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A general ordering and fixed-point principle in complete metric space

S. DANCS, M. HEGEDŰS and P. MEDVEGYEV

1. In the proof of the celebrated theorem of BISHOP and PHELPS [1] on the density of the set of support points of a bounded closed convex set in a Banach space, a lemma [1, Lemma 1], which can be considered as an ordering principle using essentially the completeness of the space [3], played a central role. The lemma has many generalizations, a maximal one being perhaps the one which is due to EKELAND [6]. The generalizations of the lemma have surprisingly many applications in various branches of mathematics as a survey paper of EKELAND [7] and, without any completness, the papers of BRØNDSTED [3], KIRK [8] and SULLIVAN [9] show.

The purpose of this paper is to show that the different generalizations of the lemma can be considered fundamentally as different forms of a general ordering, fixed point or inductive principle based on the completeness of the metric space. The importance of the different forms are essential from a very pragmatic (and, of course, very significant) point of view: which form fits better the considered problem (see other principles of analysis like e.g. the Hahn—Banach theorem which has many equivalent forms, too).

In the second section of this paper we deal with the equivalence of some well-known forms of the principle, in the third one we give two other forms and a very simple new proof of the principle. In section 4 we show that our new forms seem to fit better the proof of Menger's Theorem than the form of Caristi's fixed point theorem. In section 5 we give an application in measure theory which illustrates the fact that the principle could be a central tool in the theory of measure spaces.

2. Four equivalent forms of the principle. Throughout this section (X, d) will denote a complete metric space, and $\varphi: X \to R \cup \{+\infty\}$ a l.s.c. function, $\not\equiv +\infty$, bounded from below. Firstly we recall four theorems.

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Theorem 2.1. *If $f: X \to X$ is a map satisfying the inequality*

$$d(x, f(x)) \leqq \varphi(x) - \varphi(f(x)) \quad \text{for all} \quad x \in X, \tag{2.1}$$

then f has a fixed point in X.

Theorem 2.2. *There is a point $\bar{x}$ in the space X, for which the inequality*

$$d(\bar{x}, x) > \varphi(\bar{x}) - \varphi(x) \tag{2.2}$$

holds for all $x \in X \setminus \{\bar{x}\}$.

Theorem 2.3. *If $\tilde{x}$ is an arbitrary point of the space X, then there exists a point $\bar{x}$ in X, such that the inequalities*

$$d(\tilde{x}, \bar{x}) \leqq \varphi(\tilde{x}) - \varphi(\bar{x}), \tag{2.3}$$

$$d(\bar{x}, x) > \varphi(\bar{x}) - \varphi(x) \quad \text{for all} \quad x \in X \setminus \{\bar{x}\} \tag{2.4}$$

hold.

Theorem 2.4. *Let ε be an arbitrary positive number and u a point in X such that*

$$\varphi(u) \leqq \inf_{x \in X} \varphi(x) + \varepsilon. \tag{2.5}$$

Then for arbitrary $\lambda > 0$ there exists a point v in X such that the following inequalities hold:

$$\varphi(v) \leqq \varphi(u), \tag{2.6}$$

$$d(u, v) \leqq \lambda, \tag{2.7}$$

$$\varphi(x) > \varphi(v) - (\varepsilon/\lambda)\, d(v, x) \quad \text{for all} \quad x \in X \setminus \{v\}. \tag{2.8}$$

Theorems 2.3 and 2.4 are due to EKELAND [6, 7].

Theorem 2.1 appeared firstly in the paper of CARISTI and KIRK [8] as a theorem of Caristi. A slightly different form of Theorem 2.2 is a corollary of Theorem 2.4 in the paper of EKELAND [7], and is called a weak statement contrary to the strong statement of his Theorem 2.4. The weakness of Theorem 2.2 is, of course, illusory according to the equivalence of the statements. The equivalence (or one or another part of the implications) of the above mentioned theorems are contained, explicitly or implicitly, in EKELAND [7], SULLIVAN [9], BRØNDSTED [3], and so our very simple proofs can be found partly in these papers.

Next we turn to the proof of the equivalences of the above theorems. The logical scheme of our proof is as follows:

$$\begin{array}{ccccc} \text{Theorem 2.1} & \Leftrightarrow & \text{Theorem 2.2} & \Rightarrow & \text{Theorem 2.3} \\ & & \Uparrow & & \Downarrow \\ & & & \text{Theorem 2.4.} & \end{array}$$

Theorem 2.1 $\Rightarrow$ Theorem 2.2. If there would not exist an $\bar{x}$ satisfying (2.2), then for all $x \in X$ there would be a point $f(x) \neq x$ in the space X such that $d(x, f(x)) \leqq \varphi(x) - \varphi(f(x))$, contrary to Theorem 2.1.

Theorem 2.2 $\Rightarrow$ Theorem 2.1. If a point $\bar{x}$ satisfies (2.2), then $\bar{x}$ is a fixed point of each self-map f satisfying (2.1) since otherwise the inequality $d(\bar{x}, f(\bar{x})) > \varphi(\bar{x}) - \varphi(f(\bar{x}))$ would hold, contradicting (2.1).

Theorem 2.2 $\Rightarrow$ Theorem 2.3. The lower semicontinuity of φ implies that the set $S = \{x \in X \mid d(\tilde{x}, x) \leqq \varphi(\tilde{x}) - \varphi(x)\}$ is closed, hence the metric space (S, d) is complete. Applying Theorem 2.2 for the space S we get a point $\bar{x}$ with $d(\tilde{x}, \bar{x}) \leqq \varphi(\tilde{x}) - \varphi(\bar{x})$ and $d(\bar{x}, x) > \varphi(\bar{x}) - \varphi(x)$, for all $x \in S \setminus \{\bar{x}\}$. For Theorem 2.3 we have to show that the last inequality holds in $X \setminus S$, as well. If for $x \in X \setminus S$ the inequality $d(\bar{x}, x) \leqq \varphi(\bar{x}) - \varphi(x)$ would be true, then adding it to the inequality $d(\tilde{x}, \bar{x}) \leqq \varphi(\tilde{x}) - \varphi(\bar{x})$ we would get $d(\tilde{x}, x) \leqq \varphi(\tilde{x}) - \varphi(x)$, contrary to $x \notin S$.

Theorem 2.3 $\Rightarrow$ Theorem 2.4. Applying Theorem 2.3 with the metric $(\varepsilon/\lambda)d$ and $\tilde{x} = u$, we have a point $v = \bar{x}$ such that $(\varepsilon/\lambda)d(v, x) > \varphi(v) - \varphi(x)$ for all $x \in X \setminus \{v\}$, and $(\varepsilon/\lambda)d(u, v) \leqq \varphi(u) - \varphi(v)$. Hence we immediately get (2.6) and (2.8). The inequality $\varphi(u) \leqq \inf \varphi(x) + \varepsilon$ implies $\varphi(u) - \varphi(v) \leqq \varepsilon$; thus $(\varepsilon/\lambda)d(u, v) \leqq \varepsilon$, which gives (2.7), too.

Theorem 2.4 $\Rightarrow$ Theorem 2.2. Taking $\varepsilon = \lambda$ the implication is evident from (2.8).

Remarks. From the proof of the first equivalence one may observe, that the set of the fixed points of the selfmaps satisfying the assumption of Theorem 2.1 coincides with the set of points $\bar{x}$ satisfying (2.2) in Theorem 2.2. This obvious observation shows that all f in Theorem 2.1 have common fixed points.

It is interesting, that the fixed points in Theorem 2.1 can be localized similarly like in Theorem 2.3 or 2.4.

3. Two new forms of the principle. Firstly we will state two equivalent theorems which can be considered as new versions of the principle. We shall prove the first proposition directly, and this proof of the principle seems to be the simplest we have learned till now.

Theorem 3.1. *Let (X, d) be a complete metric space and Φ be a map $X \to 2^X$, which satisfies the following conditions:*

(3.1) *$\Phi(x)$ is a closed set for all $x \in X$.*

(3.2) *$x \in \Phi(x)$ for all $x \in X$.*

(3.3) *$x_2 \in \Phi(x_1) \Rightarrow \Phi(x_2) \subseteqq \Phi(x_1)$ for all $x_1, x_2 \in X$.*

(3.4) *For all sequences $x_1, x_2, \ldots, x_n, \ldots$ in X, that are generalized Picard-iterations, i.e. fulfil*

$$x_2 \in \Phi(x_1),\ x_3 \in \Phi(x_2),\ \ldots,\ x_n \in \Phi(x_{n-1}),\ \ldots$$

the distances $d(x_n, x_{n+1})$ tend to zero as $n \to +\infty$.

Then the map Φ *has a fixed point* $\bar{x}$ *in* X *in the sense* $\Phi(\bar{x})=\{\bar{x}\}$. *(In localized version: For arbitrary* $\tilde{x}\in X$ *there is a fixed point in* $\Phi(\tilde{x})$.*)*

Theorem 3.2. *Let* (X, d) *be a complete metric space with a continuous partial ordering* $\preceq$. *If for each increasing sequence* $x_1\preceq x_2\preceq\dots\preceq x_n\preceq\dots$ *in* X *the distances* $d(x_n, x_{n+1})$ *tend to zero, then there is a maximal element in* X. *(In localized version: For all* $\tilde{x}\in X$ *there is a maximal element in the set* $\{x\in X \mid \tilde{x}\preceq x\}$.*)*

Direct proof of Theorem 3.1. If the distance d satisfies condition (3.4) then the equivalent distance $d'=d/(1+d)$ also does, so we can suppose d is bounded on X. Let us denote the diameter of a subset $A\subset X$ by $\delta(A)$. Because of (3.2) $\Phi(x)\neq\emptyset$ for all $x\in X$, and we can construct a generalized Picard-iteration such that $x_1=\tilde{x}$, $x_n\in\Phi(x_{n-1})$ and

$$d(x_n, x_{n-1}) \geqq \delta(\Phi(x_{n-1}))/2-1/2^{n-1}.$$

Hence from conditions (3.3) and (3.4) we have

$$\Phi(x_{n-1}) \supseteqq \Phi(x_n) \quad\text{and}\quad \delta(\Phi(x_n)) \to 0 \quad\text{as}\quad n\to+\infty.$$

Using the completeness of the space, the non-empty closed sets $\Phi(x_n)$ $(n=1, 2, \dots)$ have a unique common point $\bar{x}$, i.e. $\bigcap_{n=1}^{\infty}\Phi(x_n)=\{\bar{x}\}$. The point $\bar{x}$ is fixed, since on the one hand $\bar{x}\in\Phi(x_n)$ and (3.3) imply $\Phi(\bar{x})\subseteqq\bigcap_{n=1}^{\infty}\Phi(x_n)=\{\bar{x}\}$, and on the other hand from (3.2) we have $\{\bar{x}\}\subseteqq\Phi(\bar{x})$. The localization is trivial from $x_1=\tilde{x}$.

Theorem 3.1 $\Rightarrow$ Theorem 3.2. Let $\Phi(x)=\{y \mid x\preceq y\}$. The relation $y\in\Phi(x)$ is equivalent to $x\preceq y$, hence the reflexivity and the transitivity of the ordering imply (3.2) and (3.3), respectively. From the continuity of the ordering we can conclude that the set $\Phi(x)$ is closed. If $x_1\preceq x_2\preceq\dots\preceq x_n\preceq\dots$, then $x_1, x_2, \dots, x_n, \dots$ is a generalized Picard-iteration, hence all the conditions of Theorem 3.1 are fulfilled, therefore there is a fixed point $\bar{x}$ of Φ, which is obviously maximal in X.

Theorem 3.2 $\Rightarrow$ Theorem 3.1. Define now an ordering $\preceq$ by, $x\preceq y$, iff $y\in\Phi(x)$. From this step on the proof is entirely analogous to the previous one.

Theorem 3.1 $\Rightarrow$ Theorem 2.3. Let $\Phi(x)=\{y \mid d(x, y)\leqq\varphi(x)-\varphi(y)\}$. Since φ is l.s.c., $\Phi(x)$ is closed. Condition (3.2) is satisfied evidently. The summing up of the inequalities $d(x_1, x_2)\leqq\varphi(x_1)-\varphi(x_2)$ and $d(x_2, x_3)\leqq\varphi(x_2)-\varphi(x_1)$ gives (3.3) at once. Similarly, taking the inequalities $d(x_{n-1}, x_n)\leqq\varphi(x_{n-1})-\varphi(x_n)$ $(n=2, 3, \dots)$ and summing them up we have $\sum_{n=2}^{\infty}d(x_{n-1}, x_n)<+\infty$, using the boundedness of φ from below. Applying Theorem 3.1 we have a fixed point $\bar{x}$, and by the definition of Φ the point $\bar{x}$ satisfies (2.2).

The localized version of Theorem 3.1 implies that of Theorem 2.3 in a similar way.

KIRK [8] and SZILÁGYI [10] observed, that forms 2.1 and 2.4 of the principle (Theorems 2.1, 2.4) characterize the completeness of the metric space in some sense. Similarly, we shall prove an analogous result for our forms of the principle.

Theorem 3.3. *If the metric space* (X, d) *is noncomplete, then there is a* Φ *which satisfies conditions* (3.1)—(3.4) *but has no fixed point.*

Proof. From the assumption there is a sequence $X=N_0 \supseteqq N_1 \supseteqq \dots \supseteqq N_n \supseteqq \dots$ of non-empty closed sets in X so that $\delta(N_n) \to 0$, but $\bigcap_{n=1}^{\infty} N_n = \emptyset$. Define the map Φ in the following way:

$$\Phi(x) = N_{i+1} \cup \{x\}, \quad \text{if} \quad x \in N_i \quad \text{and} \quad x \notin N_{i+1}.$$

The map Φ satisfies the assumptions of Theorem 3.1, but has no fixed point, since if $\bar{x}$ were a fixed point of Φ, we would have $\Phi(\bar{x}) = \{\bar{x}\} = N_{i_0} \cup \{\bar{x}\}$, implying $N_{i_0} = \emptyset$, contrary to the assumption.

4. Application in metric convexity. KIRK [8] observed that using the fixed point theorem of Caristi (Theorem 2.1.) it is possible to give a simple proof for Menger's Theorem, a famous theorem on metric convexity. Here we show, that other versions, namely Theorems 3.1 and 3.2 seem to fit even better to prove Menger's Theorem.

Firstly we introduce some notions and notations from distance geometry [2]. Let (Y, d) be a metric space. If for some point $a, b, c \in Y$ we have $d(a, b) = d(a, c) + d(c, b)$, then we say the point c is between the points a and b and use the notation $a\,c\,b$. Similarly the symbol $a_1 a_2 \dots a_s$ means that $d(a_1, a_s) = d(a_1, a_2) + \dots + d(a_{s-1}, a_s)$. It is evident, that the set $\{x\colon a\,x\,b\}$ is closed and it easy to see that the betweenness relation is transitive: $a\,c\,b$ and $a\,d\,c$ imply $a\,d\,b$ (or $a\,d\,c\,b$), more generally $a_i\,b\,a_{i+1}$ and $a_1 a_2 \dots a_s$ imply $a_1 \dots a_i\,b\,a_{i+1} \dots a_s$ and obviously $a_1 a_2 \dots a_s$ implies $a_{i_1} a_{i_2} \dots a_{i_j}$ $(1 \leqq i_1 \leqq i_2 \leqq \dots \leqq i_j \leqq s)$.

The metric space (Y, d) is called convex if for any two points $a, b \in Y$ there is a point c different from a and b such that $a\,c\,b$. The space is called a metric segment space if for any two points $a, b \in Y$ there is an isometric map

$$\varphi\colon [0, d(a, b)] \to \{x\colon a\,x\,b\} \quad \text{for which} \quad \varphi(o) = a \quad \text{and} \quad \varphi(d(a, b)) = b.$$

It is obvious that if the space Y is a metric segment space then it is convex. The converse statement is not generally true but it is true if the space is complete, as it is stated by the following theorem.

Theorem (Menger). *If the metric space* (Y, d) *is complete and convex then it is a metric segment space.*

First proof. It is sufficient to show, that for all $\lambda\in(0, d(a, b))$ there exists an x_λ such that $a\, x_\lambda\, b$ and $d(a, x_\lambda)=\lambda$, since the map $\lambda\to x_\lambda$ is isometric in this case.

Let $\lambda\in(0, d(a, b))$ be a fixed number and put

$$Y_1=\{y\in Y\,|\,d(a, y)\leqq\lambda\}\quad\text{and}\quad Y_2=\{y\in Y\,|\,d(b, y)\leqq d(a, b)-\lambda\}.$$

We shall apply Theorem 3.1 for the complete metric space $(Y_1\times Y_2, \varrho)$, where

$$\varrho((u_1, u_2), (v_1, v_2))=d(u_2, v_2)+d(u_1, v_1).$$

Define the map Φ in the following way:

$$\Phi(y_1, y_2)=\{(u_1, u_2)\in Y_1\times Y_2\,|\,a\, y_1\, u_1\, u_2\, y_2\, b\}.$$

The map Φ satisfies the conditions of Theorem 3.1. The assumptions (3.1) and (3.2) are obviously fulfilled, while (3.3) follows from the transitivity of betweenness: $(u_1, u_2)\in\Phi((y_1, y_2))$ and $(v_1, v_2)\in\Phi((u_1, u_2))$ mean that $a\, y_1\, u_1\, u_2\, y_2\, b$ and $a\, u_1\, v_1\, v_2\, u_2\, b$, thus from transitivity $a\, y_1\, u_1\, v_1\, v_2\, u_2\, y_2$, hence $a\, y_1\, v_1\, v_2\, y_2\, b$, i.e. $(v_1, v_2)\in\Phi((y_1, y_2))$.

If $(y_1^{(n)} y_2^{(n)})\in\Phi((y_1^{(n-1)}, y_2^{(n-1)}))$ is a generalized Picard-iteration then from the transitivity we have $a\, y_1^{(1)}\, y_1^{(2)}\dots y_1^{(n)}\, y_2^{(n)}\, y_2^{(n-1)}\dots y_2^{(1)}\, b$ as before. Hence

$$d(a, y_1^{(1)})+\dots+d(y_1^{(n)}, y_2^{(n)})+\dots+d(y_2^{(1)}, b)=d(a, b),$$

i.e.

$$\varrho((a, b), (y_1^{(1)}, y_2^{(1)}))+\dots+$$
$$+\varrho((y_1^{(n-1)}, y_2^{(n-1)}), (y_1^{(n)}, y_2^{(n)}))+d(y_1^{(n)}, y_2^{(n)})=d(a, b),$$

which yields at once that $\varrho[(y_1^{(n-1)}, y_2^{(n-1)}), (y_1^{(n)}, y_2^{(n)})]$ as $n\to\infty$, i.e. the assumption (3.4) is also satisfied.

According to the theorem we have a fixed point $(\bar y_1, \bar y_2)$, i.e. $\Phi(\bar y_1, \bar y_2)=\{(\bar y_1, \bar y_2)\}$. Now we use the convexity of the space Y to prove that $\bar y_1=\bar y_2$. Assume $\bar y_1\neq\bar y_2$; then there is a w such that $w\neq\bar y_1$, $w\neq\bar y_2$ and $\bar y_1\, w\, \bar y_2$, hence from transitivity we have $a\, \bar y_1\, w\, \bar y_2\, b$ and since $d(a, w)\leqq\lambda$ or $d(w, b)\leqq d(a, b)-\lambda$ holds, $(w, \bar y_2)$ or $(\bar y_1, w)$ is an element of $\Phi(\bar y_1, \bar y_2)$, contradicting the fixed point property. Finally, we get $\bar y_1=\bar y_2=y$. Since $y\in Y_1$, $y\in Y_2$ and $a\, y\, b$, we have $d(a, y)=\lambda$.

Second proof (Sketch). Let $\mathscr{H}$ be the set of isometric maps f to $\{x\,|\,a\, x\, b\}$ having closed domains in $[0, d(a, b)]$ and with $f(0)=a, f(d(a, b))=b$. The set $\mathscr{H}$ is not empty, since it contains the map f_0, for which $\mathrm{dom}\,(f_0)=\{0, d(a, b)\}$ and $f_0(0)=a, f_0(d(a, b))=b$. Each element of $\mathscr{H}$ can be identified with its domain or range. Let us denote by $\mathscr{K}$ the set of closed subsets of the interval $[0, d(a, b)]$, and introduce the Hausdorff-metric h on $\mathscr{K}$. It is well known that the space $(\mathscr{K}, h)$ is complete. From the properties of the Hausdorff-metric one can prove that $\mathcal{H}$ is a closed subset of $\mathcal{K}$. Let us order the elements of $\mathcal{H}$ (or equivalently, the adequate elements of $\mathcal{K}$) according to the set inclusion of the domain of maps. It is easy to see that this ordering is continuous for the metric h in $\mathcal{K}$ and also that it satisfies the last assumption of Theorem 3.2, since if $\operatorname{dom}(f_n)$ $(n=1, 2, \dots)$ is an increasing sequence, then $\sum_{n=1}^{\infty} h(\operatorname{dom}(f_n), \operatorname{dom}(f_{n+1})) \leqq d(a, b)$. The theorem gives a maximal element $\bar{f}$ in $\mathcal{H}$ (with maximal domain in $\mathcal{K}$). If $\operatorname{dom}(\bar{f}) = [0, d(a, b)]$, then Menger's Theorem is proved, otherwise $[0, d(a, b)] \setminus \operatorname{dom}(\bar{f})$ is an open set and contains an open interval (z_1, z_2), $z_1, z_2 \in \operatorname{dom}(\bar{f})$. Now using the convexity we have a point w with $w \neq z_1$, $w \neq z_2$ and $z_1 w z_2$, and so the map $\bar{f}: \bar{f} = \bar{f}$ on $\operatorname{dom}(\bar{f})$ and $f(w) = d(a, w)$ is isometric, contrary to the maximality of $\bar{f}$.

5. Application in measure theory. In the theory of measure and integral there are a lot of ordered complete metric spaces, which satisfy the assumptions of Theorem 3.2. So it is easy to show applications, and therefore our application can only be considered as an illustrative example, but it is worth noting that our proof is easier than the proof of [5] (p. I. 335.).

Firstly we mention some well-known facts from measure theory. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $\mathrm{M}(X, \mathcal{M}, \mu)$ be the space of classes of μ-equivalent real functions on X. Ordering the space $M(X, \mathcal{M}, \mu)$ by

$$f \leqq g \quad \text{iff} \quad f(x) \leqq g(x) \ \mu\text{-a.e.}$$

one may ask whether the lattice $(M, \leqq)$ is complete, i.e. whether all subsets $B \subseteqq M$ having an upper bound in the ordering have a least upper bound $f_0 = \sup B \in M(X, \mathcal{M}, \mu)$. The following famous theorem answers the question affirmatively. We shall deal with a finite measure, and the σ-finite case can be derived from this by standard arguments.

Theorem. *If $(X, \mathcal{M}, \mu)$ is a finite measure space, then $M(X, \mathcal{M}, \mu)$ is a complete lattice.*

Proof. The set M is Frechet-space with the quasi-norm

$$\|f\| = \int_X \frac{|f(x)|}{1+|f(x)|}\, d\mu.$$

A crucial property of this space is, that whenever f_n converges to f_0 then it has a subsequence f_{n_k} $(k=1, \dots)$, which converges μ-a.e. to f_0, and the ordering is continuous.

Let $B \subseteqq M$ be an order-bounded set, and let g be an upper bound of B. If C is the set of the least upper bounds of the finite subsets of B, then $\sup B = \sup C$ obviously, so we can assume that whenever $f_1, f_2 \in B$ then $f_3 = \sup(f_1, f_2)$ is also in B. Let $\bar{B}$ denote the closure of B in the Frechet-space M. We shall prove, that $\bar{B}$ has a maximal element f_0, and f_0 is the least upper bound of B. The metric space $(\bar{B}, \|\cdot\|)$ is complete and the ordering introduced before is continuous.

If f_n $(n=1, 2, \ldots)$ is an increasing sequence in the order bounded set $\bar{B}$, then f_n is convergent a.e., consequently it converges in the quasi-norm, too. According to the above, Theorem 3.2 is applicable and we have a maximal element f_0. Now we shall prove that f_0 is an upper bound for B. Since $f_0 \in \bar{B}$, we have a sequence $f_n \in B$ such that $f_n \to f_0$ both in the quasi-norm and a.e. Hence if $f \in B$, we have $\sup(f_0, f) = \sup(\lim_n f_n, f) = \lim_n [\sup(f_n, f)] \in \bar{B}$. As f_0 is maximal in $\bar{B}$, $f \leqq \sup(f_0, f) = f_0$ holds. Finally let $\tilde{f}$ be an upper bound for B, i.e. $\tilde{f} \geqq f$ for all $f \in B$. Since $f_0 = \lim f_n (f_n \in B), \tilde{f} \geqq \lim_n f_n = f_0$, i.e. f_0 is the least upper bound of B.

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(S. D. and P. M.)
NATIONAL MANAGEMENT DEVELOPMENT CENTRE
KÖNYVES K. KRT. 48—52
1476 BUDAPEST, HUNGARY

(M. H.)
KARL MARX UNIVERSITY OF ECONOMICS
DIMITROV TÉR 8
1093 BUDAPEST, HUNGARY