\mathbf{X}\models\mathrm{Cl}(g)$ in case $\mathbf{X}$ is Hausdorff and $g^{\mathbf{X}}$ is a full operation on X. Of course, this observation applies to the graphs of operations as well.

Example 1.7. Consider the class of topological abelian groups $\langle A, +, -, 0\rangle$. Then a discrete abelian group is *torsion* if and only if it is a model of the single topological atomic formula $m!v \xrightarrow{m\in\omega} 0$.

Next we shall consider *preservation properties.*

Lemma 1.8. (i) *Suppose* $\mathbf{X}\subseteq\mathbf{Y}$, τ *is a term and* $x\colon Vb\to X$. *Then* τ *is defined for* x *in* $\mathbf{X}$ *if and only if* τ *is defined for* x *in* $\mathbf{Y}$. *Moreover, in this case* $\tau^{\mathbf{X}}[x]=\tau^{\mathbf{Y}}[x]$.

(ii) *Suppose* $\varphi\colon \mathbf{X}\to\mathbf{Y}$, τ *is a term and* $x\colon Vb\to X$. *If* τ *is defined for* x, *then* τ *is defined for* $\varphi\circ x$ *and* $\varphi\tau^{\mathbf{X}}[x]=\tau^{\mathbf{Y}}[\varphi\circ x]$.

(iii) *Suppose* $\mathbf{Y}=\prod\langle\mathbf{X}_i\mid i\in I\rangle$, τ *is a term and* $x\colon Vb\to Y$. *Then* τ *is defined for* x *if and only if for every* $i\in I$, τ *is defined for* $\eta_i\circ x$. *Moreover, in this case* $\tau^{\mathbf{Y}}[x](i)=\tau^{\mathbf{X}_i}[\pi_i\circ x]$ *for all* $i\in I$.

Lemma 1.9. (i) *Suppose* $\mathbf{X}\subseteq\mathbf{Y}$, Φ *is an atomic formula and* $x\colon Vb\to X$. *Then* $\mathbf{X}\models\Phi[x]$ *iff* $\mathbf{Y}\models\Phi[x]$.

(ii) *Suppose* $\varphi\colon \mathbf{X}\to\mathbf{Y}$, Φ *is an atomic formula and* $x\colon Vb\to X$. *Then* $\mathbf{X}\models\Phi[x]$ *implies* $\mathbf{Y}\models\Phi[\varphi\circ x]$.

(iii) *Suppose* $\mathbf{Y}=\prod\langle\mathbf{X}_i\mid i\in I\rangle$, Φ *is an atomic formula and* $x\colon Vb\to Y$. *Then* $\mathbf{Y}\models\Phi[x]$ *iff* $\mathbf{X}_i\models\Phi[\pi_i\circ x]$ *for all* $i\in I$.

Corollary 1.10. *Topological quasi atomic formulas are preserved by* **I**, **S** *and* **P**.

Now we prove a technical lemma which characterizes continuous homomorphisms with a fixed domain $\mathbf{Y}$, where we have $Y\subseteq Vb$. Choose $a\colon Vb\to Y$ where

$a(v)=v$ for $v\in Y$. Next define Σ to be the *set* of all algebraic and relational atomic formulas Φ with variables from Y where $\mathbf{Y}\models\Phi[a]$, and all topological atomic formulas $\tau_d \overset{d\in D}{\to} \sigma$ with variables from Y, where D is a subset of the power set of Y directed by inclusion and $\mathbf{Y}\models\tau_d \overset{d\in D}{\to} \sigma[a]$.

Lemma 1.11. *A map $\varphi\colon Y\to X$ is a continuous homomorphism if and only if for each $\Phi\in\Sigma$, $\mathbf{X}\models\Phi[\varphi\circ a]$.*

Proof. One direction follows from Lemma 1.9 (ii), so assume $\varphi\colon Y\to X$ and $\mathbf{X}\models\Phi[\varphi\circ a]$ for each $\Phi\in\Sigma$. We claim that $\varphi\colon \mathbf{Y}\to\mathbf{X}$. Indeed, consider $g\in\mathbf{Op}$ and $\langle x_0, \dots, x_{n-1}\rangle\in\operatorname{dom}(g^{\mathbf{Y}})$. Then

$$\mathbf{Y}\models gx_0\dots x_{n-1}\approx gx_0\dots x_{n-1}[a]$$

and therefore

$$\mathbf{X}\models gx_0\dots x_{n-1}\approx gx_0\dots x_{n-1}[\varphi\circ a].$$

It follows that $\langle \varphi x_0, \dots, \varphi x_{n-1}\rangle\in\operatorname{dom}(g^{\mathbf{X}})$. Considering the atomic formula $gx_0\dots x_{n-1}\approx x_n$, where $g^{\mathbf{Y}}(x_0, \dots, x_{n-1})=x_n$, we establish that

$$\varphi g^{\mathbf{Y}}(x_0, \dots, x_{n-1}) = g^{\mathbf{X}}(\varphi x_0, \dots, \varphi x_{n-1}).$$

Operations and relations are treated similarly so that we are left with showing that φ is continuous. So suppose $W\subseteq X$ is closed and $z\in Y$ is a limit point of $\varphi^{-1}(W)$. Let D be the neighborhood system of y. D is directed by inclusion. For each $U\in D$ there exists $\delta U\in U\cap\varphi^{-1}(W)$, and $\delta U \overset{U\in D}{\to} z$. In other words, $\mathbf{Y}\models\delta U \overset{U\in D}{\to} z[a]$, so that $\mathbf{X}\models\delta U \overset{U\in D}{\to} z[\varphi\circ a]$. Thus $\varphi\circ\delta U \overset{U\in D}{\to} \varphi z$, where $\varphi\circ\delta U\in W$ for all $U\in D$. Since W is closed, $\varphi z\in W$ and therefore $z\in\varphi^{-1}(W)$.

Theorem 1.12. $\mathbf{Y}\in\mathbf{ISP}\mathscr{K}$ *if and only if* $\mathbf{Y}\models\mathrm{Th}_{\mathrm{tqa}}\mathscr{K}$.

Proof. Assume $\mathbf{Y}\models\mathrm{Th}_{\mathrm{tqa}}\mathscr{K}$. We shall establish conditions (i)—(iv) of the Separation Principle. Without loss of generality we may assume that $Y\subseteq Vb$. Let $x, y\in Y$, where $x\neq y$, and define Σ as in Lemma 1.11. Then

$$\mathbf{Y}\not\models \bigwedge\{\Phi\,|\,\Phi\in\Sigma\}\Rightarrow x\approx y[a]$$

and therefore there exist $\mathbf{X}\in K$ and $b\colon Vb\to X$ such that

$$\mathbf{X}\not\models \bigwedge\{\Phi|\Phi\in\Sigma\}\Rightarrow x\approx y[b].$$

Define $\varphi\colon Y\to X$ by $\varphi(v)=b(v)$, $v\in Y$. By the choice of variables we may assume that $b=\varphi\circ a$. Since $\mathbf{X}\models\Phi[\varphi\circ a]$ for each $\Phi\in\Sigma$, $\varphi\colon\mathbf{Y}\to\mathbf{X}$ by Lemma 1.11. Since $\varphi(x)\neq\varphi(y)$, condition (i) of the Separation Principle is established. The other conditions are proven by the same argument. Now check that the assertion is also true when $\mathbf{Y}$ is trivial.

Corollary 1.13. *A class $\mathcal{K}$ of (compact) topological structures is a (compact) topological quasi variety if and only if $\mathcal{K}$ is closed under* $\mathbf{I}, \mathbf{S}_{(c)}$, *and* $\mathbf{P}$.

Example 1.14. Since a subspace of a compact topological space is not necessarily compact, by Theorem 1.8, the class of *compact* topological structures is *not* in general a topological quasi variety. This forced us to introduce the notion of a *compact topological quasi variety* in the metalanguage.

As mentioned in the introduction, MAL'CEV [17] shows that a class $\mathcal{K}$ of *algebras* is a quasi variety if and only if it is closed under the formation of isomorphic images, subalgebras and reduced products, and ANDRÉKA, BURMEISTER and NÉMETI [1] have generalized this to *partial algebras*. It turns out that there is no corresponding result for topological quasi varieties because they are not closed under ultrapowers.

Lemma 1.15. *Let* $\mathbf{X}$ *be any topological structure and let* U *be a nonprincipal ultrafilter on an infinite set* I. *Then the quotient topology on* X_U^I *is the indiscrete topology.*

Proof. Let $\varphi: X^I \to X_U^I$ be the canonical mapping, $M \subseteq X^I$ a basic open set in the product space, and let $x \in X^I$. Choose $y \in M$ which differs from x on a finite set. Since U is nonprincipal, $\{i \in I \mid x(i) = y(i)\} \in U$. Thus $x/U = y/U \in \varphi(M)$ so that $\varphi(M) = X_U^I$. The assertion follows at once.

Now a definition of the ultraproduct $\mathbf{X}_U^I$ can only be considered adequate if the canonical homomorphism $\varphi: \mathbf{X}^I \to \mathbf{X}_U^I$ is continuous. Thus the topology on X_U^I has to be trivial. Hence, if $\mathbf{X}$ is a non-trivial Hausdorff structure, then $\mathbf{X}_U^I$ is not Hausdorff. It follows from Example 3.2 that no topological quasi variety containing a non-trivial Hausdorff structure is closed under ultrapowers.

2. Compact (topological) quasi varieties equivalent to (algebraic) quasi varieties

In this section we shall investiate a method of generating compact topological quasi varieties which has been recently developed in duality theory. This method yields many interesting examples of compact quasi varieties which play an important role in the literature. The foundations of (topological) duality theory were laid in DAVEY [6]. In a recent paper DAVEY and WERNER [7] give an expansive exposition of duality theory which contains some important advances yielding new applications. The idea of central interest to us is their notion of *full duality*. Unfortunately DAVEY and WERNER [7] contains some claims concerning full duality whose proofs are not correct in case partial operations are involved. This then yields some applications that are not justified. In this section we shall develop a theory of full duality which hopefully is both correct and substantially contributes to better understanding of this notion. Moreover, our approach yields new full duality results which vindicate and generalize the unestablished claims of Davey and Werner. For this purpose we shall (almost) completely adopt the notation and terminology of DAVEY and WERNER [7] and shall briefly review the setting of their investigation.

To begin with, we shall simultaneously deal with two distinct similarity types: A type t of *algebras* determined by a set Op of operation symbols, and a type $\mathbf{t}$ of topological structures determined by a set $\mathbf{Op}$ of operation symbols, a set $\mathbf{POp}$ of partial operation symbols and a set $\mathbf{Rl}$ of relation symbols. Given is a finite, non-empty set P, an algebra $\mathfrak{P}=\langle P, f^{\mathfrak{P}}\rangle_{f\in Op}$ of similarity type t, and a topological structure $\mathbf{P}=\langle P, f^{\mathbf{P}} g^{\mathbf{P}}, r^{\mathbf{P}}\rangle_{f\in \mathbf{Op},\, g\in \mathbf{POp},\, r\in \mathbf{Rl}}$ of similarity type $\mathbf{t}$, where P is endowed with the discrete topology. In addition we require that $\mathbf{P}$ and $\mathfrak{P}$ satisfy the two equivalent conditions of the next lemma:

Lemma 2.1. *The following are equivalent:*

(i) *Each operation, non-empty partial operation and non-empty relation of* $\mathbf{P}$ *determines a subalgebra of a power of* $\mathfrak{P}$.

(ii) *Each operation of* $\mathfrak{P}$ *is a continuous homomorphism from a power of* $\mathbf{P}$ *into* $\mathbf{P}$.

The purpose of this requirement is to secure Lemma 2.2.

We now consider the *(algebraic) quasi variety* $\mathscr{L}=\mathbf{ISP}\mathfrak{P}$ as a category, where for each $\mathfrak{A}, \mathfrak{B}\in\mathscr{L}$, $\mathscr{L}(\mathfrak{A}, \mathfrak{B})$ denotes the set of homomorphisms f: $\mathfrak{A}\to\mathfrak{B}$. Simultaneously we consider the *compact (topological) quasi variety* $\mathscr{R}_0=\mathbf{IS}_c\mathbf{P}\mathbf{P}$ as a category, where for each $\mathbf{X}, \mathbf{Y}\in\mathscr{R}_0$, $\mathscr{R}_0(\mathbf{X}, \mathbf{Y})$ denotes the set of continuous homomorphisms φ: $\mathbf{X}\to\mathbf{Y}$. Notice, since P is a compact Hausdorff space, $\mathbf{X}\in\mathbf{S}_c\mathbf{P}\mathbf{P}$ if and only if $\mathbf{X}$ is a closed substructure of a direct power of $\mathbf{P}$.

Lemma 2.2. (DAVEY and WERNER [7]). (i) *For each* $\mathfrak{A}\in\mathscr{L}$, $\mathscr{L}(\mathfrak{A}, \mathfrak{P})$ *determines a compact substructure of* $\mathbf{P}^A$.

(ii) *For each* $\mathbf{X}\in\mathscr{R}_0$, $\mathscr{R}_0(\mathbf{X}, \mathbf{P})$ *determines a subalgebra of* $\mathfrak{P}^X$.

For each $\mathfrak{A}\in\mathscr{L}$, let $D(\mathfrak{A})$ be the compact substructure of $\mathbf{P}^A$ determined by $D(A)=\mathscr{L}(\mathfrak{A}, \mathfrak{P})$, and for each $\mathbf{X}\in\mathscr{R}_0$, let $E(\mathbf{X})$ be the subalgebra of $\mathfrak{P}^X$ determined by $E(X)=\mathscr{R}_0(\mathbf{X}, \mathbf{P})$. For each $f\in\mathscr{L}(\mathfrak{A}, \mathfrak{B})$ and $g\in D(B)$ define $D(f)(g)= =g\circ f$, and for each $\varphi\in\mathscr{R}_0(\mathbf{X}, \mathbf{Y})$ and $\psi\in E(Y)$ define $E(\varphi)(\psi)=\psi\circ\varphi$.

Lemma 2.3. (DAVEY and WERNER [7]). *D is a contravariant functor, i.e.*

(i) $D(\mathfrak{A})\in\mathscr{R}_0$,

(ii) $D(f)\colon D(\mathfrak{B}) \to D(\mathfrak{A})$,

(iii) $D(f\circ g) = D(g)\circ D(f)$.

Lemma 2.4 (DAVEY and WERNER [7]). *E is a contravariant functor, i.e.*

(i) $E(\mathbf{X})\in\mathscr{L}$,

(ii) $E(\varphi)\colon E(\mathbf{Y}) \to E(\mathbf{X})$,

(iii) $E(\varphi\circ\psi) = E(\psi)\circ E(\varphi)$.

This setting suggests a natural concept of duality. E is called a *dual representation* for $\mathscr{L}$ if E is *onto objects,* i.e. for each $\mathfrak{A}\in\mathscr{L}$ there exists $\mathbf{X}\in\mathscr{R}_0$ such that $\mathfrak{A}\cong E(\mathbf{X})$. DAVEY and WERNER [7] set up a situation where such a dual representation is achieved *canonically.* For each $\mathfrak{A}\in\mathscr{L}$ and $a\in A$ define the projection $e_{\mathfrak{A}}(a)=\pi_a\colon D(\mathfrak{A})\to\mathbf{P}$, and for each $\mathbf{X}\in\mathscr{R}_0$ and $x\in X$ define the projection $\varepsilon_{\mathbf{X}}(x)=$ $=\pi_x\colon E(\mathbf{X})\to\mathfrak{P}$.

Lemma 2.5 (DAVEY and WERNER [7]). *D and E are adjoint to each other, and $e_{\mathfrak{A}}$ and $\varepsilon_{\mathbf{X}}$ are embeddings in $\mathscr{L}$ and $\mathscr{R}_0$ respectively, i.e.*

(i) $e_{\mathfrak{A}}\colon \mathfrak{A} \,\|\!\!\longrightarrow ED(\mathfrak{A})$.

(ii) $\varepsilon_{\mathbf{X}}\colon \mathbf{X}\,\|\!\!\longrightarrow DE(\mathbf{X})$ *is a topological embedding onto a closed subspace of* $\mathbf{P}^{E(X)}$.

(iii) *For each* $h\in\mathscr{L}(\mathfrak{A},\mathfrak{B})$ *and* $\varphi\in\mathscr{R}_0(\mathbf{X},\mathbf{Y})$, *the following diagrams commute:*

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{\;h\;} & \mathfrak{B} \\ {\scriptstyle e_{\mathfrak{A}}}\big\Downarrow & & \big\Downarrow{\scriptstyle e_{\mathfrak{B}}} \\ ED(\mathfrak{A}) & \xrightarrow[ED(h)]{} & ED(\mathfrak{B}) \end{array} \qquad \begin{array}{ccc} \mathbf{X} & \xrightarrow{\;\varphi\;} & \mathbf{Y} \\ {\scriptstyle \varepsilon_{\mathbf{X}}}\big\Downarrow & & \big\Downarrow{\scriptstyle \varepsilon_{\mathbf{Y}}} \\ DE(\mathbf{X}) & \xrightarrow[DE(\varphi)]{} & DE(\mathbf{Y}) \end{array}$$

(iv) *There is a one-to-one correspondence between* $\mathscr{L}(\mathfrak{A},E(\mathbf{X}))$ *and* $\mathscr{R}_0(\mathbf{X},D(\mathfrak{A}))$ *defined by the commuting diagrams*

$$\begin{array}{ccc} \mathfrak{A} & \|\!\!\xrightarrow{\;e_{\mathfrak{A}}\;} & ED(\mathfrak{A}) \\ & {\scriptstyle g}\searrow & \big\downarrow{\scriptstyle E(\varphi)} \\ & & E(\mathbf{X}) \end{array} \qquad \begin{array}{ccc} \mathbf{X} & \|\!\!\xrightarrow{\;\varepsilon_{\mathbf{X}}\;} & DE(\mathbf{X}) \\ & {\scriptstyle \varphi}\searrow & \big\downarrow{\scriptstyle D(g)} \\ & & D(\mathfrak{A}) \end{array}$$

i.e. $g=E\big(D(g)\circ\varepsilon_{\mathbf{X}}\big)\circ e_{\mathfrak{A}}$ and $\varphi=D\big(E(\varphi)\circ e_{\mathfrak{A}}\big)\circ\varepsilon_{\mathbf{X}}$.

In this setting DAVEY and WERNER [7] define their notions of duality and full duality.

Definition 2.6 (DAVEY and WERNER [7]). (D,E) is called a *duality* if for every $\mathfrak{A}\in\mathscr{L}$, $e_{\mathfrak{A}}\colon \mathfrak{A}\,\|\!\!\twoheadrightarrow ED(\mathfrak{A})$ is an isomorphism.

Clearly, if (D,E) is a duality, then E is a dual representation for $\mathscr{L}$. Moreover, in this case for each $\mathfrak{A}\in\mathscr{L}$ there is a *canonical* choice for a representative in $\mathscr{R}_0$, namely $D(\mathfrak{A})$. Thus all members of the (algebraic) quasi variety $\mathbf{ISP}\mathfrak{P}$ are *uniformly* represented as algebras of continuous functions.

The notion of "full duality" concerns the "uniqueness" of the representation. For this purpose DAVEY and WERNER [7] consider full subcategories $D(\mathscr{L})\subseteq\mathscr{S}\subseteq\mathscr{R}_0$.

Definition 2.7 (Davey and Werner [7]). Let (D, E) be a duality and let $D(\mathscr{L}) \subseteq \mathscr{S} \subseteq \mathscr{R}_0$ be a full subcategory. (D, E) is called *full* between $\mathscr{L}$ and $\mathscr{S}$ if for every $\mathbf{X} \in \mathscr{S}$, $\varepsilon_{\mathbf{X}}$: $\mathbf{X} \Vdash\!\!\rightarrow DE(\mathbf{X})$ is a topological isomorphism.

Now it appears to us that this relativized notion of full duality is rather misleading because in a sense it is completely superfluous. This situation then tends to distract from the real "issue of full duality". To say more precisely what we mean, let $D(\mathscr{L}) \subseteq \mathscr{S} \subseteq \mathscr{R}_0$ be a full subcategory. $E_{\mathscr{S}}$ denotes the restriction of E to $\mathscr{S}$, and is called a *category anti-equivalence* between $\mathscr{S}$ and $\mathscr{L}$ if it is onto objects, *full* (i.e. for any $\mathbf{X}, \mathbf{Y} \in \mathscr{S}$ and any h: $E(\mathbf{Y}) \rightarrow E(\mathbf{X})$, there exists φ: $\mathbf{X} \rightarrow \mathbf{Y}$ such that $h = E(\varphi)$) and *faithful* (i.e. for any $\varphi, \psi \in \mathscr{S}(\mathbf{X}, \mathbf{Y})$, if $E(\varphi) = E(\psi)$ then $\varphi = \psi$). In this setting the last condition is actually redundant:

Lemma 2.8. *E is faithful.*

Proof. Suppose $E(\varphi) = E(\psi)$, where φ, ψ: $\mathbf{X} \rightarrow \mathbf{Y}$, and let χ: $\mathbf{Y} \Vdash\!\!\rightarrow \mathbf{P}^I$. Then for each $i \in I$, $\pi_i \circ \chi \in E(Y)$. Thus for all $x \in X$,

$$E(\varphi)(\pi_i \circ \chi)(x) = E(\psi)(\pi_i \circ \chi)(x),$$

$$\pi_i \chi \varphi(x) = \pi_i \chi \psi(x),$$

$$\chi \varphi(x) = \chi \psi(x),$$

$$\varphi(x) = \psi(x).$$

This establishes that $\varphi = \psi$.

Similarly we say that D is a category anti-equivalence between $\mathscr{L}$ and $\mathscr{S}$ if it is onto objects, full and faithful. Again we may forget about the last condition.

Lemma 2.9. *D is faithful.*

Proof. Similar to the proof of Lemma 2.8.

Lemma 2.10. *Suppose (D, E) is a duality and $D(\mathscr{L}) \subseteq \mathscr{S} \subseteq \mathscr{R}_0$ is a full subcategory. If (D, E) is full between $\mathscr{L}$ and $\mathscr{S}$, then D: $\mathscr{L} \rightarrow \mathscr{S}$ and $E_{\mathscr{S}}$: $\mathscr{S} \rightarrow \mathscr{L}$ are both category anti-equivalences which are inverse to each other in the sense that*

$$ED(\mathfrak{A}) \cong \mathfrak{A} \quad \textit{for all} \quad \mathfrak{A} \in \mathscr{L}; \quad DE(\mathbf{X}) \cong \mathbf{X} \quad \textit{for all} \quad \mathbf{X} \in \mathscr{S}.$$

Proof. By Definitions 2.6, 2.7 and Lemmas 2.8, 2.9 both D and E are onto objects and faithful. To show that D is full, let $\mathfrak{A}, \mathfrak{B} \in \mathscr{L}$ and φ: $D(\mathfrak{B}) \rightarrow D(\mathfrak{A})$. Define $h = e_{\mathfrak{B}}^{-1} \circ E(\varphi) \circ e_{\mathfrak{A}}$. Then h: $\mathfrak{A} \rightarrow \mathfrak{B}$ and by Lemma 2.5 (iii),

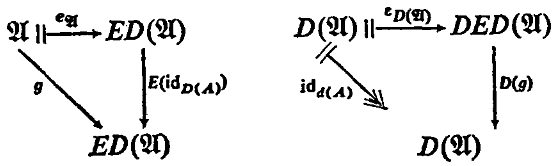
$$E(\varphi) = e_{\mathfrak{B}} \circ h \circ e_{\mathfrak{A}}^{-1} = ED(h).$$

It follows from Lemma 2.8 that $\varphi = D(h)$. This establishes that D is full, and to show that E is full we argue similarly using Lemma 2.9.

Now if E is a dual representation for $\mathscr{L}$ then the "natural notion of uniqueness" is category anti-equivalence. Again in the Davey—Werner setting more is required: Each $\mathfrak{A}\in\mathscr{L}$ can be *canonically* recaptured from its canonical representation $D(\mathfrak{A})$. This is an extremely tight connection so that the next (purely category theoretical) observation is not surprising.

Lemma 2.11. *Suppose (D, E) is a duality and $D(\mathscr{L})\subseteq\mathscr{S}\subseteq\mathscr{R}_0$ is a full subcategory. Then (D, E) is full between $\mathscr{L}$ and $\mathscr{S}$ if and only if $\mathbf{I}\mathscr{S}=\mathbf{I}D(\mathscr{L})$.*

Proof. If (D, E) is full between $\mathscr{L}$ and $\mathscr{S}$, then $\mathbf{I}\mathscr{S}=\mathbf{I}D(\mathscr{L})$ by definition. Conversely, consider $A\in\mathscr{L}$. In Lemma 2.5 (iv) set $\mathbf{X}=D(\mathfrak{A})$ and $\varphi=\mathrm{id}_{D(A)}$. Then we obtain the following commuting diagrams:



By Lemma 2.4, $E(\mathrm{id}_{D(A)})=\mathrm{id}_{ED(A)}$ and therefore $g=\mathrm{id}_{ED(A)}\circ e_{\mathfrak{A}}=e_{\mathfrak{A}}$. It follows that $D(g)$ is bijective and hence $\varepsilon_{D(\mathfrak{A})}$ is bijective. Now we obtain at once from Lemma 2.5 (iii) that for any $\mathbf{X}\in\mathbf{I}D(\mathscr{L})$, $\varepsilon_{\mathbf{X}}\colon \mathbf{X} \Vdash\!\!\rightarrow DE(\mathbf{X})$ is a topological isomorphism.

This observation reveals that any duality is full in exactly one way. The "issue of full duality" appears to be the task of *identifying* the category $\mathbf{I}D(\mathscr{L})$, i.e., to give a *comprehensible description* of the topological structures belonging to this category *in terms of their topology and their structure*. It does not appear that the Davey—Werner definition of full duality is helpful in this respect. In an attempt to carry out this task we shall first give a description of the category $\mathbf{I}D(\mathscr{L})$ which is completely independent from the category theoretical construction, i.e. which does not involve the adjoint contravariant functors D and E. For this purpose we introduce the notion of hull-kernel closed subset of a power of P. This notion plays an important role in sheaf representation and has been investigated in a much more general context in Krauss and Clark [16]. However, for our purposes it suffices to consider a limited version which we shall give a self contained treatment. For each non-empty set S, let $\mathfrak{F}_S$ be the subalgebra of $\mathfrak{P}^{(P^S)}$ ($\mathbf{ISP}\mathfrak{P}$-freely) generated by the set $\{\pi_s \mid s\in S\}$ of projections. For $\sigma, \tau\in F_S$ define

$$\mathrm{Eq}(\sigma, \tau) = \{x\in P^S \mid \sigma(x) = \tau(x)\}$$

and for an arbitrary subset $X\subseteq P^S$ define the *hull-kernel closure* $\bar{X}$ of X by

$$\bar{X} = \bigcap\{\mathrm{Eq}(\sigma, \tau) \mid X \subseteq \mathrm{Eq}(\sigma, \tau)\}.$$

In other words, $y \in \bar{X}$ if and only if $\sigma(y)=\tau(y)$ whenever $\sigma, \tau \in F_S$ and $\sigma(x)=\tau(x)$ for all $x \in X$. In particular, each member of $\pi_X F_S$ has a unique extension in $\pi_{\bar{X}} F_S$ so that $\pi_X \mathfrak{F}_X \cong \pi_{\bar{X}} \mathfrak{F}_S$. X is called *hull-kernel closed* if $X=\bar{X}$.

Lemma 2.12. *For any* $X, Y \subseteq P^S$,

(i) $X \subseteq \bar{X}$,

(ii) $\bar{\bar{X}}=\bar{X}$,

(iii) $X \subseteq Y$ *implies* $\bar{X} \subseteq \bar{Y}$.

Lemma 2.13. *Every hull-kernel closed subset of* P^S *is closed in the product topology.*

First we shall give a characterization of hull-kernel closed sets which will reveal the role of partial operations in the topological structures of similarity type $\mathbf{t}$.

Lemma 2.14. *Suppose* $S \neq \emptyset$ *and* $X \subseteq P^S$. *If for every* $f\colon \pi_X \mathfrak{F}_S \to \mathfrak{P}$ *there exists* $x \in X$ *such that* $f=\pi_x$, *then* X *is hull-kernel closed.*

Proof. If $y \in \bar{X}$, we can define $f\colon \pi_X \mathfrak{F}_S \to \mathfrak{P}$ by $f(\pi_X \sigma)=\sigma(y)$. Choose $x \in X$ so that $f=\pi_x$. Then for any $s \in S$,

$$x(s) = \pi_s(x) = \pi_x(\pi_s) = f(\pi_X(\pi_s)) = \pi_s(y) = y(s)$$

so that $y=x \in X$.

Now suppose $\mathfrak{A} \subseteq \mathfrak{P}^I$ and $f\colon \mathfrak{A} \to \mathfrak{P}$. We can view f as an I-place partial operation on P. For each non-empty S we can canonically lift f to an I-place partial operation $\bar{f}$ on P^S. The domain of $\bar{f}$ is defined by

$$\bar{A} = \{x \in (P^S)^I \mid \pi_s \circ x \in A \quad \text{for all} \quad s \in S\},$$

and $\bar{f}\colon \bar{A} \to P^S$ is defined by $\bar{f}(x)(s)=f(\pi_s \circ x)$. We call $X \subseteq P^S$ closed under $\bar{f}$ if $\bar{f}(x) \in X$ whenever $x \in X^I \cap \bar{A}$.

Lemma 2.15. *Suppose* $S \neq \emptyset$ *and* $X \subseteq P^S$. *Then* X *is hull-kernel closed if and only if* X *is closed under every* $\bar{f}$, *where* $\mathfrak{A} \in \mathbf{SP}\mathfrak{P}$ *and* $f\colon \mathfrak{A} \to \mathfrak{P}$.

Proof. Assume X is hull-kernel closed, $\mathfrak{A} \subseteq \mathfrak{P}^I$, $f\colon \mathfrak{A} \to \mathfrak{P}$ and $x \in X^I \cap \bar{A}$. Since $\mathfrak{F}_S$ is generated by $\{\pi_s \mid s \in S\}$, each member of F_S is of the form $\tau^{\mathfrak{F}_S}(\pi_{s_0}, \ldots, \pi_{s_{n-1}})$, where τ is an n-place term. So suppose

$$X \subseteq E(\tau^{\mathfrak{F}_S}(\pi_{s_0}, \ldots, \pi_{s_{n-1}}), \sigma^{\mathfrak{F}_S}(\pi_{s_0}, \ldots, \pi_{s_{n-1}})).$$

Then

$$\tau^{\mathfrak{A}}(\pi_{s_0} \circ x, \ldots, \pi_{s_{n-1}} \circ x)\colon I \to P$$

and therefore for each $i \in I$,

$$\tau^{\mathfrak{A}}(\pi_{s_0} \circ x, \ldots, \pi_{s_{n-1}} \circ x)(i) = \tau^{\mathfrak{A}}(\pi_{s_0} x(i), \ldots, \pi_{s_{n-1}} x(i)) =$$

$$= \tau^{\mathfrak{P}}(x(i)(s_0), \ldots, x(i)(s_{n-1})) = \tau^{\mathfrak{F}_S}(\pi_{s_0}, \ldots, \pi_{s_{n-1}})(x(i)) =$$

$$= \sigma^{\mathfrak{F}_S}(\pi_{s_0}, \ldots, \pi_{s_{n-1}})(x(i)) = \sigma^{\mathfrak{A}}(\pi_{s_0} \circ x, \ldots, \pi_{s_{n-1}} \circ x)(i).$$

Thus

$$\tau^{\mathfrak{F}_S}(\pi_{s_0}, \ldots, \pi_{s_{n-1}})(\bar{f}(x)) = \tau^{\mathfrak{P}}(\bar{f}(x)(s_0), \ldots, \bar{f}(x)(s_{n-1})) =$$

$$= \tau^{\mathfrak{P}}(f(\pi_{s_0} \circ x), \ldots, f(\pi_{s_{n-1}} \circ x)) = f(\tau^{\mathfrak{A}}(\pi_{s_0} \circ x, \ldots, \pi_{s_{n-1}} \circ x)) =$$

$$= f(\sigma^{\mathfrak{A}}(\pi_{s_0} \circ x, \ldots, \pi_{s_{n-1}} \circ x)) = \sigma^{\mathfrak{F}_S}(\pi_{s_0}, \ldots, \pi_{s_{n-1}})(\bar{f}(x)).$$

This establishes that $\bar{f}(x) \in \bar{X} = X$.

Conversely, suppose for all $\mathfrak{A} \in \mathbf{SP}\mathfrak{P}$ and $f\colon \mathfrak{A} \to \mathfrak{P}$, X is closed under $\bar{f}$. Consider $\pi_X \mathfrak{F}_S \subseteq \mathfrak{P}^X$ and $f\colon \pi_X \mathfrak{F}_S \to \mathfrak{P}$. Define $x \in P^S$ by $x(s) = f(\pi_X(\pi_s))$. We shall show that $x \in X$ and $f = \pi_x$. So consider $\overline{\pi_X F_S} \subseteq (P^S)^X$ and $\bar{f}\colon \overline{\pi_X F_S} \to P^S$. Let $\tau(y) = y$ for all $y \in X$. Clearly $\pi_s \circ \tau \in \pi_X F_S$ for all $s \in S$ and therefore $\tau \in \overline{\pi_X F_S}$. By hypothesis, $\bar{f}(\tau) \in X$ and for each $s \in S$,

$$\bar{f}(\tau)(s) = f(\pi_s \circ \tau) = f(\pi_X(\pi_s)) = x(s).$$

Thus $x = \bar{f}(\tau) \in X$. Finally, for each $s \in S$,

$$f(\pi_X(\pi_s)) = x(s) = \pi_s(x) = \pi_x(\pi_X(\pi_s)).$$

Since $\mathfrak{F}_S$ is generated by $\{\pi_s \mid s \in S\}$, $f = \pi_x$. By Lemma 2.14, X is hull-kernel closed.

Corollary 2.16. *Every hull-kernel closed set $X \subseteq P^S$, where $S \neq \emptyset$, determines a substructure $\mathbf{X} \subseteq \mathbf{P}^S$.*

Proof. Suppose $g \in \mathbf{POp}$. By the requirement Lemma 2.1 (i), $g^{\mathbf{P}} \subseteq P^{n+1}$ determines a subalgebra of $\mathfrak{F}^{n+1}$. Thus $D = \mathrm{dom}(g^{\mathbf{P}})$ determines a subalgebra $\mathfrak{D} \subseteq \mathfrak{P}^n$ and $g^{\mathbf{P}}\colon \mathfrak{D} \to \mathfrak{P}$. Thus by Lemma 2.15, X is closed under $g^{\mathbf{P}^S}\colon \bar{D} \to P^S$.

Lemma 2.17. *For every $\mathfrak{A} \in \mathscr{L}$, $D(A) \subseteq P^A$ is hull-kernel closed.*

Proof. Consider $\mathfrak{F}_A \subseteq \mathfrak{P}^{(P^A)}$. To simplify notation, we take $\mathfrak{P} = \langle P, + \rangle$, where $+$ is binary. Then for any $a, b \in A$, $\pi_a, \pi_b, \pi_{a+b} \in F_A$. Thus for any $f \in D(A)$,

$$\pi_{a+b}(f) = f(a+b) = f(a) + f(b) = \pi_a(f) + \pi_b(f) = (\pi_a + \pi_b)(f).$$

This shows that $D(A) \subseteq \mathrm{Eq}(\pi_{a+b}, \pi_a + \pi_b)$. It follows that for any $g \in \overline{D(A)}$, $\pi_{a+b}(g) = (\pi_a + \pi_b)(g)$ and therefore $g(a+b) = g(a) + g(b)$. Thus $g \in D(A)$ and $\overline{D(A)} \subseteq D(A)$.

Let $\mathbf{S}_{hk}\mathbf{PP}$ be the class of topological structures $\mathbf{X}$, where for some $S\neq\emptyset$, $X\subseteq P^S$ is hull-kernel closed, and let $\mathcal{R}_{hk}=\mathbf{IS}_{hk}\mathbf{PP}$.

Theorem 2.18. *$D(\mathcal{L})\subseteq\mathcal{R}_{hk}\subseteq\mathcal{R}_0$ is a full subcategory.*

Proof. Use Lemma 2.13, Corollary 2.16 and Lemma 2.17.

Next we shall characterize duality in terms of hull-kernel closed sets.

Lemma 2.19. *For each $f\in D(F_S)$ there exists $x\in P^S$ such that $f=\pi_x$.*

Proof. Define $x\in P^S$ by $x(s)=f(\pi_s)$. Then for any $s\in S$,

$$f(\pi_s) = x(s) = \pi_s(x) = \pi_x(\pi_s).$$

Since $\{\pi_s \mid s\in S\}$ generates $\mathfrak{F}_S$, $f=\pi_x$.

Lemma 2.20. *$\mathfrak{A}\in\mathbf{ISP}\mathfrak{P}$ if and only if there exist non-empty S and hull-kernel closed $X\subseteq P^S$ such that $\mathfrak{A}\cong\pi_X\mathfrak{F}_S$.*

Proof. Suppose $\mathfrak{A}\in\mathbf{ISP}\mathfrak{P}$. Then there exist non-empty S and $f\colon \mathfrak{F}_S\twoheadrightarrow\mathfrak{A}$, and there exist I and $g\colon \mathfrak{A}\,||\!\!\longrightarrow\mathfrak{P}^I$. For each $i\in I$ consider

$$\mathfrak{F}_S\xrightarrow{f}\mathfrak{A}\,||\!\!\xrightarrow{g}\mathfrak{P}^I\xrightarrow{\pi_i}\mathfrak{P}.$$

Let $h_i\to\pi_i\circ g\circ f$. Then $h_i\in D(F_S)$ and by Lemma 2.19, there exist $x_i\in P^S$ such that $h_i=\pi_{x_i}$. Let $X=\{x_i \mid i\in I\}$. Then for any $\sigma,\tau\in F_S$ the following assertions are equivalent:

$$f(\sigma) = f(\tau).$$

$$\text{For all } i\in I,\ \pi_i gf(\tau)=\pi_i gf(\tau).$$

$$\text{For all } i\in I,\ h_i(\sigma)=h_i(\tau).$$

$$\text{For all } i\in I,\ \pi_{x_i}(\sigma)=\pi_{x_i}(\tau).$$

$$\pi_X(\sigma)=\pi_X(\tau).$$

Thus $\mathfrak{A}\cong\pi_X\mathfrak{F}_S\cong\pi_{\bar{X}}\mathfrak{F}_S$, and we may assume that $X=\bar{X}$.

Lemma 2.21. *For any non-empty S and closed $\mathbf{X}\subseteq\mathbf{P}^S$, $\pi_X\mathfrak{F}_S\subseteq E(\mathbf{X})$.*

Proof. Check directly that for each $s\in S$, $\pi_X(\pi_s)\colon X\to P$ is continuous. Since $\mathfrak{F}_S$ is generated by $\{\pi_s \mid s\in S\}$, by induction for each $\tau\in F_S$, $\pi_X(\tau)\colon X\to P$ is continuous. The remainder of the assertion follows from the requirement Lemma 2.1(iii).

Now in Lemma 2.5(iv) set $\mathfrak{A}=\pi_X\mathfrak{F}_S$ and $i\colon \pi_X\mathfrak{F}_S \Vert\!\!\longrightarrow E(\mathbf{X})$ the injection mapping to obtain the following commuting diagrams:

$$\begin{array}{ccc} \pi_X\mathfrak{F}_S & \Vert\!\!\xrightarrow{e_{\pi_X\mathfrak{F}_S}} & ED(\pi_X\mathfrak{F}_S) \\ & {\scriptstyle i}\searrow & \downarrow{\scriptstyle E(\delta_{\mathbf{X}})} \\ & & E(\mathbf{X}) \end{array} \qquad \begin{array}{ccc} \mathbf{X} & \Vert\!\!\xrightarrow{\varepsilon_{\mathbf{X}}} & DE(\mathbf{X}) \\ & {\scriptstyle \delta_{\mathbf{X}}}\searrow & \downarrow{\scriptstyle D(i)} \\ & & D(\pi_X\mathfrak{F}_S) \end{array}$$

Clearly $\delta_{\mathbf{X}}(x)=\pi_x\colon \pi_X\mathfrak{F}_S\to\mathfrak{P}$.

Lemma 2.22. *$\delta_{\mathbf{X}}\colon \mathbf{X}\Vert\!\!\longrightarrow D(\pi_X\mathfrak{F}_S)$ is a topological embedding onto a closed subspace of $P^{\pi_X F_S}$.*

Proof. Adjust the proof of Lemma 2.5 (ii).

Lemma 2.23. *$\delta_{\mathbf{X}}\colon \mathbf{X}\Vert\!\!\longrightarrow D(\pi_X\mathfrak{F}_S)$ is a topological isomorphism if and only if X is hull-kernel closed.*

Proof. If $\delta_{\mathbf{X}}$ is a surjective mapping then, by Lemma 2.14, X is hull-kernel closed. Conversely, assume X is hull-kernel closed. By Lemma 2.22 it suffices to show that $\delta_{\mathbf{X}}$ is a surjective mapping. Indeed, let $f\in D(\pi_X F_S)$ and consider

$$\mathfrak{F}_S \xrightarrow{\pi_X} \pi_X\mathfrak{F}_S \xrightarrow{f} \mathfrak{P}.$$

By Lemma 2.19, there exists $x\in P^S$ such that $f\circ\pi_X=\pi_x$. We shall show that $x\in X$. Consider $\sigma,\tau\in F_S$, where $X\subseteq\mathrm{Eq}(\sigma,\tau)$. Then $\pi_X(\sigma)=\pi_X(\tau)$ and therefore $f(\pi_X(\sigma))=f(\pi_X(\tau))$. Thus $\pi_x\sigma=\pi_x\tau$, i.e., $\sigma(x)=\tau(x)$. This establishes that $x\in\mathrm{Eq}(\sigma,\tau)$, so $x\in\overline{X}=X$. Now we obtain for any $s\in S$,

$$f(\pi_X(\pi_s)) = \pi_x(\pi_s) = \pi_x(\pi_X(\pi_s)).$$

Since $\{\pi_s \mid s\in S\}$ generates $\mathfrak{F}_S$, $f=\pi_x=\delta_{\mathbf{X}}(x)$.

Corollary 2.24. *If X is hull-kernel closed, then $E(\mathbf{X})=\pi_X\mathfrak{F}_S$ if and only if $e_{\pi_X\mathfrak{F}_S}\colon \pi_X\mathfrak{F}_S\Vert\!\!\longrightarrow ED(\pi_X\mathfrak{F}_S)$ is an isomorphism.*

Proof. Returning to the definition of $\delta_{\mathbf{X}}$ we obtain from Lemma 2.23 the following commuting diagrams:

$$\begin{array}{ccc} \pi_X\mathfrak{F}_S & \Vert\!\!\xrightarrow{e_{\pi_X\mathfrak{F}_S}} & ED(\pi_X\mathfrak{F}_S) \\ & {\scriptstyle i}\searrow & \Vert\!\!\downarrow{\scriptstyle E(\delta_{\mathbf{X}})} \\ & & E(\mathbf{X}) \end{array} \qquad \begin{array}{ccc} \mathbf{X} & \Vert\!\!\xrightarrow{\varepsilon_{\mathbf{X}}} & DE(\mathbf{X}) \\ & {\scriptstyle \delta_{\mathbf{X}}}\searrow & \downarrow{\scriptstyle D(i)} \\ & & D(\pi_X\mathfrak{F}_S) \end{array}$$

The assertion follows at once.

Now we obtain the promised characterization of duality in terms of hull-kernel closed sets:

Theorem 2.25. *(D, E) is a duality if and only if for every non-empty S and every hull-kernel closed $X \subseteqq P^S$, $E(\mathbf{X}) = \pi_X \mathfrak{F}_S$.*

Proof. One direction follows immediately from Corollary 2.24, and for the other use Lemmas 2.5, 2.20 and Corollary 2.24.

Notice that in case (D, E) is a duality, then in particular for any non-empty S, $E(\mathbf{P}^S) = \mathfrak{F}_S$. Next we give a description of the dual category $\mathbf{I}D(\mathscr{L})$ which does not depend on the category theoretical construction:

Theorem 2.26. *Suppose (D, E) is a duality and $D(\mathscr{L}) \subseteqq \mathscr{S} \subseteqq \mathscr{R}_0$ is a full subcategory. Then (D, E) is full between $\mathscr{L}$ and $\mathscr{S}$ if and only if $\mathscr{S} = \mathscr{R}_{hk}$.*

Proof. By Theorem 2.25, for every non-empty S and every hull-kernel closed $X \subseteqq P^S$, $\varepsilon_{\mathbf{X}} = \delta_{\mathbf{X}}$. Thus by Lemma 2.24, (D, E) is full between $\mathscr{L}$ and $\mathscr{R}_{hk}$. The remainder of the assertion follows from Lemma 2.11.

Now the "dilemma of full duality" becomes apparent: The *definition* of hull-kernel closed structures $\mathbf{X} \in \mathbf{S}_{hk}\mathbf{PP}$ does *not* involve the topology of the space X and the structure of $\mathbf{X}$ (in terms of the similarity type $\mathbf{t}$!) but involves the $\mathbf{ISP}\mathfrak{P}$-free structures of similarity type t ! The code to translate between the two similarity types is given by Lemma 2.1. Unfortunately this code is so involved that the description "topological isomorphs of hull-kernel closed substructures of powers of $\mathbf{P}$" considered as a description of the category $\mathscr{R}_{hk}$ *in terms of the topology and the structure of its members* is so circuitous that it is practically incomprehensible. Now the authors have not been able to find a single example of a duality result in this setting where a comprehensible description of the dual category $\mathscr{R}_{hk}$ is given *in terms of the topology and the structure of its members* unless $\mathscr{R}_{hk} = \mathscr{R}_0$. Thus the "issue of full duality" appears to be the question:

"When is $\mathscr{R}_{hk} = \mathscr{R}_0$?"

which by Theorem 2.26 translates into the question:

"When is the duality full between $\mathscr{L}$ and $\mathscr{R}_0$?"

In fact every "full duality result" the authors are aware of implicitly involves showing $\mathscr{R}_{hk} = \mathscr{R}_0$. To make this explicit we shall conclude this section by describing those circumstances of duality which yield $\mathscr{R}_{hk} = \mathscr{R}_0$. Moreover, we shall see that under these circumstances we will always obtain the stronger conclusion $\mathbf{S}_{hk}\mathbf{PP} = \mathbf{S}_c\mathbf{PP}$, i.e. *closed subspaces of powers of $\mathbf{P}$ are hull-kernel closed.* Since all of this is relevant only in case (D, E) is a duality, we shall state an important result of Davey and WERNER [7] which yields duality in all cases considered by them. The key is their *interpolation condition:*

(IC) For all non-empty finite T and $\mathbf{X} \subseteq \mathbf{P}^T$, every ψ: $\mathbf{X} \to \mathbf{P}$ is the restriction of a T-ary term function of $\mathfrak{P}$ (i.e. $\pi_X \mathfrak{F}_T = E(\mathbf{X})$).

Theorem 2.27 (DAVEY and WERNER [7]). *If* $\mathbf{Op} \cup \mathbf{Rl}$ *is finite and* (IC) *holds, then* (D, E) *is a duality.*

Most applications of this theorem can be obtained from a special case which was proven independently by the authors and R. McKenzie. A $(k+1)$-ary term τ is called a *near-unanimity term* for $\mathfrak{P}$ if for any $a, b \in P$,

$$\tau^P(a, b, \dots, b) = \tau^P(b, a, b, \dots, b) = \dots = \tau^P(b, \dots, b, a) = b.$$

This notion was introduced by BAKER and PIXLEY [2]. They prove that (IC) holds in case $\mathfrak{P}$ has a $(k+1)$-ary near-unanimity term and $\mathbf{P}$ is chosen to have *all* subalgebras of $\mathfrak{P}^k$ as relations.

Corollary 2.28 (Clark, Krauss and McKenzie). *Suppose* $\mathfrak{P}$ *has a* $(k+1)$*-ary near unanimity term and let* $\mathbf{P} = \langle P, r \rangle_{r \in S\mathfrak{P}^k}$. *Then* (D, E) *is a duality.*

Now we shall investigate circumstances yielding $\mathcal{R}_{hk} = \mathcal{R}_0$ under the hypothesis of Theorem 2.27, covering all "full duality results" that have come to our attention. We start with a strengthening of Theorem 2.27:

Theorem 2.29. *If* $\mathbf{Op} \cup \mathbf{Rl}$ *is finite and* (IC) *holds, then for every non-empty* S *and every closed* $\mathbf{X} \subseteq \mathbf{P}^S$, $E(\mathbf{X}) = \pi_X \mathfrak{F}_S$.

Proof. It is easy to verify that this is what DAVEY and WERNER [7] actually prove, although they don't state it.

Corollary 2.30. *If* $\mathbf{Op} \cup \mathbf{Rl}$ *is finite and* (IC) *holds, then* $\mathcal{R}_{hk} = \mathcal{R}_0$ *if and only if* $S_{hk}\mathbf{PP} = S_c\mathbf{PP}$, *i.e. "closed sets are hull-kernel closed".*

Proof. Suppose $\mathcal{R}_{hk} = \mathcal{R}_0$ and $\mathbf{X} \in S_c\mathbf{PP}$. By Theorem 2.26, $\varepsilon_{\mathbf{X}}$: $\mathbf{X} \Vdash\!\!\rightarrow DE(\mathbf{X})$ and therefore, by Theorem 2.29, $\varepsilon_{\mathbf{X}}$: $\mathbf{X} \Vdash\!\!\rightarrow D(\pi_X \mathfrak{F}_S)$. Thus $\varepsilon_{\mathbf{X}} = \delta_{\mathbf{X}}$ and X is hull-kernel closed by Lemma 2.23.

Thus in the setting determined by the hypothesis of Theorem 2.27, a duality (D, E) is *full* between $\mathcal{L}$ and $\mathcal{R}_0$ if and only if "closed sets are hull-kernel closed." Now DAVEY and WERNER [7] give sufficient conditions for this to occur which we shall look at next. Actually we shall take a little detour and look back at Lemma 2.15. This tells us that taking *all* homomorphisms from subalgebras of *arbitrary* powers of $\mathfrak{P}$ into $\mathfrak{P}$ as (infinitary) partial operations of $\mathbf{P}$ would even yield $S_{hk}\mathbf{PP} = SP\mathbf{P}$. Now to begin with this would force us to introduce *infinitary* partial operations and to consider a similarity type with a *proper* class of partial operation symbols, clearly going beyond the formal scope of our setting. Moreover, the hypothesis of Theorem 2.27 actually commits us to *finitely* many *(finitary!)* partial operation symbols! On the other hand, notice that once we have obtained a duality from Theorem 2.27, then adding finitary $\mathfrak{P}$-homomorphisms to $\mathbf{Op} \cup \mathbf{POp}$ *while keeping* $\mathbf{POp}$ *finite* preserves (IC) and does not affect $\mathrm{S}_{hk}\mathbf{P}\mathfrak{P}$ whereas in general it will "cut down" $\mathrm{S}_c\mathbf{P}\mathbf{P}$. If we finally succeed to obtain $\mathrm{S}_{hk}\mathbf{P}\mathbf{P}=\mathrm{S}_c\mathbf{P}\mathbf{P}$ achieving full duality this way, then in general we will have blown up the similarity type $\mathbf{t}$ to a size which is too unwieldy for applications, so that the issue of "reducing the similarity type by removing redundancies" arises. There may also be some practical advantages to "rearranging the similarity type" by treating certain relations as partial operations or operations. Frequently all of this can be accomplished without disturbing full duality. Let F_1, F_2 be sets of operations, G_1, G_2 sets of partial operations and R_1, R_2 sets of relations on P respectively, where $F_2 \subseteq F_1$, $G_2 \subseteq G_1$ and $R_1 \subseteq R_2$. Suppose that $\mathbf{P}_1=\langle P, f, g, r\rangle_{f\in F_1, g\in G_1, r\in R_1}$ and $\mathbf{P}_2=\langle P, f, g, r\rangle_{f\in F_2, g\in G_2, r\in R_2}$ both satisfy the conditions of Lemma 2.1. We say that $F_1 \cup G_1 \cup R_1$ *generates* R_2 if for every non-empty finite T and $\mathbf{X}\subseteq \mathbf{P}_1^T$, if $\psi\colon \mathbf{X}\to\mathbf{P}_1$ then ψ preserves all relations of R_2 (and hence $\psi\colon \mathbf{X}\to\mathbf{P}_2$).

Lemma 2.31. *Suppose* $R_1 \cup G_1 \cup R_1$ *generates* R_2.
(i) *If* $\mathbf{P}_2$ *satisfies* (IC), *then* $\mathbf{P}_1$ *satisfies* (IC).
(ii) *If* $\mathrm{S}_{hk}\mathbf{P}\mathbf{P}_2=\mathrm{S}_c\mathbf{P}\mathbf{P}_2$, *then* $\mathrm{S}_{hk}\mathbf{P}\mathbf{P}_1=\mathrm{S}_c\mathbf{P}\mathbf{P}_1$.

Altogether these observations suggest the following

Full Duality Strategy 2.32. *Step 1:* Choose $\mathbf{t}$ such that $\mathbf{POp}\cup\mathbf{Rl}$ is finite and (IC) holds.

Step 2: Increase $\mathbf{Op}\cup\mathbf{POp}$ keeping $\mathbf{POp}$ finite until $\mathrm{S}_{hk}\mathbf{P}\mathbf{P}=\mathrm{S}_c\mathbf{P}\mathbf{P}$.

Step 3: Decrease $\mathbf{Rl}$, possibly treating certain relations as partial operations or operations, until $\mathbf{Op}\cup\mathbf{POp}\cup\mathbf{Rl}$ is a minimal generating set of the set of relations chosen in Step 1.

All full duality results we have looked at can actually be obtained following the Full Duality Strategy (where in some cases one or more steps may be skipped) and we shall present selected samples later in this section. First we shall give several tests to check whether Step 2 of this strategy has been successfully completed.

Lemma 2.33. *If* $\mathbf{POp}=\emptyset$, *then the following three conditions are equivalent:*
(i) $\mathrm{S}_{hk}\mathbf{P}\mathbf{P}=\mathrm{S}_c\mathbf{P}\mathbf{P}$.
(ii) *Every substructure of a finite power of* $\mathbf{P}$ *is hull-kernel closed.*
(HK) *For every non-empty finite* T, $\mathbf{X}\subseteq\mathbf{P}^T$ *and* $y\in P^T-X$, *there are two* T*-ary term functions of* $\mathfrak{P}$ *which agree on* X *but not at* y.

Proof. (HK) is just a full statement of (ii), and (i) implies (ii) trivially. So assume (HK) and let $\mathbf{X}\subseteq \mathbf{P}^S$ be closed, $z\in P^S-X$. Then there is a basic clopen set

$$U_T = \{x\in P^S | \pi_T z = \pi_T x\},$$

where $T\subseteq S$ is finite, $z\in U_T$ and $X\cap U_T=\emptyset$. It follows that $y=\pi_T z\notin \pi_T X$. Now there exists an embedding h: $\mathfrak{F}_T \Vdash\!\!\rightarrow \mathfrak{F}_S$ which sends the projection π_t, considered as a generator of $\mathfrak{F}_T$, to the projection π_t considered as a generator of $\mathfrak{F}_S$. Since $\mathbf{POp}=\emptyset$, $\pi_T X$ determines a substructure of $\mathbf{P}^T$. By hypothesis there exist σ, $\tau\in F_T$ such that σ and τ agree on $\pi_T X$ but not at y. Thus $h(\sigma)$ and $h(\tau)$ agree on X but not at z. This shows that X is hull-kernel closed and (i) is established.

Corollary 2.34. *Suppose* $\mathbf{POp}=\emptyset$, $\mathbf{Rl}$ *is finite and* (IC) *holds. Then the duality* (D, E) *is full between* $\mathscr{L}$ *and* $\mathscr{R}_0$ *if and only if* (HK) *holds.*

Proof. Use Theorem 2.26, Corollary 2.30 and Lemma 2.33.

Davey and Werner [7] proceed somewhat differently introducing the condition.

$(E3)_F$ If T is finite and $\mathbf{X}\subseteq\mathbf{Y}\subseteq\mathbf{P}^T$, where $X\neq Y$, then there are distinct φ, ψ: $\mathbf{X}\to\mathbf{P}$ which agree on X.

Now it turns out that in the presence of (IC) the conditions $(E3)_F$ and (HK) are equivalent:

Lemma 2.35. $(E3)_F$+(IC)⇒(HK)⇒$(E3)_F$.

Proof. Assume $(E3)_F$ and (IC) and let $\mathbf{X}\subseteq\mathbf{P}^T$ and $y\in P^T-X$ where T is non-empty finite. Let $\mathbf{Y}$ be the substructure of $\mathbf{P}^T$ generated by $X\cup\{y\}$. By $(E3)_F$ there are distinct $\varphi, \psi\in E(Y)$ which agree on X, and therefore not at y. By (IC), φ and ψ are restrictions of T-ary term functions of $\mathfrak{P}$.

Next, assume (HK) and let $\mathbf{X}\subsetneqq\mathbf{Y}\subseteq\mathbf{P}^T$, where T is finite. Consider any $y\in Y-X$. By (HK) there are σ, $\tau\in F_T$ which agree on X but not at y. By Lemma 2.21, $\pi_Y\sigma$, $\pi_Y\tau\in E(Y)$. This establishes $(E3)_F$.

Now we obtain an adjusted version of the Second Full-Duality Theorem of Davey and Werner [7]:

Corollary 2.36. *Suppose* $\mathbf{POp}=\emptyset$, $\mathbf{Rl}$ *is finite and* (IC) *holds. Then the duality* (D, E) *is full between* $\mathscr{L}$ *and* $\mathscr{R}_0$ *if and only if* $(E3)_F$ *holds.*

Proof. Use Corollary 2.34 and Lemma 2.35.

In those cases where $\mathbf{POp}\neq\emptyset$ we shall obtain Step 2 of the Full Duality Strategy by a completely different approach. A finite non-trivial algebra $\mathfrak{P}$ is called *filtral* if all congruences on subdirect products of subalgebras of $\mathfrak{P}$ are induced by filters on the index set. Using Jónsson [14] it is not hard to verify that $\mathfrak{P}$ is filtral if and only if it generates a congruence distributive variety and its non-trivial subalgebras are all simple (cf. Krauss [15]). So let $\mathfrak{P}$ be a filtral algebra, let K be the set of non-trivial isomorphisms between non-trivial subalgebras of $\mathfrak{P}$ and let E be the set of elements of P which determine a trivial subalgebra of $\mathfrak{P}$. Consider $\mathbf{P}=\langle P, \eta, e\rangle_{\eta\in K,\, e\in E}$.

Theorem 2.37. $\mathrm{S}_{hk}\mathrm{P}\mathbf{P}=\mathrm{S}_c\mathrm{P}\mathbf{P}$.

Proof. Let $\mathbf{X}\subseteqq\mathbf{P}^S$ be closed, $\mathfrak{A}\subseteqq\mathfrak{P}^I$ and $f\colon \mathfrak{A}\to\mathfrak{P}$. By Lemma 2.15 we must show that X is closed under $\bar{f}$. So consider $x\in X^I$, where $\pi_s\circ x\in A$ for all $s\in S$. We have to show $\bar{f}(x)\in X$, where $\bar{f}(x)(s)=f(\pi_s\circ x)$. If f has constant value $e\in E$ then $f(x)=\bar{e}\in X$, since $\mathbf{X}\subseteqq\mathbf{P}^S$. Otherwise, since $\mathfrak{P}$ is filtral, there is an ultrafilter U on I such that for $a, b\in A$,

$$f(a) = f(b) \quad\text{iff}\quad \mathrm{Eq}\,(a, b)\in U, \quad h(a) = p \quad\text{iff}\quad a^{-1}(p)\in U.$$

Since f and h have the same kernel, there exists an isomorphism $\eta\colon h(\mathfrak{A}) \mapsto f(\mathfrak{A})$ such that $f=\eta\circ h$. It follows that $\bar{f}=\eta^{\mathbf{X}}\circ\bar{h}$, where $\eta^{\mathbf{X}}$ is the canonical extension of η to X. Since $\mathbf{X}\subseteqq\mathbf{P}^S$, X is closed under $\eta^{\mathbf{X}}$, so that $\bar{f}(s)\in X$ iff $\bar{h}(x)\in X$. Since X is closed, for $\bar{h}(x)\in X$ it suffices that for any finite $T\subseteqq S$, $\pi_T\bar{h}(s)\in\pi_T X$. For each $t\in T$ set

$$\bar{h}(x)(t) = h(\pi_t\circ x) = p_t\in P,$$

and let

$$M = \bigcap_{t\in T} (\pi_t\circ x)^{-1}(p_t).$$

Then $M\in U$. Consider any $i\in M$. Then for any $t\in T$,

$$x_i(t) = (\pi_t\circ x)(i) = p_t = \bar{h}(x)(t).$$

This establishes that $\pi_T\bar{h}(x)\in\pi_T X$.

Gathering what we have found we can now state a very general result with many immediate applications. If the finite nontrivial algebra $\mathfrak{P}$ has a $k+1$-ary *near unanimity term* we perform *Step 1* by taking all members of $\mathrm{S}\mathfrak{P}^k$ as relations for $\mathbf{P}$ according to Corollary 2.28. Now Mitschke [18] has shown that such an algebra always generates a congruence distributive variety. Thus $\mathbf{P}$ is also filtral just in case its *non-trivial subalgebras are all simple.* In this case we can do *Step 2* by adding the set K of non-trivial isomorphisms between non-trivial subalgebras of $\mathfrak{P}$ as partial operations of $\mathbf{P}$ and the set E of elements which determine a trivial subalgebra of $\mathfrak{P}$ as constants of $\mathbf{P}$ according to Theorem 2.37. *Step 3* remains as a clean-up operation which uses Lemma 2.31 and relies on more special properties of the algebra $\mathfrak{P}$.

Theorem 2.38. *Suppose $\mathfrak{P}$ has a $k+1$-ary near unanimity term and only simple non-trivial subalgebras. Let F, G and R be sets of operations, partial operations and* *relations on P respectively, where $E \subseteqq F$, $K \subseteqq G$ and $R \subseteqq S\mathfrak{P}^k$. If*

$$\boldsymbol{P} = \langle P, r, g, f \rangle_{r \in R, g \in G, f \in F}$$

satisfies the conditions of Lemma 2.1 and $R \cup G \cup F$ generates $S\mathfrak{P}^k$, then (D, E) is a full duality between **ISP**$\mathfrak{P}$ *and* $\mathbf{IS}_c\mathbf{P}\boldsymbol{P}$.

A non-trivial finite algebra $\mathfrak{P}$ is called a dual discriminator algebra if the dual discriminator on P, defined by

$$d^P(x, y, z) = \begin{cases} x & \text{if} \quad x = y \\ z & \text{if} \quad x \neq y \end{cases}$$

is a term function of $\mathfrak{P}$. This notion is due to FRIED and PIXLEY [9]. In this case the dual discriminator serves as a 3-ary near unanimity term on $\mathfrak{P}$ and forces nontrivial subalgebras to be simple, so Theorem 2.38 applies.

Corollary 2.39. *If $\mathfrak{P}$ is a dual discriminator algebra and*

$$\boldsymbol{P} = \langle P, r, \eta, e \rangle_{r \in S\mathfrak{P}^2,\ \eta \in K,\ e \in E}$$

then (D, E) is a full duality between **ISP**$\mathfrak{P}$ *and* $\mathbf{IS}_c\mathbf{P}\boldsymbol{P}$.

DAVEY and WERNER [7] give many examples of full duality applying their Second Full Duality Theorem. Three of these applications are not correct (quasi primal algebras, weakly associative lattices and median algebras) because in those cases $\boldsymbol{POp} \neq \emptyset$. These erroneous arguments also appear in WERNER [23]. Now it turns out that all three examples are dual discriminator varieties and we can still establish their claims as consequences of Theorem 2.38. As additional applications of Theorem 2.38 we consider primal algebras, quasi primal algebras, distributive lattices (where $\mathfrak{P}$ is a dual discriminator algebra which is not quasi primal) and De Morgan algebras (where $\mathfrak{P}$ is a filtral near unanimity algebra which is not dual discriminator). Finally we give an application of Corollary 2.34 considering semi lattices with unit (where $\mathfrak{P}$ is neither filtral nor near unanimity). The reader will easily convince himself that the remaining examples in DAVEY and WERNER [7] can be treated similarly.

Example 2.40: Primal algebras. A non-trivial finite algebra $\mathfrak{P}$ is called *primal* if every (finitary) operation on $\mathfrak{P}$ is a term function. Clearly a primal algebra has a 3-ary near-unanimity term. Moreover, it is simple and has neither proper subalgebras nor nontrivial automorphisms. Let $\boldsymbol{P} = \langle P \rangle$, i.e. take $\boldsymbol{Op} \cup \boldsymbol{POp} \cup \boldsymbol{Rl} = \emptyset$.

Theorem 2.41. *If $\mathfrak{P}$ is a primal algebra, then (D, E) is a full duality between* **ISP**$\mathfrak{P}$ *and* $\mathbf{IS}_c\mathbf{P}\boldsymbol{P}$.

Proof. $H \cup E = \emptyset$ and the subalgebras of $\mathfrak{P}^2$ are $\mathfrak{P}^2$ and the diagonal of $\mathfrak{P}^2$. It follows at once that $\emptyset$ generates $S\mathfrak{P}^2$. The assertion follows from Theorem 2.38.

Example 2.42: Quasi primal algebras. A non-trivial finite algebra $\mathfrak{Q}$ is called *quasi primal* if the *ternary discriminator* on Q, defined by

$$t^{Q}(x, y, z) = \begin{cases} x & \text{if} \quad x \neq y \\ z & \text{if} \quad x = y \end{cases}$$

is a term function of $\mathfrak{Q}$. Since $d^{Q}(x, y, z) = t^{Q}(x, t^{Q}(x, y, z), z)$, a quasi primal algebra is a dual discriminator algebra and Theorem 2.38 applies. To carry out Step 3 of the Full Duality Strategy in this case, several options appear to be available and we choose the setting of Davey and Werner [7]. Let H be the set of *all* isomorphisms between *all* subalgebras of $\mathfrak{Q}$ together with the empty mapping and let $\mathbf{Q} = \langle Q, \eta, e\rangle_{\eta \in H,\, e \in E}$.

Theorem 2.43. *(D, E) is a full duality between* $\mathbf{ISP}\mathfrak{Q}$ *and* $\mathbf{IS}_c\mathbf{PQ}$.

Proof. The subalgebras of $\mathfrak{Q}^2$ are exactly the direct products of subalgebras of $\mathfrak{Q}$ and the isomorphisms between subalgebras of $\mathfrak{Q}$. It easily follows that $H \cup E$ generates $\mathbf{S}\mathfrak{Q}^2$. The assertion follows from Theorem 2.38.

Example 2.44: Distributive lattices. Let $\mathfrak{D} = \langle \{0, 1\}, \wedge, \vee\rangle$ be the two-element lattice generating the (quasi) variety of distributive lattices. Then

$$(x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$$

defines the dual discriminator on $\{0, 1\}$ and Theorem 2.38 applies. Let $\mathbf{D} = \langle \{0, 1\}, \leqq, 0, 1\rangle$.

Theorem 2.45. *(D, E) is a full duality between* $\mathbf{ISP}\mathfrak{D}$ *and* $\mathbf{IS}_c\mathbf{PD}$.

Proof. $\mathfrak{D}$ has no non-trivial automorphisms and it is easy to verify that $\{\leqq\} \cup \cup \{0, 1\}$ generates $\mathbf{S}\mathfrak{D}^2$. The assertion follows from Theorem 2.38.

Example 2.46: Weakly associative lattices. $\mathfrak{W} = \langle W, \wedge, \vee\rangle$ is called a *weakly associative lattice* if it satisfies the lattice axioms with the exception that the associative laws are replaced by the weak associative laws

$$((x \wedge z) \vee (y \wedge z)) \vee z = z, \quad ((x \vee z) \wedge (y \vee z)) \wedge z = z.$$

This notion is due to Fried and Grätzer [8]. A weakly associative lattice has the *unique bound property* if any two elements have unique upper and lower bounds. Fried and Pixley [9] show that a non-trivial finite weakly associative lattice is a filtral algebra if and only if it is a dual discriminator algebra if and only if it has the unique bound property. So let $\mathfrak{W} = \langle W, \wedge, \vee\rangle$ be a non-trivial finite weakly associative lattice with the unique bound property. Then Theorem 2.38 applies. Let H be the set of *all* isomorphisms between *all* subalgebras of $\mathfrak{W}$ together with the empty mapping, let $\leqq$ be the ordering on any fixed two-element subalgebra of $\mathfrak{W}$ and let $\boldsymbol{W}=\langle W, \leqq, \eta, e\rangle_{\eta\in H,\, e\in E}$.

Theorem 2.47. *(D, E) is a full duality between* $\mathbf{ISP}\mathfrak{W}$ *and* $\mathbf{IS}_c\mathbf{P}\boldsymbol{W}$.

Proof. Werner [23] shows that $\{\leqq\}\cup H\cup E$ generates $\mathbf{S}\mathfrak{W}^2$. The assertion follows from Theorem 2.38.

Example 2.48: Median algebras. Let $\mathfrak{M}_2=\langle\{0, 1\}, d\rangle$, where d is the dual discriminator on $\{0, 1\}$. Then Theorem 2.38 applies. The only automorphism of $\mathfrak{M}_2$ is defined by $0'=1$ and $1'=0$. Let $\boldsymbol{M}_2=\langle\{0, 1\}, \leqq, ', 0, 1\rangle$.

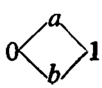
Theorem 2.49. *(D, E) is a full duality between* $\mathbf{ISP}\mathfrak{M}_2$ *and* $\mathbf{IS}_c\mathbf{P}\boldsymbol{M}_2$.

Proof. Werner [23] shows that $\{\leqq\}\cup\{'\}\cup\{0, 1\}$ generates $\mathbf{S}\mathfrak{M}_2^2$. The assertion follows from Theorem 2.38.

Example 2.50: DeMorgan algebras. The (quasi) variety of DeMorgan algebras is generated by the algebra $\mathfrak{M}=\langle\{0, a, b, 1\}, \wedge, \vee, 0, 1, \sim\rangle$ where $\langle\{0, a, b, 1\}, \wedge, \vee, 0, 1\rangle$ is a bounded lattice with a and b incomparable and $\sim$ is the unary operation defined by $\sim 0=1$, $\sim 1=0$, $\sim a=a$, $\sim b=b$. It is easy to check that $\mathfrak{M}$ has only simple subalgebras and

$$m(x, y, z) = (x\wedge y)\vee(x\wedge z)\vee(y\wedge z)$$

is a 3-ary near unanimity term for $\mathfrak{M}$. This time $m^{\mathfrak{M}}$ is not the dual discriminator on $\mathfrak{M}$, and Blok and Pigozzi [4] check that the dual discriminator is not a term function of $\mathfrak{M}$ at all. Now let $\leqq$ be the partial ordering



on $\{0, a, b, 1\}$ and let α be the automorphism of $\mathfrak{M}$ that interchanges a and b. Let $\boldsymbol{M}=\langle\{0, a, b, 1\}, \leqq, \alpha\rangle$.

Theorem 2.51. *(D, E) is a full duality between* $\mathbf{ISP}\mathfrak{M}$ *and* $\mathbf{IS}_c\mathbf{P}\boldsymbol{M}$.

Proof. Davey and Werner [7] verify that $\{\leqq\}\cup\{\alpha\}$ generates all 45 subalgebras of $\mathfrak{M}^2$. The assertion follows from Theorem 2.38.

Example 2.52: Semi lattices with unit. Let $\mathfrak{S}=\langle\{0, 1\}, \wedge, 1\rangle$ be the two-element semi lattice generating the (quasi) variety of semi lattices with unit. Now take $\boldsymbol{S}=\langle\{0, 1\}, \wedge, 1\rangle$.

Theorem 2.53. *(D, E) is a full duality between* $\mathbf{ISP}\mathfrak{S}$ *and* $\mathbf{IS}_c\mathbf{P}\boldsymbol{S}$.

Proof. Davey and Werner [7] verify (IC) and $(E3)_F$, and we verify (HK) directly: Choose T finite, $\boldsymbol{X} \subseteqq \boldsymbol{S}^T$, $y \in S^T - X$. Let

$$z = \bigwedge\{x \in X \mid x \geqq y\}.$$

Now define $\sigma, \tau \in F_T$, by

$$\sigma = \bigwedge\{\pi_t \mid z(t) = 1\}, \quad \tau = \bigwedge\{\pi_t \mid y(t) = 1\}.$$

Since $y \notin X$, $z > y$ so there exists $t \in T$ such that $z(t)=1$ and $y(t)=0$. Thus $\sigma(y)=0$ but $\tau(y)=1$. However for any $x \in X$, if $x \geqq y$ then $x \geqq z$ so that $\sigma(x)=\tau(x)=1$, whereas if $x \ngeqq y$ then $x \ngeqq z$ so that $\sigma(x)=\tau(x)=0$. This establishes (HK). Now use Corollary 2.34.

3. Axioms for topological quasi atomical theories

In this section we shall present examples of (compact) topological quasi varieties and investigate their topological quasi atomical theories.

Example 3.1. A topological space is T_0 if distinct points have distinct neighborhood systems. Consider the empty similarity type $\mathbf{t}$, i.e. $\boldsymbol{Op} \cup \boldsymbol{POp} \cup \boldsymbol{Rl} = \emptyset$, and let $P_0 = \{0, 1\}$ be the *Sierpinski space* with open sets $\emptyset$, $\{0\}$, $\{0, 1\}$. Then for any topological space $\boldsymbol{Y}$ the following are equivalent:

(i) $\boldsymbol{Y} \in \mathbf{ISPP}_0$,

(ii) $\boldsymbol{Y} \models [u_d \xrightarrow{d \in \{0\}} v_0 \wedge v_d \xrightarrow{d \in \{0\}} u_0] \Rightarrow u \approx v$,

(iii) $\boldsymbol{Y}$ is a T_0-space.

Proof. The equivalence of (ii) and (iii) is obvious and (i) implies (ii) by Corollary 1.10. Finally assume (iii). We shall verify conditions (i) and (iv) of the Separation Principle. Suppose $x, y \in Y$, where $x \neq y$. Then, say, there exists an open neighborhood U of x, where $y \notin U$. Define

$$\varphi(z) = \begin{cases} 0 & \text{if } z \in U, \\ 1 & \text{if } z \notin U. \end{cases}$$

This establishes condition (i) of the Separation Principle. Suppose D is a subset of the power set of Y directed by inclusion, $\delta\colon D \to Y$ is a net in Y and $y \in Y$, where $\delta d \overset{d \in D}{\nrightarrow} y$. Then there exists an open neighborhood U of Y such that δ is not eventually in U. Now define φ as before and condition (iv) of the Separation Principle is established. (i) of the assertion now follows from the Separation Principle.

This example shows that the class of T_0-*spaces* is a topological quasi variety in *any* similarity type.

Example 3.2. A topological space is Hausdorff if and only if limits of convergent nets are unique. Thus a topological structure $\mathbf{X}$ (of any similarity type $\mathbf{t}$!) is Hausdorff if and only if for any directed set $\langle D, \leqq\rangle$ and any net $v\colon D\to Vb$, $\mathbf{X}$ is a model of

$$[v_d \xrightarrow{d\in D} u \wedge v_d \xrightarrow{d\in D} w] \Rightarrow u \approx w.$$

Thus the class of topological *Hausdorff* structures is a topological quasi variety in *any* similarity type. While our axioms are simple and uniform they constitute a proper class, and we claim that this is essential.

Lemma 3.3. *For each regular cardinal $\varkappa$ there is a compact topological space $X_\varkappa$ which is not Hausdorff but whose subspaces of cardinality less than $\varkappa$ are zero-dimensional (i.e. have a basis of clopen sets) Hausdorff.*

Proof. For $X_\varkappa$ take the set $\varkappa+2$ with subbasis consisting of all sets $U\cup\{\varkappa\}$ and $U\cup\{\varkappa+1\}$ where $U\subseteqq\varkappa$ is an open interval.

Corollary 3.4. *Let Σ be the topological quasi-atomical theory of Hausdorff spaces. Then every subset Σ_0 of Σ has a model which is not Hausdorff.*

Proof. Let $\varkappa$ be a regular (e.g., successor) cardinal larger than the number of variables occurring in the formulas of Σ_0. We verify that $\mathbf{X}_\varkappa\models\Sigma_0$. Let $\Phi\in\Sigma_0$ and choose $a\colon Vb\to X_\varkappa$. Let $\mathbf{Y}$ be the subspace of $\mathbf{X}_\varkappa$ determined by the images under a of the variables that occur in Φ. Without loss of generality we may assume that $a\colon Vb\to Y$. Since $\mathbf{Y}$ has smaller cardinality than $\varkappa$, $\mathbf{Y}\models\Phi[a]$ by Lemma 3.3. It follows from Lemma 1.9 (i) that $\mathbf{X}_\varkappa\models\Phi[a]$.

Example 3.5. Consider the empty similarity type $\mathbf{t}$, i.e. $\mathbf{Op}\cup\mathbf{POp}\cup\mathbf{Rl}=\emptyset$, and let P be a finite set with at least two elements carrying the *discrete topology*. Then for any topological space $\mathbf{Y}$ the following are equivalent:

(i) $\mathbf{Y}\in\mathbf{ISP}\mathbf{P}$,

(ii) $\mathbf{Y}\models\mathrm{Th}_{\mathrm{tqa}}\mathbf{P}$,

(iii) $\mathbf{Y}$ is a zero-dimensional Hausdorff space.

Proof. The equivalence of (i) and (ii) follows from Corollary 1.10, and (i) implies (iii) trivially. Finally assume (iii). We shall verify conditions (i) and (iv) of the Separation Principle. Suppose $x, y\in Y$, where $x\neq y$. Since $\mathbf{Y}$ is a zero-dimensional Hausdorff space, there exists a clopen neighborhood U of x, where $y\notin U$. Choose $a, b\in P$, where $a\neq b$ and define

$$\varphi(z)=\begin{cases} a & \text{if} \quad z\in U,\\ b & \text{if} \quad z\notin U.\end{cases}$$

This establishes condition (i) of the Separation Principle. To establish condition (iv) we proceed just as in Example 3.1.

This example shows that the class of *totally disconnected Hausdorff* structures is a topological quasi variety in *any* similarity type.

The difference between Examples 3.1 and 3.2 on the one side and Example 3.5 on the other forces us to broach the subject of "axiomatizability" of a topological quasi atomical theory. Intuitively speaking, a theory is axiomatizable in case it is "intelligible" and, roughly speaking, this means that axioms for the theory can be "explicitly written down" in some sense. Well-known technical expliciations of this notion then follow suit, which in the case of infinitary languages (like ours!) require rather sophisticated machinery. Without getting involved in all of this (and we shall not!), it is clear that in Examples 3.1 and 3.2 we have explicitly written down axioms for the topological quasi atomical theories of T_0*-spaces* and of *Hausdorff* spaces respectively, whereas in Example 3.5 we have *not* explicitly written down axioms for the theory of *zero-dimensional Hausdorff* spaces. The reason is simple: We have not been able to. Since further inquiries into this matter require considerations going beyond the scope of this paper, we shall leave it at that.

Example 3.6. Let $\mathbf{t}$ be the similarity type of topological Abelian groups $\langle G, +, -, 0\rangle$, and let $\mathbf{C}$ be the circle group of real numbers modulo the integers with the quotient topology. By Pontryagin's Duality (PONTRYAGIN [21]), for any compact topological structure $\mathbf{Y}$, $\mathbf{Y} \in \mathbf{IS}_c\mathbf{PC}$ if and only if $\mathbf{Y}$ is a compact Abelian group.

Of course, the axioms for Abelian groups (trivially) "axiomatize" the topological quasi atomical theory of topological Abelian groups. However, that the class of compact Abelian groups is generated (as a compact topological quasi variety) by $\mathbf{C}$ is a highly nontrivial observation.

Now we shall turn to the examples of Section 2. Each "full duality result" obtained from Corollaries 2.34, 2.36 and Theorem 2.38 yields two category anti-equivalences

$$D: \mathbf{ISP}\mathfrak{P} \to \mathbf{IS}_c\mathbf{P}\boldsymbol{P}, \quad E: \mathbf{IS}_c\mathbf{P}\boldsymbol{P} \to \mathbf{ISP}\mathfrak{P}$$

between the (algebraic) quasi variety $\mathbf{ISP}\mathfrak{P}$ and the compact (topological) quasi variety $\mathbf{IS}_c\mathbf{P}\boldsymbol{P}$ which are inverse to each other (Lemma 2.10). The primary goal of this kind of "unique representation" is to gain insight into the quasi variety $\mathbf{ISP}\mathfrak{P}$ from ones knowledge of the compact quasi variety $\mathbf{IS}_c\mathbf{P}\boldsymbol{P}$ (at least this appears to be the original motivation for "duality results"!). An obvious prerequisite for success is that one "knows" which topological structures $\boldsymbol{X}$ belong to $\mathbf{IS}_c\mathbf{P}\boldsymbol{P}$. Now in a sense one does because the definition of the quasi variety $\mathbf{IS}_c\mathbf{P}\boldsymbol{P}$ contains a clear description of its members. However, this description is not very helpful to "decide" whether a given topological structure $\boldsymbol{X}$ does belong to $\mathbf{IS}_c\mathbf{P}\boldsymbol{P}$ or not. In fact the Compact Hausdorff Separation Principle (Corollary 1.3), which really just spells out the description "topological isomorph of a compact substructure of a power of $\boldsymbol{P}$", rarely is helpful in this task. What is needed is a description of the members of

$\mathbf{IS}_c\mathbf{PP}$ in terms of their topology and structure which does not involve "constructions". Now in Section 1 we have done exactly that. We have found the "right" language to characterize the members of $\mathbf{IS}_c\mathbf{PP}$ as the *compact models of the topological quasi atomical theory of* $\mathbf{P}$. But in general this is not an "intelligible" description in the sense that it is not possible to "decide" whether a given topological quasi atomic formula Φ of this language does belong to $\mathrm{Th}_{tqa}\mathbf{P}$ or not. What is required is an "*axiomatization*" *of the topological quasi atomical theory* of $\mathbf{P}$. Now we have already run into trouble with that task failing to axiomatize the topological quasi atomical theory of zero dimensional Hausdorff spaces in Example 3.5. Fortunately what is required in the examples arising in the setting of Section 2 is something more special: We need to find an "axiomatization" of the topological quasi atomical theory of the *compact* topological quasi variety $\mathbf{IS}_c\mathbf{PP}$, and this we can do.

Example 3.7: Boolean spaces. Returning to the setting of Example 3.5 we show that we can axiomatize the compact topological quasi variety $\mathbf{IS}_c\mathbf{PP}$ of Boolean spaces. Let BL denote the class of all formulas

$$\{\Phi \mid \Phi \in \Sigma\} \Rightarrow x \approx y$$

where $\mathbf{Y}$ is a compact space, Σ is defined as in Lemma 1.11 and x and y are two points of $\mathbf{Y}$ which are not separated by clopen sets. Then for any topological space $\mathbf{Y}$ the following are equivalent:

(i) $\mathbf{Y} \in \mathbf{IS}_c\mathbf{PP}$;

(ii) $\mathbf{Y}$ is a compact model of BL;

(iii) $\mathbf{Y}$ is a Boolean space.

Proof. The equivalence of (i) and (iii) follows from Example 3.5. Next assume (iii). To prove (ii) it is sufficient, by Lemma 1.10 and (i), to verify that $\mathbf{P} \models BL$. Accordingly consider

$$\{\Phi \mid \Phi \in \Sigma\} \Rightarrow x \approx y$$

in BL as above, $b\colon Vb \to P$ such that $\mathbf{P} \models \Phi[b]$ for each $\Phi \in \Sigma$. By Lemma 1.11 $\varphi\colon \mathbf{Y} \to \mathbf{P}$, where $\varphi(v) = b(v)$. If $bx \neq by$, then $\varphi^{-1}(bx)$ and $\varphi^{-1}(by)$ are clopen sets separating x and y. Thus $bx = by$ and therefore $\mathbf{P} \models (x \approx y)[b]$.

Conversely assume (iii) fails. Since $\mathbf{Y}$ is compact there must be distinct members x and y of $\mathbf{Y}$ which are not separated by clopen sets. But then

$$\{\Phi \mid \Phi \in \Sigma\} \Rightarrow x \approx y$$

is a formula of BL not satisfied by $\mathbf{Y}$, and we conclude that (ii) fails.

As in Example 3.2, the size of the axiom system cannot be reduced.

Corollary 3.8. *Let Σ be the topological quasi-atomical theory of Boolean spaces. Then every subset Σ_0 of Σ has a compact model that is not Boolean.*

Proof. The proof is exactly the same as in Corollary 3.4 since every zero-dimensional Hausdorff space can be embedded in a Boolean space.

The axiom system *BL* will be incorporated into our subsequent examples. Before we continue we should like to make a few comments on the nature of this axiomatization of the topological quasi atomical theory of Boolean spaces. Although it appears to be a "reasonable" axiomatization of the (infinitary) *first-order* theory under consideration, it does not appear to be a *mathematically useful* characterization of Boolean spaces. Now the usual definition of Boolean spaces obviously translates into a simple *higher order* definition in the language under consideration. Thus item (ii) of Example 3.7 may be viewed as a (mathematically useless) *first-order* axiomatization of the compact quasi variety $\mathbf{IS}_c\mathbf{PP}$, whereas item (iii) may be viewed as a (mathematically useful) *higher-order* axiomatization. This pattern will reoccur in all subsequent examples.

Corollary 3.9 (Hu [13]). *If* $\mathfrak{P}$ *is a primal algebra, then* (D, E) *is a full duality between* $\mathbf{ISP}\mathfrak{P}$ *and the category of Boolean spaces.*

Proof. Use Theorem 2.41 and Example 3.7.

Example 3.10: Boolean H-spaces. Return to the setting of Example 2.42. Actually, we shall consider a more general setting where $\mathfrak{Q}$ is an arbitrary non-trivial finite algebra, H is the set of *all* isomorphisms between *all* subalgebras of $\mathfrak{Q}$ together with the empty mapping, and E is the set of elements of Q determining a trivial subalgebra of $\mathfrak{Q}$.

Now take $\mathbf{Q}=\langle Q, \eta, e\rangle_{\eta\in H,\, e\in E}$ and consider each $\eta\in H$ as a partial operation symbol, and each $e\in E$ as an individual constant to determine the similarity type $\mathbf{t}$. Davey and Werner [7] call a topological $\mathbf{t}$-structure $\mathbf{X}=\langle X, \eta^{\mathbf{X}}, e^{\mathbf{X}}\rangle_{\eta\in H,\, e\in E}$ a *Boolean H-space* if

(i) $\mathbf{X}$ is a Boolean space.
(ii) Each $\eta^{\mathbf{X}}$ is a homeomorphism between closed subspaces of $\mathbf{X}$.
(iii) $(\eta\circ\gamma)^{\mathbf{X}}=\eta^{\mathbf{X}}\circ\gamma^{\mathbf{X}}$.
(iv) $(\eta\cap\gamma)^{\mathbf{X}}=\eta^{\mathbf{X}}\cap\gamma^{\mathbf{X}}$.
(v) If η is the identity on Q, then $\eta^{\mathbf{X}}$ is the identity on X.
(vi) $\emptyset^{\mathbf{X}}=\emptyset$.
(vii) If $e\in E$ and η is the identity on $\{e\}$, then $\eta^{\mathbf{X}}$ is the identity on $\{e^{\mathbf{X}}\}$.
(viii) If $e_0, e_1\in E$ and $\eta e_0=e_1$, then $\eta^{\mathbf{X}}e_0^{\mathbf{X}}=e_1^{\mathbf{X}}$.

We need a topological fact whose verification is straightforward.

Lemma 3.11. *Let X and Y be compact Hausdorff spaces,* $X_0\subseteq X$ *and* $g\colon X_0\to Y$. *Then (the graph of) g is closed in* $X\times Y$ *if and only if* X_0 *and* $g(X_0)$ *are both closed and g is continuous.*

Using this lemma we observe that the definition of Boolean H-space translates directly into topological quasi equational axioms:

Lemma 3.12. *Let $\mathfrak{Q}$ be a nontrivial finite algebra. Then a compact $\mathfrak{t}$-structure $\mathbf{X}$ is a Boolean H-space if and only if it is a model of*

(i)′ BL.

(ii)′ $\mathrm{Cl}(\eta)$, $\eta u \approx \eta v \Rightarrow u \approx v$, *where* $\eta \in H$ *and* $\mathrm{Cl}(\eta)$ *is defined in Example 1.6.*

(iii)′ $\eta\delta u \approx v \Leftrightarrow \gamma u \approx v$, *where* $\eta, \delta, \gamma \in H$ *and* $\eta \circ \delta = \gamma$.

(iv)′ $\gamma u \approx v \Leftrightarrow [\eta u \approx v \wedge \delta u \approx v]$, *where* $\gamma, \eta, \delta \in H$ *and* $\gamma = \eta \cap \delta$.

(v)′ $\eta v \approx v$, *where* η *is the identity on* Q.

(vi)′ $\emptyset u \not\approx v$.

(vii)′ $\eta u \approx v \Leftrightarrow [u \approx e \wedge v \approx e]$, *where* η *is the identity on* $\{e\}$.

(viii)′ $\eta e_0 \approx e_1$, *where* $e_0, e_1 \in E$ *and* $\eta e_0 = e_1$.

Proof. The equivalence of (i) and (i)′ is Example 3.7 and the equivalence of (ii) and (ii)′ follows from Lemma 3.11. The remaining equivalences are straightforward.

Our goal is to prove that the topological quasi variety generated by $\mathbf{Q}$ is exactly the class of Boolean H-spaces. This will require a somewhat detailed examination of the consequences of the axioms (i)′–(viii)′ for Boolean H-spaces. While some parts of our argument may be found in Davey and Werner's proof of SEP, a correct proof of the Compact Hausdorff Separation Principle appears to require that we reproduce our argument in full. To do so we introduce some more specialized notation. If $\mathfrak{A} \subseteq \mathfrak{Q}$ and $\mathbf{X}$ is a Boolean H-space, we let "1_A" denote the identity on A and "X_A" the domain of $1_A^{\mathbf{X}}$. Moreover, we write "$1_\emptyset$" for the empty map: $1_\emptyset = \emptyset \in H$, and "$X_\emptyset$" for the domain of $1_\emptyset^{\mathbf{X}}$: $X_\emptyset = \emptyset \subseteq X$.

Lemma 3.13. *Let $\mathbf{X}$ be a Boolean H-space, $\eta \in H$, $\mathfrak{A}$ and $\mathfrak{B}$ subalgebras of $\mathfrak{Q}$.*

(i) $1_A^{\mathbf{X}}$ *is the identity on* X_A.

(ii) *If* η *has domain* A, *then* $\eta^{\mathbf{X}}$ *has domain* X_A.

(iii) $X_{A \cap B} = X_A \cap X_B$.

(iv) $\mathfrak{A} \subseteq \mathfrak{B}$ implies $X_A \subseteq X_B$.

(v) $(\eta^{-1})^{\mathbf{X}} = (\eta^{\mathbf{X}})^{-1}$.

Proof. (i) $1_A \cap 1_Q = 1_A$ so $1_A^{\mathbf{X}} \cap 1_Q^{\mathbf{X}} = 1_A^{\mathbf{X}} \cap 1_X = 1_A^{\mathbf{X}}$. Thus $1_A^{\mathbf{X}} \subseteq 1_X$.

(ii) Let $Y \subseteq X$ be the domain of $\eta^{\mathbf{X}}$. $\eta \circ 1_A = \eta$ so $\eta^{\mathbf{X}} \circ 1_A^{\mathbf{X}} = \eta^{\mathbf{X}}$ so $Y \subseteq X_A$. Then $\eta^{-1} \circ \eta = 1_A$, so $(\eta^{-1})^{\mathbf{X}} \circ \eta^{\mathbf{X}} = 1_A^{\mathbf{X}} \supseteq 1_Y$. It follows that $1_A^{\mathbf{X}} = 1_Y$ so $X_A = Y$.

(iii) $X_{A \cap B} = \mathrm{dom}\, 1_{A \cap B}^{\mathbf{X}} = \mathrm{dom}\, (1_A \cap 1_B)^{\mathbf{X}} = \mathrm{dom}\, (1_A^{\mathbf{X}} \cap 1_B^{\mathbf{X}}) = X_A \cap X_B$.

(iv) Use (iii).

(v) Let η have domain A. Then $\eta^{-1} \circ \eta = 1_A$ so $(\eta^{-1})^{\mathbf{X}} \circ \eta^{\mathbf{X}} = 1_A^{\mathbf{X}}$. It follows that $(\eta^{\mathbf{X}})^{-1} \subseteq (\eta^{-1})^{\mathbf{X}}$. Replacing η by η^{-1}, we obtain $[(\eta^{-1})^{\mathbf{X}}]^{-1} \subseteq \eta^{\mathbf{X}}$ so $(\eta^{-1})^{\mathbf{X}} \subseteq$ $\subseteq (\eta^{\mathbf{X}})^{-1}$.

Lemma 3.14 (DAVEY and WERNER [7], 2.7, (3)). $\mathbf{Q}$ *is injective in the category of Boolean H-spaces.*

Our proof of the Compact Hausdorff Separation Principle will depend on finding "many" continuous homomorphisms from a Boolean H-space $\mathbf{X}$ into $\mathbf{Q}$. We say $x \in X$ is a *fixed point* of $\eta^{\mathbf{X}}$, $\eta \in H$, if x is in the domain of $\eta^{\mathbf{X}}$ and $\eta^{\mathbf{X}} x = x$. Now define

$$H_x = \{\eta \in H \mid x \text{ is a fixed point of } \eta^{\mathbf{X}}\}$$

and let $\mathfrak{S}_x$ be the subalgebra of $\mathfrak{Q}$ consisting of all elements fixed by each member of H_x. Notice that every morphism $\varphi\colon \mathbf{X} \to \mathbf{Q}$ takes x into S_x. Let $E^{\mathbf{X}} = \{e^{\mathbf{X}} \mid e \in E\}$ and let $\mathbf{H}x$ be the substructure of $\mathbf{X}$ generated by x:

$$Hx = \{\eta^{\mathbf{X}} x \mid \eta \in H,\ x \text{ in the domain of } \eta\} \cup E^{\mathbf{X}}.$$

Lemma 3.15. *Let $\mathbf{X}$ be a Boolean H-space, $x \in X$.*

(i) $x \in X_A$ *if and only if* $\mathfrak{S}_x \subseteq \mathfrak{A}$.

(ii) *If* $x \notin E^{\mathbf{X}}$, *then* $|S_x| > 1$.

(iii) *If* $x = e^{\mathbf{X}} \in E^{\mathbf{X}}$, *then* $S_x = \{e\}$.

(iv) *If* $a \in S_x$, *then there is a* $\varphi\colon \mathbf{H}x \to \mathbf{Q}$ *such that* $\varphi(x) = a$.

Proof. (i) If $x \in X_A$ then x is a fixed point of $1_A^{\mathbf{X}}$ so each element of S_x is a fixed point of 1_A. Thus $S_x \subseteq A$. Conversely, let $S_x \subseteq A$. By definition of S_x,

$$1_{S_x} = 1_Q \cap \bigcap \{\eta \mid \eta \in H_x\}$$

so, by (iv),

$$1_{S_x}^{\mathbf{X}} = 1_Q^{\mathbf{X}} \cap \bigcap \{\eta^{\mathbf{X}} \mid \eta \in H_x\}.$$

The right contains (x, x) so the left does as well, and $x \in X_{S_x}$. Now $S_x \subseteq A$ so, by (3.13, iv), $x \in X_{S_x} \subseteq X_A$.

(ii) Since $x \in X_{S_x}$ by (i), and $X_\emptyset = \emptyset$, we obtain $S_x \neq \emptyset$. Suppose $S_x = \{e\}$. Then $e \in E$ so by (3.10, vii) we would have $x \in X_{S_x} = X_{\{e\}} = \{e^{\mathbf{X}}\} \subseteq E^{\mathbf{X}}$.

(iii) Consider $x = e^{\mathbf{X}} \in E^{\mathbf{X}}$. Then $e^{\mathbf{X}}$ is the only fixed point of $1_e^{\mathbf{X}}$ (3.10, vii), so $S_x \subseteq \{e\}$. But if $e^{\mathbf{X}}$ is a fixed point of $\eta^{\mathbf{X}}\colon X_A \to X_B$, then

$$\emptyset \neq 1_e^{\mathbf{X}} = 1_e^{\mathbf{X}} \cap \eta^{\mathbf{X}} = (1_e \cap \eta)^{\mathbf{X}}.$$

Since $\emptyset^{\mathbf{X}} = \emptyset$, $1_e \subseteq \eta$ so $\eta e = e \in S_x$.

(iv) If $x = e^{\mathbf{X}} \in E^{\mathbf{X}}$, then $S_x = \{e\}$ so by (viii) $Hx = E^{\mathbf{X}}$ and we define $\varphi(c^{\mathbf{X}}) = c$ for $c \in E$. Now suppose $c, d \in E$ and $\eta^{\mathbf{X}} c^{\mathbf{X}} = d^{\mathbf{X}}$. Then

$$(c^{\mathbf{X}}, d^{\mathbf{X}}) \in 1_d^{\mathbf{X}} \circ \eta^{\mathbf{X}} \circ 1_c^{\mathbf{X}} = (1_d \circ \eta \circ 1_c)^{\mathbf{X}} \neq \emptyset.$$

Consequently $1_d \circ \eta \circ 1_c \neq \emptyset$ so $\eta(c) = d$ and $\varphi(\eta^{\mathbf{X}} c^{\mathbf{X}}) = \varphi(d^{\mathbf{X}}) = d = \eta(c) = \eta(\varphi c^{\mathbf{X}})$.

Next consider $x \notin E^{\mathbf{X}}$. Then $\{\eta^{\mathbf{X}}x \mid \eta \in H\} \cap E^{\mathbf{X}} = \emptyset$ so that, in view of the preceding paragraph it remains to find an H-preserving map

$$\psi: \{\eta^{\mathbf{X}}x \mid \eta \in H\} \to Q.$$

If η has domain A and $x \in X_A$, the domain of $\eta^{\mathbf{X}}$, then by (i) $a \in A$. Moreover, if $\eta^{\mathbf{X}}x = \gamma^{\mathbf{X}}x$ then x is a fixed point of $(\gamma^{\mathbf{X}})^{-1} \circ \eta^{\mathbf{X}} = (\gamma^{-1} \circ \eta)^{\mathbf{X}}$, by (3.13, v), so a is a fixed point of $\gamma^{-1} \circ \eta$ and $\eta a = \gamma a$. It follows that ψ is well defined by $\psi(\eta^{\mathbf{X}}x) = \eta a$.

Theorem 3.16. *Let $\mathfrak{Q}$ be a nontrivial finite algebra,* $\mathbf{Q} = \langle Q, \eta, e \rangle_{\eta \in H,\, e \in E}$. *For a* **t**-*structure* $\mathbf{X}$ *the following are equivalent:*

(i) $\mathbf{X} \in \mathbf{IS}_c\mathbf{PQ}$.

(ii) $\mathbf{X}$ *is a compact model of* (i)′–(viii)′.

(iii) $\mathbf{X}$ *is a Boolean H-space.*

Proof. (ii) and (iii) are equivalent by Lemma 3.12 and (i) implies (ii) by Corollary 1.10. It remains to show that a Boolean H-space $\mathbf{X}$ satisfies the conditions of the Compact Hausdorff Separation Principle, Corollary 1.3.

We first consider $x, y \in X$ where $x \neq y$. By Lemma 3.14 we need only find $\chi: \mathbf{H}x \cup \mathbf{H}y \to \mathbf{Q}$ separating x and y. If $x, y \in E^{\mathbf{X}}$, then $Hx \cup Hy = E^{\mathbf{X}}$ and we take $\chi e^{\mathbf{X}} = e$ (3.15, iv). Otherwise, assume $x \notin E^{\mathbf{X}}$. If $y \in E^{\mathbf{X}}$ take any $\chi: \mathbf{H}x \to \mathbf{Q}$ (3,15, iv). If $y \notin E^{\mathbf{X}}$ and $Hx \cap Hy = E^{\mathbf{X}}$, take any $\varphi: \mathbf{H}x \to \mathbf{Q}$, $\psi: \mathbf{H}y \to \mathbf{Q}$ and let $\chi = \varphi \cup \psi$.

Finally, suppose $x \notin E^{\mathbf{X}}$, $y \notin E^{\mathbf{X}}$ and $Hx = Hy$. Let $\eta^{\mathbf{X}}x = y$. By (3.15, i) S_x is contained in the domain of η. We claim that some member of S_x is not fixed by η. Indeed if $\eta a = a$ for all $a \in S_x$ then would have $\eta \cap 1_{S_x} = 1_{S_x}$ so $\eta^{\mathbf{X}} \cap 1^{\mathbf{X}}_{S_x} = 1^{\mathbf{X}}_{S_x}$. But $x \in X_{S_x}$ (3.15, i) and $\eta^{\mathbf{X}}x = y \neq x$. Choose $a \in S_x$ so that $\eta a \neq a$. Let $\varphi: \mathbf{H}x \to \mathbf{Q}$ take x to a (3.15, iv). Then $\varphi(y) = \varphi(\eta^{\mathbf{X}}x) = \eta\varphi(x) = \eta a \neq \varphi(x)$.

To verify the second condition, choose x not in the domain $\eta^{\mathbf{X}}$. Let η have domain A. Then $x \notin X_A$. By (3.15, i), $S_x \nsubseteq A$. Choose $a \in S_x - A$ and $\varphi: \mathbf{H}x \to \mathbf{Q}$ taking x to a. By Lemma 3.14 φ extends to a $\psi: \mathbf{X} \to \mathbf{Q}$ where $\psi(x) = a$ is not in the domain of η. As there are no relations we obtain (i) from Corollary 1.3.

Corollary 3.17. *If $\mathfrak{Q}$ is a quasi primal algebra, then (D, E) is a full duality between* $\mathbf{ISP}\mathfrak{Q}$ *and the category of Boolean H-spaces.*

Proof. Use Theorems 2.43 and 3.13.

Example 3.18. Bounded Priestley spaces. Return to the setting of Example 2.44. A topological structure $\langle X, \leqq^{\mathbf{X}} \rangle$ is called a *Priestley space* if X is a partially ordered Boolean space and for each $x, y \in X$, if $x \nleqq^{\mathbf{X}} y$ then there is a clopen increasing set containing x but not y. $\langle X, \leqq^{\mathbf{X}}, 0^{\mathbf{X}}, 1^{\mathbf{X}} \rangle$ is a *bounded Priestley space* if $\langle X, \leqq^{\mathbf{X}} \rangle$ is a Priestley space with bounds $0^{\mathbf{X}}$ and $1^{\mathbf{X}}$. Now for each choice of a bounded partially ordered space $\mathbf{Y}=\langle Y, \leqq^{\mathbf{Y}}, 0^{\mathbf{Y}}, 1^{\mathbf{Y}}\rangle$ where $Y\subseteq Vb$, and any $x, y\in Y$, where $x\not\leqq^{\mathbf{Y}} y$ and every clopen increasing set containing x also contains y, define Σ as in Lemma 1.11 and let ψ be the formula

$$\bigwedge\{\Phi \mid \Phi\in\Sigma\} \Rightarrow x \leqq y.$$

Let BPS denote the class of all such ψ. Then for any topological structure $\mathbf{Y}$ the following are equivalent:

(i) $\mathbf{Y}\in \mathbf{IS}_c\mathbf{PD}$.

(ii) $\mathbf{Y}$ is a compact model of BPS and the axioms for bounded posets.

(iii) $\mathbf{Y}$ is a bounded Priestley space.

Proof. Assume (i). To prove (ii) we must show, by Corollary 1.10, that $\mathbf{D}\models BPS$. Let $\psi\in BPS$ as described above, $b\colon Vb\to D$ where $\mathbf{D}\models\Phi[b]$ for each $\Phi\in\Sigma$. By Lemma 1.11, $\varphi\colon \mathbf{Y}\to\mathbf{D}$ where $\varphi(v)=b(v)$. Since φ is a homomorphism $\varphi^{-1}(1)$ is a clopen increasing set. Now if $\varphi(x)=1$ we conclude $x\in\varphi^{-1}(1)$ so $y\in\varphi^{-1}(1)$. It follows that $\varphi(x)\leqq\varphi(y)$, i.e., $\mathbf{D}\models(x\leqq y)[b]$.

To prove (ii) implies (iii), assume that $\mathbf{Y}$ is a compact bounded partially ordered space that is not a Priestley space. Without loss of generality we assume $Y\subseteq Vb$. Then there are $x, y\in Y$ where $x\not\leqq^{\mathbf{Y}} y$ but every clopen increasing set containing x also contains y. Defining Σ as in Lemma 1.11 we obtain

$$\bigwedge\{\Phi \mid \Phi\in\Sigma\} \Rightarrow x \leqq y$$

in BPS not satisfied by $\mathbf{Y}$. Thus (ii) fails.

Finally, it is easy to see that (iii) gives the conditions of the Compact Hausdorff Separation Principle from which we obtain (i).

Corollary 3.19 (PRIESTLEY [19], [20]). *(D, E) is a full duality between* $\mathbf{ISP}\mathfrak{D}$ *and the category of bounded Priestley spaces.*

Proof. Use Theorem 2.45 and Example 3.18.

We shall omit the axiomatization of the topological quasi atomical theories of $\mathbf{IS}_c\mathbf{PW}$ (Example 2.46) and of $\mathbf{IS}_c\mathbf{PM}_2$ (Example 2.48).

Example 3.20: DeMorgan algebras. We first observe that by omitting all references to the bounds 0 and 1, in the previous example we would obtain a system PS of axioms for *all* Priestley spaces $\langle X, \leqq^{\mathbf{X}}\rangle$. Now return to the setting of Example 2.50. Then for any topological structure $\mathbf{Y}$ the following are equivalent:

(i) $\mathbf{Y}\in\mathbf{IS}_c\mathbf{PM}$,

(ii) $\mathbf{Y}$ is a compact model of PS, the poset axioms and

$$\alpha\alpha u \approx u, \quad u \leqq v \Rightarrow \alpha v \leqq \alpha u.$$

Proof. (i) implies (ii) by Corollary 1.10 and DAVEY and WERNER [7] show that the conditions of the Compact Hausdorff Separation Principle follow immediately from (ii).

Corollary 3.21 (CORNISH and FOWLER [5]). *(D, E) is a full duality between* $\mathbf{ISP}\mathfrak{M}$ *and the category of all Priestley spaces with an order inverting homeomorphism of order two.*

Proof. Use Theorem 2.51 and Example 3.20.

Example 3.22: Boolean semi lattices with unit. Return to the setting of Example 2.52. Then for any topological structure $\mathbf{Y}=\langle Y, \wedge, 1\rangle$ the following are equivalent:

(i) $\mathbf{Y}\in\mathbf{IS}_c\mathbf{P}\mathbf{S}$;

(ii) $\mathbf{Y}$ is a compact model of BL and the axioms for commutative semi lattices with unit.

Proof. By Example 3.7 we only have to show that (ii) implies (i). Under the hypothesis of (ii) DAVEY and WERNER [7] verify the conditions of the Compact Hausdorff Separation Principle for Algebras.

Corollary 3.23 (HOFMANN, MISLOVE and STRALKA [12]). *(D, E) is a full duality between* $\mathbf{ISP}\mathfrak{S}$ *and the category of Boolean semi lattices with unit.*

Proof. Use Theorem 2.53 and Example 3.22.

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