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Ergodic theorems in von Neumann algebras

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0. Introduction. The classical individual ergodic theorem claims that if $(X, \mathfrak{S}, \mu)$ is a measure space, α is an invertible measure-preserving transformation of X then for every integrable complex function f on X the averages

$$s_n(f) = \frac{1}{n}(f+\alpha f+\ldots+\alpha^{n-1}f) \tag{1}$$

converge μ-almost everywhere to an α-invariant function (where αf is defined by $(\alpha f)(x)=f(\alpha(x))$). In a von Neumann algebra setting one may investigate the convergence of averages of type (1), when f is an element of a von Neumann algebra $\mathfrak{A}$ and α is an automorphism of $\mathfrak{A}$. The first ergodic theorems for automorphisms of von Neumann algebras were established by KOVÁCS and SZŰCS [7], [8] and give that the averages (1) converge strongly provided that $\mathfrak{A}$ has a faithful normal α-invariant state φ. Later LANCE [10] proved an almost uniform ergodic theorem. Namely, if $A\in\mathfrak{A}$ then there exists an element $\hat{A}\in\mathfrak{A}$ such that for every $\varepsilon>0$ there is a projection E in $\mathfrak{A}$ with the property

$$\varphi(I-E) < \varepsilon, \quad s_n(A)E \to \hat{A}E \tag{2}$$

in norm (shortly $s_n(A)\to\hat{A}$ φ-almost uniformly). A similar theorem was obtained by SINAĬ and ANŠELEVIČ [12] in special circumstances (for quantum lattice systems), for several parameters. The crucial point of Lance's proof is a maximal ergodic theorem: if $A\in\mathfrak{A}^+$ and $\varepsilon=\varphi(A)^{1/2}$ then there is an operator $C\in\mathfrak{A}$ such that $s_n(A)\leqq C$ for every $n\in\mathrm{N}$ and $\|C\|\leqq 2\|A\|$, $\varphi(C)\leqq 4\sqrt{\varepsilon}$. This does not have an analogue in the commutative ergodic theory but (and because) it is a simple consequence of Hopf's maximal ergodic theorem.

Further extension of the almost uniform theory has appeared in [2], [4], [14] and [15]. The main objective of this paper is to replace the invariant state with an invariant weight and to obtain a slightly weaker almost uniform convergence. In

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fact, instead of (2) we can prove

(3) $$Es_n(A)E \to E\hat{A}E$$

in norm. Yeadon [16] proved a similar convergence under the condition that there exists a faithful normal semifinite trace. We treat continuous flows and the case of several parameters, as well.

Let $\mathfrak{A}$ be a von Neumann algebra and φ a faithful semifinite normal weight on $\mathfrak{A}^+$. Then $\mathfrak{A}_0=\{A\in\mathfrak{A}: \varphi(A^*A)<+\infty$ and $\varphi(AA^*)<+\infty\}$ becomes a full left Hilbert algebra with *-algebra structure induced by $\mathfrak{A}$ and with inner-product $\langle A, B\rangle_\varphi=\varphi(B^*A)$ $(A, B\in\mathfrak{A}_0)$. Our main reference on this subject is the monograph [13], whose notation we shall follow. Denote by $\mathscr{H}$ the Hilbert space completion of $\mathfrak{A}_0$. For $B\in\mathfrak{A}_0$ one defines an $L_B\in\mathscr{B}(\mathscr{H})$ by the formula $L_BA=BA$ $(A\in\mathfrak{A}_0)$. $\mathscr{L}(\mathfrak{A}_0)=\{L_B: B\in\mathfrak{A}_0\}''$ is called the left von Neumann algebra of $\mathfrak{A}_0$. There is a faithful representation $\pi: \mathfrak{A}\to\mathscr{L}(\mathfrak{A}_0)$ defined by $\pi(A)B=AB$ $(A\in\mathfrak{A}, B\in\mathfrak{A}_0)$ such that for $A\in\mathfrak{A}^+$

$$\varphi(A)=\begin{cases}\|B\|_\varphi^2, & \text{if there exists } B\in\mathfrak{A}_0 \text{ such that } \pi(A)^{1/2}=L_B\\ +\infty, & \text{otherwise.}\end{cases}$$

Here $\|B\|_\varphi^2=\langle B, B\rangle_\varphi$. (See [13], p. 276 or [1].) So we may assume that $\mathfrak{A}$ is the left von Neumann algebra of a full (i.e., achieved) left Hilbert algebra $\mathfrak{A}_0$ and φ is the canonical weight on $\mathscr{L}(\mathfrak{A}_0)^+$.

Suppose that $\mathfrak{A}$ and φ are fixed. A linear mapping $\alpha: \mathfrak{A}\to\mathfrak{A}$ will be called a kernel provided that the following conditions hold:

(i) for $0\leqq A\leqq I$ and $A\in\mathfrak{A}$ we have $0\leqq\alpha(A)\leqq I$ and $\varphi(\alpha(A))\leqq\varphi(A)$,

(ii) for every $A\in\mathfrak{A}$ the inequality $\varphi(\alpha(A)^*\alpha(A))\leqq\varphi(A^*A)$ is valid.

Kernels proved to be useful in ergodic theory. Every Schwarz map satisfying condition (i) is a kernel. In particular, endomorphisms and completely positive maps of norm one are kernels. We are going to see that kernels have some automatic continuity.

1. The maximal ergodic theorem. The proofs of individual ergodic theorems usually need a maximal ergodic theorem. Ours involves a series of operators.

Theorem 1. *Let φ be a faithful semifinite normal weight on a von Neumann algebra $\mathfrak{A}$ and α a linear mapping $\mathfrak{A}\to\mathfrak{A}$ satisfying condition* (i). *Assume that $A_m\in\mathfrak{A}^+$ and $\varepsilon_m>0$ $(m\in\mathbf{N})$. Then there is a projection $E\in\mathfrak{A}$ such that*

(4) $$\|Es_r(A_m)E\|\leqq 2\varepsilon_m \quad (r, m\in\mathbf{N}),$$

(5) $$\varphi(I-E)\leqq 2\sum_{m=1}^{\infty}\varepsilon_m^{-1}\varphi(A_m).$$

We divide the proof into lemmas. We always assume $\mathfrak{A}$ to be a left von Neumann algebra $\mathscr{L}(\mathfrak{A}_0)$ of a full Hilbert algebra $\mathfrak{A}_0$.

Lemma 1. *Under the hypotheses of Theorem 1, for any* $n \in \mathrm{N}$ *there is a projection* $E_n \in \mathfrak{A}$ *such that*

(6) $$\|E_n s_r(A_m) E_n\| \leqq \varepsilon_m \quad (r, m \leqq n),$$

(7) $$\varphi(I - E_n) \leqq \sum_{m=1}^{\infty} \varepsilon_m^{-1} \varphi(A_m).$$

Proof. Let $\mathfrak{A}_0'$ be the right Hilbert algebra associated with $\mathfrak{A}_0$. So $\mathfrak{A}_0' \subset \mathscr{H}$ and for $\eta \in \mathfrak{A}_0'$ the formula $R_\eta \xi = L_\xi \eta$ $(\xi \in \mathfrak{A}_0)$ defines a bounded operator $R_\xi \in \mathscr{B}(\mathscr{H})$. It is well-known that

$$\{R_\eta \colon \eta \in \mathfrak{A}_0'\}' = \mathscr{L}(\mathfrak{A}_0).$$

(See [13] or [14].) Let φ' be the canonical weight on the right von Neumann algebra $\mathscr{R}(\mathfrak{A}_0') = \{R_\eta \colon \eta \in \mathfrak{A}_0'\}''$, that is, for $T \in \mathscr{R}(\mathfrak{A}_0')^+$

$$\varphi'(T) = \begin{cases} \|\eta\|^2, & \text{if there is } \eta \in \mathfrak{A}_0' \text{ such that } R_\eta = T^{1/2} \\ +\infty, & \text{otherwise.} \end{cases}$$

For $A \in \mathscr{L}(\mathfrak{A}_0)^+$ and $T \in \mathscr{R}(\mathfrak{A}_0')^+$ we define $h(A, T)$ if $\varphi(A) < +\infty$ or $\varphi'(T) < +\infty$. Namely, let

$$h(A, T) = \langle T\xi, \xi \rangle \quad \text{if } A^{1/2} = L_\xi \text{ for some } \xi \in \mathfrak{A}_0,$$

$$h(A, T) = \langle A\eta, \eta \rangle \quad \text{if } T^{1/2} = R_\eta \text{ for some } \eta \in \mathfrak{A}_0'.$$

h is *wo*-continuous and additive in each variable separately, and

$$\varphi(A) = \sup\{h(A, T) \colon 0 \leqq T \leqq I,\ T \in \mathscr{R}(\mathfrak{A}_0'),\ \varphi'(T) < +\infty\},$$

$$\varphi'(T) = \sup\{h(A, T) \colon 0 \leqq A \leqq I,\ A \in \mathscr{L}(\mathfrak{A}_0),\ \varphi(A) < +\infty\}.$$

Let $\eta \in \mathfrak{A}_0'$. Then the formula

$$(\xi_1, \xi_2) \mapsto \langle \alpha(L_{\xi_2^\# \xi_1})\eta, \eta \rangle \quad (\xi_1, \xi_2 \in \mathfrak{A}_0)$$

defines a bounded sesquilinear form on $\mathfrak{A}_0$. Since $\varphi(\alpha(L_{\xi_i^\# \xi_i^\#})) \leqq \varphi(L_{\xi_i^\# \xi_i}) = \|\xi_i\|^2 < +\infty$ there is $\mu_i \in \mathfrak{A}_0$ such that $\alpha(L_{\xi_i^\# \xi_i})^{1/2} = \mu_i$ and we have the following estimation.

$$|\langle \alpha(L_{\xi_2^\# \xi_1})\eta, \eta \rangle| \leqq \langle \alpha(L_{\xi_1^\# \xi_1})\eta, \eta \rangle^{1/2} \langle \alpha(L_{\xi_2^\# \xi_2})\eta, \eta \rangle^{1/2} =$$

$$= \langle R_{\eta^b \eta}\mu_1, \mu_1 \rangle^{1/2} \langle R_{\eta^b \eta}\mu_2, \mu_2 \rangle^{1/2} \leqq \|R_{\eta^b \eta}\| \|\mu_1\| \|\mu_2\| \leqq \|R_{\eta^b \eta}\| \|\xi_1\| \|\xi_2\|.$$

Consequently, there is a bounded operator $\bar{\alpha}(R_{\eta b_\eta})\in\mathscr{B}(\mathscr{H})$ such that

(8) $$\langle\bar{\alpha}(R_{\eta b_\eta})\xi_1, \xi_2\rangle = \langle\alpha(L_{\xi_2^\# \xi_1})\eta, \eta\rangle.$$

If $T\in\mathscr{R}(\mathfrak{A}_0')^+$ and $\varphi'(T)<+\infty$ then $\bar{\alpha}(T)\in\mathscr{B}(\mathscr{H})^+$ and $\|\bar{\alpha}(T)\|\leqq\|T\|$. Since $\bar{\alpha}(T)$ commutes with L_ξ for every $\xi\in\mathfrak{A}_0$ we have $\bar{\alpha}(T)\in\mathscr{R}(\mathfrak{A}_0')^+$. Taking $A=L_{\xi^\#\xi}$ and $T=R_{\eta b_\eta}$ ($\xi\in\mathfrak{A}_0, \eta\in\mathfrak{A}_0'$) we obtain $h(A, \bar{\alpha}(T))=h(\alpha(A), T)$ from (8). We can use this to show that $\varphi'(\bar{\alpha}(T))\leqq\varphi'(T)$ for $T\in\mathscr{R}(\mathfrak{A}_0')^+$. Namely,

$$\varphi'(\bar{\alpha}(T)) = \sup\{h(A, \bar{\alpha}(T)) \colon A \leqq I, A\in\mathscr{L}(\mathfrak{A}_0)^+, \varphi(A) < +\infty\} =$$
$$= \sup\{h(\alpha(A), T) \colon A \leqq I, A\in\mathscr{L}(\mathfrak{A}_0)^+, \varphi(A) < +\infty\} \leqq \varphi'(T).$$

$\mathscr{R}(\mathfrak{A}_0')\overline{\otimes}M_n$ is the von Neumann algebra of $n\times n$ matrices with entries from $\mathscr{R}(\mathfrak{A}_0')$. Its elements will be denoted by $(X_{r,m})$, where $X_{r,m}\in\mathscr{R}(\mathfrak{A}_0')$ ($r, m\leqq n$).

$$K = \{(X_{r,m})\in\mathscr{R}(\mathfrak{A}_0')\overline{\otimes}M_n \colon X_{r,m} \geqq 0,\ \Sigma X_{r,m} \leqq I\}$$

is an ultraweakly compact convex set. We define a real function on K in the following fashion:

$$g((X_{r,m})) = \sum_{r=1}^{n}\sum_{m=1}^{n} r[h(S_r(B_m), X_{r,m}) - \varphi'(X_{r,m})]$$

where $B_m\in\mathscr{L}(\mathfrak{A}_0)^+$ is fixed and $\varphi(B_m)<+\infty$ ($m\leqq n$). The function g is ultraweakly upper semicontinuous and attains its finite maximum value for some choice $(X_{r,m})\in K$. If $I-\Sigma X_{r,m}=Z$, $X\in\mathscr{R}(\mathfrak{A}_0')$ and $0\leqq X\leqq Z$ then from the inequality

$$g((X_{r,m})) \geqq g((X_{r,m}+\delta(r, r_0)\delta(m, m_0)X))$$

we obtain

(9) $$h(s_{r_0}(B_{m_0}), X) \leqq \varphi'(X)$$

for every $r_0, m_0\leqq n$.

Now take

$$Y_{r,m} = \begin{cases}\bar{\alpha}(X_{r+1,m}) & \text{for} \quad r \leqq n-1 \\ 0 & \text{for} \quad r = n.\end{cases}$$

The properties of $\bar{\alpha}$ give that $(Y_{r,m})\in K$ and hence $g((X_{r,m}))\geqq g((Y_{r,m}))$. It follows that

$$\sum_{m=1}^{n}\sum_{r=1}^{n}[h(B_m, X_{r,m}) - \varphi'(X_{r,m})] \geqq \sum_{m=1}^{n}\sum_{r=1}^{n}(r-1)[\varphi'(X_{r,m}) - \varphi'(\bar{\alpha}(X_{r,m}))].$$

Replace B_m with $\varepsilon_m^{-1}A_m$. So

(10) $$\sum_{m=1}^{n}\sum_{r=1}^{n}\varepsilon_m^{-1}h(A_m, X_{r,m}) \geqq \sum_{m=1}^{n}\sum_{r=1}^{n}\varphi'(X_{r,m}),$$

and by (9)

(11) $$h(s_r(A_m), X) \leqq \varepsilon_m\varphi'(X) \quad (r, m \leqq n).$$

Let $E_0=\{\eta\in\mathfrak{A}_0'\colon R_\eta^*R_\eta\leqq\lambda Z$ for some $\lambda>0\}$. E_0 is a linear subspace of $\mathscr{H}$. If $\eta\in E_0$ and $\omega\in\mathfrak{A}_0'$ then $R_\omega\eta\in\mathfrak{A}_0'$ and by [13], p. 249, for $Z=R_{R_\omega\eta}$ we have $T^*T=R_\eta^*R_\omega^*R_\omega R_\eta\leqq\|R_\omega\|^2R_\eta^*R_\eta$. So E_0 is stable under the operators R_η $(\eta\in\mathfrak{A}_0')$ and if E_n denotes the orthogonal projection onto the closure of E_0, then $E_n\in\mathscr{L}(\mathfrak{A}_0)$.

If $\eta\in E_0$ then $\langle s_r(A_m)\eta,\eta\rangle=h(s_r(A_m),R_\eta^*R_\eta)\leqq\varepsilon_m\varphi'(R_\eta^*R_\eta)=\varepsilon_m\|\eta\|^2$ according to (11). Therefore we may conclude that $\|E_ns_r(A_m)E_n\|\leqq\varepsilon_m$.

Let F be a projection in $\mathscr{L}(\mathfrak{A}_0)$ such that $F\leqq I-E_n$ and $\varphi(F)<+\infty$. Then

$$\varphi(F)=\sup\{h(F,Z_1+\sum X_{r,m})\colon 0\leqq Z_1\leqq Z,\varphi'(Z_1)<+\infty\}\leqq$$
$$\leqq 0+h(F,\sum X_{r,m})\leqq\varphi'(\sum X_{r,m})\leqq\sum\varepsilon_m^{-1}h(A_m,X_{r,m})\leqq$$
$$\leqq\sum_m\varepsilon_m^{-1}h(A_m,\sum_m X_{r,m})\leqq\sum_{m=1}^n\varepsilon_m^{-1}\varphi(A_m).$$

Since φ is semifinite and lower w-semicontinuous we have $\varphi(I-E_n)=\sum_{m=1}^n\varepsilon_m^{-1}\varphi(A_m)$ and the proof is complete.

Lemma 2. *Under the conditions of Theorem* 1 *there is a* $C\in\mathscr{L}(\mathfrak{A}_0)^+$ *such that* $C\leqq I$ *and*

(12) $$Cs_n(A_m)C\leqq\varepsilon_mC\quad(n,m\in\mathbf{N}),$$

(13) $$\varphi(I-C)\leqq\sum_{m=1}^\infty\varepsilon_m^{-1}\varphi(A_m).$$

Proof. Let E_n be the projection guaranteed by Lemma 1. There is a convergent subsequence (E_{n_k}) of (E_n) and $E_{n_k}\xrightarrow{\text{wo}}C$ for some $C\in\mathscr{L}(\mathfrak{A}_0)$. Evidently $0\leqq C\leqq I$ and by the semicontinuity $\varphi(I-C)\leqq\sum_{m=1}^\infty\varepsilon_m^{-1}\varphi(A_m)$. From $E_ns_r(A_m)E_n\leqq\varepsilon_mE_n$ $(r,m\leqq n)$ a routine argument gives that $Cs_r(A_m)C\leqq\varepsilon_mC$ for every $r,m\in\mathbf{N}$.

Proof of Theorem 1. Take $C\in\mathfrak{A}^+$ with properties (12) and (13) in Lemma 2 and let $\int_0^1\lambda dP(\lambda)$ be the spectral resolution of C. For $E=I-P(1/2)$ we have $I-E=P(1/2)\leqq2(I-C)$ and (5) follows from (13). On the other hand,

$$Es_r(A_m)E=DCs_r(A_m)CD\leqq\varepsilon_mDCD\leqq2\varepsilon_mE$$

where $D=\int_{1/2}^1\lambda^{-1}dP(\lambda)$. This completes the proof.

The first maximal ergodic theorem similar to Theorem 1 was obtained by YEADON [16] for a trace instead of a general weight and for a single operator instead of a sequence. A version for state and for a sequence appeared in GOLDSTEĬN's paper [4]. Here we utilized several of their ideas. If $A_m=0$ for $m>1$ then the theorem claims the existence of a projection possessing the properties $\|Es_r(A)E\| \leqq 2\varepsilon$ and $\varphi(I-E) \leqq 2\varepsilon^{-1}\varphi(A)$. In the commutative case this is equivalent to the inequality

$$\mu(\{x: \sup s_n(f)(x) > \lambda\}) \leqq \frac{c}{\lambda}\|f\|_1,$$

which frequently occurs in commutative ergodic theory (see, for example, [3], p. 705).

As a matter of fact, Theorem 1 implies Lance's maximal ergodic theorem. Namely,

$$s_r(A) \leqq 2Es_r(A)E + 2(I-E)s_r(A)(I-E) \leqq 4\varepsilon E + 2\|A\|(I-E) = C_\varepsilon.$$

If $\|A\| \leqq 1$ and $\varepsilon = \varphi(A)^{1/2}$ then $\|C_\varepsilon\| \leqq 2$ and $\varphi(C_\varepsilon) \leqq 8\varphi(A)^{1/2}$. We notice that if $\varphi(I) = +\infty$ then the assertion of Lance's maximal ergodic theorem is false even in the commutative case.

2. An individual ergodic theorem. In this paragraph we are going to use Theorem 1 to deduce the following

Theorem 2. *Let φ be a faithful semifinite normal weight on a von Neumann algebra $\mathfrak{A}$ and α a kernel on $\mathfrak{A}$. Assume that $A \in \mathfrak{A}$ and $\varphi(A^*A) < +\infty$, $\varphi(AA^*) < +\infty$. Then there is $\hat{A} \in \mathfrak{A}$ such that for every $\varepsilon > 0$ there exists a projection E in $\mathfrak{A}$ satisfying the following conditions*

$$\varphi(I-E) < \varepsilon \quad \textit{and} \quad \left\|E(s_n(A) - \hat{A})E\right\| \to 0.$$

Moreover $\varphi(\hat{A}^*\hat{A}) < +\infty$ and $\varphi(\hat{A}\hat{A}^*) < +\infty$.

We notice that Lance proved $s_n(A) \to \hat{A}$ ultrastrongly in [10].

Lemma 3. *Suppose that $B \in \mathfrak{A}^{sa}$ and $\varphi(B^2) < +\infty$. Then there is a decomposition $B = C + D - E$ where $C \in \mathfrak{A}^{sa}$, $D \in \mathfrak{A}^+$, $E \in \mathfrak{A}^+$, $\|C\| \leqq \varphi(B^2)^{1/2}$, $\varphi(D) \leqq \varphi(B^2)^{1/2}$, $\varphi(E) \leqq \varphi(B^2)^{1/2}$ and $\|C\|$, $\|D\|$, $\|E\| \leqq \|B\|$.*

Proof. Let $\int_{-\infty}^{\infty} \lambda dP(\lambda)$ be the spectral resolution of B. Take $C = \int_{-\varepsilon}^{\varepsilon} \lambda dP(\lambda)$, $D = \int_{\varepsilon}^{\infty} \lambda dP(\lambda)$ and $E = -\int_{-\infty}^{-\varepsilon} \lambda dP(\lambda)$ where $\varepsilon = \varphi(B^2)^{1/2}$. Then $D, E \leqq \varepsilon^{-1}B^2$ and all the requirements are fulfilled.

Lemma 4. *For $B \in \mathfrak{A}$ we have $\left\|s_n(B - s_k(B))\right\| \leqq kn^{-1}\|B\|$ if $n > k$.*

Proof. It is straightforward from the identity

$$ns_n(B - s_k(B)) = \sum_{i=0}^{k-2} \alpha^i(B) - \sum_{i=0}^{k-2} \frac{i+1}{k}\alpha^i(B) - \sum_{i=0}^{k-2} \frac{k-1-i}{k}\alpha^{n+i}(B) \quad (k > 1).$$

Proof of Theorem 2. For $\xi \in \mathfrak{A}_0$ let $V\xi = \mu$ where $\mu \in \mathfrak{A}_0$ and $\alpha(L_\xi) = L_\mu$. Since $\varphi(\alpha(L_\xi)^* \alpha(L_\xi)) \leqq \varphi(L_{\xi^\# \xi}) = \|\xi\|^2$ such a μ exists and V can be extended to a contraction on $\mathscr{H}$. By the mean ergodic theorem for a contraction ([11], p. 144) there is a projection $P \in \mathscr{B}(\mathscr{H})$ such that $n^{-1} \sum_{i=0}^{n-1} V^i \xi \to P\xi$ for every $\xi \in \mathfrak{A}_0$. If $f \in \mathscr{L}(\mathfrak{A}_0)_*$ and $f(B) = \langle B\eta_1, \eta_2 \rangle$ for some $\eta_1, \eta_2 \in \mathfrak{A}_0'$ then

$$\Phi(f) = \lim \langle s_n(A)\eta_1, \eta_2 \rangle = \lim \left\langle R_{\eta_1} n^{-1} \sum_{i=0}^{n-1} V^i \xi_0, \eta_2 \right\rangle =$$

$$= \langle R_{\eta_1} P\xi_0, \eta_2 \rangle = \langle P\xi_0, \eta_2 \eta_1^b \rangle$$

where $A = L_{\xi_0}$. Since $|\Phi(f)| \leqq \|A\| \, \|\eta_1\| \, \|\eta_2\| = \|S\| \, \|f\|$ there is an element $\hat{A} \in \mathscr{L}(\mathfrak{A}_0)$ with the property

$$\langle \hat{A}\eta_1, \eta_2 \rangle = \langle R_{\eta_1} P\xi_0, \eta_2 \rangle$$

for every $\eta_1, \eta_2 \in \mathfrak{A}_0'$. Similarly,

$$\langle (\hat{A})^* \eta_1, \eta_2 \rangle = \langle R_{\eta_1} P\xi_0^\#, \eta_2 \rangle.$$

Hence $\hat{A}\eta = R_\eta P\xi_0$ and $(\hat{A})^*\eta = R_\eta P\xi_0^\#$ for every $\eta \in \mathfrak{A}_0'$. By [13], p. 252, we may conclude $P\xi_0 \in \mathfrak{A}_0'' = \mathfrak{A}_0$ and $\hat{A} = L_{P\xi_0}$. Consequently, $s_n(A) \xrightarrow{w} \hat{A}$ and $\alpha(\hat{A}) = \hat{A}$. According to the mean ergodic theorem,

$$\xi_0 - P\xi_0 = \xi_0 - k^{-1} \sum_{i=0}^{k-1} V^i \xi_0 + \xi_k \quad (k \in \mathbf{N}),$$

where $\|\xi_k\| = \delta_k$ and $\delta_k \to 0$. By the left representation L we have

$$A - \hat{A} = A - s_k(A) + B_k$$

where $\|B_k\|_2 = \varphi(B_k^* B_k)^{1/2} = \|\xi_k\| = \delta_k$. If $A = A^*$ then $B_k = B_k^*$, and by splitting into selfadjoint and skewadjoint parts we arrive at the decomposition

$$A - \hat{A} = A - s_k(A) + B_k^1 + iB_k^2 \tag{14}$$

and here $B_k^1, B_k^2 \in \mathscr{L}(\mathfrak{A}_0)^{sa}$ and $\|B_k^1\|_2, \|B_k\|_2 \leqq \delta_k$. Apply Lemma 3 for B_k^1 and B_k^2. So

$$A - \hat{A} = A - s_k(A) + C_k^1 + D_k^1 + E_k^1 + i(C_k^2 + D_k^2 - E_k^2) \tag{15}$$

and $\|C_k^i\| \leqq \delta_k$, $\varphi(D_k^i) \leqq \delta_k$, $\varphi(E_k^i) \leqq \delta_k$ $(i = 1, 2)$. Choose a subsequence (δ_{m_k}) of (δ_k) such that $\delta_{m_k} < 16 \cdot k^{-1} 2^{-k} \cdot \varepsilon$ and use Theorem 1. Taking $\{A_m\} = \bigcup_{k=1}^{\infty} \{D_{m_k}^1, E_{m_k}^1, D_{m_k}^2, E_{m_k}^2\}$ and putting $1/k$ in the role of ε corresponding to $D_{m_k}^1, E_{m_k}^1, D_{m_k}^2$ and $E_{m_k}^2$, we obtain a projection E such that

$$\|Es_n(D^i_{m_k})E\| \leqq 2k^{-1} \quad (i = 1, 2,\ k\in \mathbf{N},\ n\in \mathbf{N}),$$

$$\|Es_n(E^i_{m_k})E\| \leqq 2k^{-1} \quad (i = 1, 2,\ k\in \mathbf{N},\ n\in \mathbf{N}),$$

$$\varphi(I-E) \leqq 4\cdot 2 \sum_{k=1}^{\infty} k\delta_{m_k} \leqq \varepsilon.$$

In order to prove $\|E(s_n(A)-\hat{A})E\|\to 0$ we can estimate in the following way:

$$\|Es_n(A-\hat{A})E\| \leqq \left\|s_n(A-s_{n_k}(A))\right\|+2(\delta_{m_k}+4k^{-1}) \leqq 2n^{-1}m_k\|A\|+10k^{-1}.$$

(Lemma 4 was used to estimate the first term.) This inequality shows the required result.

We notice that the proof has given a little more than what was formulated in the theorem. Since the mean ergodic theorem is valid even for power-bounded operators instead of property (ii) of kernels, the weaker condition

(ii$_0$) there is a $C>0$ such that for every $n\in \mathbf{N}$ and $A\in \mathfrak{A}$, $\varphi(\alpha^n(A^*)\alpha^n(A))\leqq$ $\leqq C\varphi(A^*A)$ fulfils

would have been sufficient. However, in the really interesting cases, when α is an automorphism or a completely positive map, condition (i) implies condition (ii).

3. Results on several kernels. Let $\mathfrak{A}$ be a von Neumann algebra and φ a faithful semifinite normal weight on $\mathfrak{A}^+$. If α_i: $\mathfrak{A}\to\mathfrak{A}$ is a kernel for $i\leqq k$ then

$$s^i_n(A) = \frac{1}{n}\sum_{l=0}^{n-1} \alpha^l_i(A)$$

converges in some sense to a limit $\Phi^i(A)\in \mathfrak{A}$ provided that $\varphi(A^*A)$ and $\varphi(AA^*)$ are finite. The joint behaviour of several kernels in von Neumann algebras was investigated by CONZE and DANG-NGOC [2]. This paragraph generalizes some results from [2], where φ is assumed to be a state.

Theorem 3. *Let $\mathfrak{A}, \varphi, \alpha_i, \Phi^i, s^i_n$ $(i\leqq k)$ be as above. If $A_m\in \mathfrak{A}^+$ and $\varepsilon_m>0$ $(m\in \mathbf{N})$ then there is a projection E in $\mathfrak{A}$ such that*

$$\|Es^k_{n_k}\dots s^1_{n_1}(A_m)E\| \leqq C(k, A_m)\varepsilon_m, \tag{16}$$

$$\varphi(I-E) \leqq 2^{k+1} \sum_{m=1}^{\infty} \varepsilon_m^{-k}\varphi(A_m), \tag{17}$$

where $C(1, A_m)=2$ and $C(k+1, A_m)=2C(k, A_m)+4\|A_m\|$.

Proof. For $k=1$ this is Theorem 1. By induction there is a projection E_m such that

$$\|E_m s^{k-1}_{n_{k-1}}\dots s^1_{n_1}(A_m)E_m\| \leqq C(k-1, A_m)\varepsilon_m,\ \varphi(I-E_m) \leqq 2^k\varepsilon_m^{-k+1}\varphi(A_m).$$

Apply Theorem 1 with α_k, $I-E_m$ and ε_m $(m\in\mathbf{N})$. We obtain a projection E with the following properties:

$$\|Es_{n_k}^k(I-E_m)E\| \leqq 2\varepsilon_m,$$

$$\varphi(I-E) \leqq 2\sum_{m=1}^{\infty}\varepsilon_m^{-1}\varphi(I-E_m) \leqq 2^{k+1}\sum_{m=1}^{\infty}\varepsilon_m^{-k}\varphi(A_m).$$

Hence E satisfies (17), and we verify (16).

$$Es_{n_k}^k\dots s_{n_1}^k(A_m)E \leqq$$

$$\leqq Es_{n_k}^k\big(2E_m s_{n_{k-1}}^{k-1}\dots s_{n_1}^1(A_m)E_m+2(I-E_m)s_{n_{k-1}}^{k-1}\dots s_{n_1}^1(A_m)(I-E_m)\big) \leqq$$

$$\leqq 2C(k-1, A_m)\varepsilon_m I+2\|A_m\|\, Es_{n_k}^k(I-E_m)E \leqq 2C(k-1, A_m)\varepsilon_m I+4\varepsilon_m\|A_m\|\, I.$$

Corollary. *Let* $\mathfrak{A}, \varphi, \alpha_i, \Phi^i, s_n^i$ $(i\leqq k)$ *be the same as above. Suppose that* $\varphi(I)=1$ *and* $A\in\mathfrak{A}^+$. *Then there exists an operator* $C\in\mathfrak{A}$ *such that*

$$s_{n_k}^k\dots s_{n_1}^1(A) \leqq C$$

for every $n_1, n_2, \dots, n_k\in\mathbf{N}$. *Moreover,*

$$\|C\| \leqq \delta\|A\|, \quad \varphi(C) \leqq \varepsilon\varphi(A)^{1/k+1}\|A\|^{k/k+1}$$

where ε *and* δ *are constant (depending only on* k*).*

Proof. Assume that $\|A\|=1$ and apply Theorem 3 in the case $A_1=A$, $\varepsilon_1=\varphi(A)^{1/k+1}$ and $A_m=0$ for $m>1$. Then

$$Es_{n_k}^k\dots s_{n_1}^1(A)E \leqq C(k, A)\varphi(A)^{1/k+1},$$

$$\varphi(I-E) \leqq 2^{k+1}\varphi(A)\varphi(A)^{-k/k+1} = 2^{k+1}\varphi(A)^{1/k+1}.$$

Therefore $s_{n_k}^k\dots s_{n_1}^1(A)\leqq 2C(k, A)\varphi(A)^{1/k+1}+2(I-E)=C_1$ and

$$\|C_1\| \leqq 2C(k, A)\varphi(A)^{1/k+1}+2 \leqq 2C(k, A)+2,$$

$$\varphi(C_1) \leqq 2C(k, A)\varphi(A)^{1/k+1}+2^{k+2}\varphi(A)^{1/k+1}.$$

Now we have $\|C_1\|\leqq\delta$ and $\varphi(C_1)\leqq\varepsilon\varphi(A)^{1/k+1}$.

In the general case one can obtain an operator C_1 for $A\|A\|^{-1}$ as above and take $C=\|A\|C_1$. So C satisfies both requirements.

Theorem 4. *Let* $\mathfrak{A}, \varphi, \alpha_i, \Phi^i, s_n^i$ $(i\leqq k)$ *be as above. Suppose that* $A\in\mathfrak{A}$ *and* $\varphi(A^*A), \varphi(AA^*)$ *are finite. Then for every* $\varepsilon>0$ *there is a projection* $E\in\mathfrak{A}$ *such that* $\varphi(I-E)<\varepsilon$ *and*

$$\left\|E\big(s_{n_k}^k\dots s_{n_1}^1(A)-\Phi^k\dots\Phi^1(A)\big)E\right\| \to 0$$

if $n_1\to\infty, \dots, n_k\to\infty$ *independently.*

Proof. We follow the lines of the proof of Theorem 2 but use Theorem 3 instead of Theorem 1. For the sake of simplicity we assume that $k=2$.

Similarly to (15) we have the following decompositions on the basis of Lemma 3:

$$A = \big(A - s_l^1(A)\big) + \Phi^1(A) + C_l^1 + D_l^1 - E_l^1 + i(C_l^2 + D_l^2 - E_l^2),$$
$$\Phi^1(A) = \big(\Phi^1(A) - s_l^2\Phi^1(A)\big) + \Phi^2\Phi^1(A) + C_l^3 + D_l^3 + E_l^3 + i(C_l^4 + D_l^4 - E_l^4).$$

Here $0 \leqq D_l^i, E_l^i, \|C_l^i\| \leqq \delta_l$, $\varphi(D_l^i) \leqq \delta_l$, $\varphi(E_l^i) \leqq \delta_l$ $\|D_l^i\| \leqq 2\|A\|$, $\|E_l^i\| \leqq 2\|A\|$ $(i = 1, 2, 3, 4)$ and $\delta_l \to 0$. For every $l \in \mathbf{N}$,

$$s_{n_2}^2 s_{n_1}^1(A) - \Phi^2\Phi^1(A) = s_{n_2}^2 s_{n_1}^1\big(A - s_l^1(A)\big) + s_{n_2}^2\big(\Phi^1(A) - s_l^2\Phi^1(A)\big) +$$
$$+ s_{n_2}^1 s_{n_1}^1(C_l^1 + D_l^1 - E_l^1 + iC_l^2 + iD_l^2 - iE_l^2) + s_{n_2}^2(C_l^3 + D_l^3 - E_l^3 + iC_l^4 + iD_l^4 - iE_l^4).$$

Choose a subsequence (δ_{m_l}) of (δ_l) such that $\delta_{m_l} < l^{-k}2^{-l}2^{-k-5}\varepsilon$ and apply Theorem 3 to the elements $D_{m_l}^i, E_{m_l}^i$ with $1/l$ in the role of ε $(l \in \mathbf{N}, i \leqq 4)$. So we have a projection E such that

$$\varphi(I - E) \leqq 2^{k+1}8 \sum_{l=1}^{\infty} l^k l^{-k} 2^{-l} 2^{-k+5}\varepsilon \leqq \varepsilon$$

and

$$\|E s_n^2 s_{n_4}^1(X_{m_l}^i) E\| \leqq \frac{2}{l}$$

where $n_1, n_2, l \in \mathbf{N}$, $i \leqq 4$ and $X = D, E$. Use Lemma 4 and the inequalities above to obtain the estimate

$$\big\|E\big(s_{n_2}^2 s_{n_1}^1(A) - \Phi^2\Phi^1(A)\big)E\big\| \leqq \frac{2m_l\|A\|}{n_1} + \frac{2m_l\|A\|}{n_2} + 4\delta_{m_l} + \frac{16}{l}$$

which concludes our proof.

Theorem 4 is a discrete Dunford—Schwartz—Zygmund type ergodic theorem for non-commuting kernels (cf. [17]). A continuous version will be contained in the next paragraph.

4. Continuous flows. First we establish an automatic continuity of kernels.

Lemma 5. *If $\alpha: \mathfrak{A} \to \mathfrak{A}$ is a kernel then there is a w-continuous kernel $\alpha^c: \mathfrak{A} \to \mathfrak{A}$ such that $\alpha(A) = \alpha^c(A)$ if $\varphi(A^*A)$ and $\varphi(AA^*)$ are finite.*

Proof. Let $\mathfrak{A}_0 = \{A \in \mathfrak{A}: \varphi(A^*A), \varphi(AA^*) < +\infty\}$. We show that α is weakly continuous on the unit ball of $\mathfrak{A}_0$. By Remark 2.2.3 in [6] it follows that $\alpha|\mathfrak{A}_0$ extends to a w-continuous mapping of $\mathfrak{A}$, which is obviously a kernel.

First we prove that if $V: \mathcal{H} \to \mathcal{H}$ is defined by $\alpha(L_\xi) = L_{V\xi}$ $(\xi \in \mathfrak{A}_0)$ and $\eta \in \mathfrak{A}_0'$

then $V^*(\eta\eta^b)\in\mathfrak{A}_0'$. Take $\bar\alpha(R_{\eta^b\eta})$ from (8). Then

$$\langle\xi_1,\bar\alpha(R_{\eta^b\eta})\xi_2\rangle=\langle\bar\alpha(R_{\eta^b\eta})\xi_1,\xi_2\rangle=\langle\alpha(L_{\xi_2^\#\xi_1})\eta,\eta\rangle=$$
$$=\langle\xi_2^\#\xi_1,V^*(\xi\xi^b)\rangle=\langle\xi_1,L_{\xi_1}V^*(\xi\xi^b)\rangle.$$

So $L_\xi V^*(\eta\eta^b)=\bar\alpha(R_{\eta^b\eta})\xi=\bar\alpha(R_{\eta^b\eta})^*\xi$ for every $\xi\in\mathfrak{A}_0$. According to [13], p. 248, $V^*(\eta\eta^b)\in\mathfrak{A}_0'$ and $R_{V^*(\eta^b\eta)}=\bar\alpha(R_{\eta\eta^b})$. Moreover, since $\varphi'(\bar\alpha(R_{\eta\eta^b}))<+\infty$, there is $\eta_1\in\mathfrak{A}_0'$ such that $V^*(\eta\eta^b)=\eta_1\eta_1^b$.

Let (L_{ξ_γ}) be a directed net in the unit ball of $\{L_\xi\colon\xi\in\mathfrak{A}_0\}$ converging weakly to 0. We have

$$\langle\alpha(L_{\xi_\gamma})\eta,\eta\rangle=\langle\xi_\gamma,V^*(\eta\eta^b)\rangle=\langle L_{\xi_\gamma}\eta_1,\eta_1\rangle\to 0.$$

By polarization $\langle\alpha(L_{\xi_\gamma})\eta,\mu\rangle\to 0$ for every $\eta,\mu\in\mathfrak{A}_0'$ and we have obtained $\alpha(L_{\xi_\gamma})\to 0$ weakly.

In this paragraph we deal with one-parameter semigroups of kernels. Namely, for $t\in\mathbf{R}^+$ let $\alpha_t\colon\mathfrak{A}\to\mathfrak{A}$ be a kernel so that $\alpha_0=$identity and $\alpha_t\circ\alpha_s=\alpha_{t+s}$ $(t,s\in\mathbf{R}^+)$. We assume the following continuity property:

(iii) $t\mapsto\varphi(\alpha_t(A)^*\alpha_t(A))$ is continuous if $\varphi(A^*A)$ and $\varphi(AA^*)$ are finite.

If α_t's are endomorphisms and φ is α_t-invariant for every $t\in\mathbf{R}^+$ then (iii) is always fulfilled.

Define $V_t\in\mathscr{B}(\mathscr{H})$ by $\alpha_t(L_\xi)=L_{V_t\xi}$ $(\xi\in\mathfrak{A}_0)$. Then (V_t) is a one-parameter semigroup of contractions, $t\mapsto V_t\xi$ is continuous for every $\xi\in\mathfrak{A}_0$. We need the following technical lemma.

Lemma 6. *Let* (α_t) *be a one-parameter semigroup of kernels with property* (iii). *Then for* $\xi\in\mathfrak{A}_0$ *the integral*

$$\sigma_T(L_\xi)=\frac{1}{T}\int_0^T\alpha_t(L_\xi)\,dt\quad(T>0)$$

exists in weak sense. In addition,* $\mu=\frac{1}{T}\int_0^T V_t\xi\,dt\in\mathfrak{A}_0$ *and* $L_\mu\sigma_T(L_\xi)$.

Proof. Let $\zeta_T=\frac{1}{T}\int_0^T V_t\xi\,dt$ for $\xi\in\mathfrak{A}_0$. If $\eta\in\mathfrak{A}_0'$ then

$$R_\eta\zeta_T(\xi)=\frac{1}{T}\int_0^T R_\eta V_t\xi\,dt=\frac{1}{T}\int_0^T\alpha_t(L_\xi)\eta\,dt.$$

There is a unique operator $\sigma_T(L_\xi)\in\mathscr{B}(\mathscr{H})$ such that

$$\sigma_T(L_\xi)\eta=\frac{1}{T}\int_0^T\alpha_t(L_\xi)\,dt\quad(\eta\in\mathfrak{A}_0').\tag{18}$$

Similarly $\sigma_T(L_{\xi^\#})\eta=\frac{1}{T}\int_0^T \alpha_t(L_{\xi^\#})\eta\,dt$ and it is easy to see that $\sigma_T(L_{\xi^\#})=\sigma_T(L_\xi)^*$. Using [13], p. 252, we may conclude $\zeta_T(\xi)\in\mathfrak{A}_0''=\mathfrak{A}_0$ and $L_{\zeta_T(\xi)}=\sigma_T(L_\xi)$. The rest of the assertion is given by (18).

An important consequence of the above lemmas is that for any kernel α we have

$$\alpha(\sigma_T(A))=\frac{1}{T}\int_0^T \alpha(\alpha_t(A))\,dt \tag{19}$$

if $\varphi(A^*A)$ and $\varphi(AA^*)$ are finite.

Now we are in a position to prove the maximal ergodic theorem for a one-parameter semigroup of kernels.

Theorem 5. *Let* $\mathfrak{A}, \varphi, (\alpha_t), \sigma_T$ *be as above. If* $A_m\in\mathfrak{A}^+$ *and* $\varepsilon_m>0$ $(m\in\mathrm{N})$ *then there is a projection* E *in* $\mathfrak{A}$ *such that*

$$\varphi(I-E)\leqq 2\sum_{m=1}^{\infty}\varepsilon_m^{-1}\varphi(A_m),\quad \|E\sigma_T(A_m)E\|\leqq\varepsilon_m\quad (m\in\mathrm{N}, T\in R^+).$$

Proof. For $\delta>0$ we define $A_m^\delta=\frac{1}{\delta}\int_0^\infty \alpha_t(A_m)\,dt$ and $s_n^\delta(A)=\frac{1}{n}\sum_{i=0}^{n-1}\alpha_{i\delta}(A)$. Then $s_n^\delta(A_m^\delta)=\sigma_{n\delta}(A_m)$ according to (19) and $\varphi(A_m^\delta)\leqq\varphi(A_m)$. Now apply Lemma 2 to $A_m^\delta, \varepsilon_m, \alpha_\delta$. So we obtain $C_\delta\in\mathfrak{A}_1^+$ with the properties

$$C_\delta\sigma_{n\delta}(A_m)C_\delta=C_\delta s_n^\delta(A_m^\delta)C_\delta\leqq\varepsilon_m C_\delta,\ \varphi(I-C)\leqq\sum_{m=1}^{\infty}\varepsilon_m^{-1}\varphi(A_m).$$

Choose a sequence (δ_k) such that $\delta_k\searrow 0$ and $C_{\delta_k}\to C$ weakly for some $C\in\mathfrak{A}_1^+$. Then

$$\varphi(I-C)\leqq\sum_{m=1}^{\infty}\varepsilon_m^{-1}\varphi(A_m),\quad C\sigma_{n\delta_k}(A_m)C\leqq\varepsilon_m C\quad (k, m, n\in\mathrm{N}).$$

By straightforward estimation,

$$C\sigma_T(A_m)C\leqq C(\sigma_T(A_m)-\sigma_{n\delta_k}(A_m))C+C\sigma_{n\delta_k}(A_m)C\leqq$$
$$\leqq 2T^{-1}|T-n\delta_k|\,\|A_m\|\,C^2+\varepsilon_m C.$$

Since $|T-n\delta_k|$ can be chosen arbitrary small we infer $C\sigma_T(A_m)C\leqq\varepsilon_m C$ $(m\in\mathrm{N})$.

If $\int_0^1\lambda dP(\lambda)$ is the spectral resolution of C then take $E=I-P(1/2)$ again as in the proof of Theorem 1.

Theorem 6. *Let* $\mathfrak{A}, \varphi, (\alpha_t), \sigma_T$ *be as above. If* $\varphi(A^*A)$ *and* $\varphi(AA^*)$ *are finite then there exists an operator* $\Phi(A)\in\mathfrak{A}$ *with the following property. For* $\varepsilon>0$ *there is a projection* $E\in\mathfrak{A}$ *such that* $\varphi(I-E)<\varepsilon$ *and*

$$\|E(\sigma_T(A)-\Phi(A))E\|\to 0\quad T\to+\infty.$$

Proof. By Lemma 6, $\sigma_T(A)$ exists as a weak* integral and $t\mapsto V_t\xi$ is continuous for $\xi\in\mathfrak{A}_0$. Since $\mathfrak{A}_0$ is dense in $\mathscr{H}$, now $t\mapsto V_t x$ is strongly integrable over every finite interval (cf. [3], p. 685). Hence $\zeta_T(x)=\frac{1}{T}\int_0^T V_t x\,dt$ is defined for every $x\in\mathscr{H}$. Now we apply Theorem 1 from [3], p. 687, and obtain that $\zeta_T x\to Px$ for every $x\in\mathscr{H}$ as $T\to+\infty$. From this point on one can follow the lines of the proof of Theorem 2. One can show that if $\xi\in\mathfrak{A}_0$ then $P\xi\in\mathfrak{A}_0$ and $\sigma_T(L_\xi)\eta\to L_{P\xi}\eta$ for every $\eta\in\mathfrak{A}_0'$. Let $\Phi(L_\xi)=L_{P\xi}$. From the equality $\xi-P\xi=\xi-\zeta_k(\xi)+\xi_k$ defining ξ_k by left representation we have

$$A-\Phi(A) = A-\sigma_k(A)+A_k \quad (k\in\mathbf{N})$$

where $A=L_\xi$, $A_k=L_{\xi_k}$ and $\|A_k\|_2=\varphi(A_k^*A_k)^{1/2}=\|\xi_k\|=\delta_k\to 0$ as $k\to+\infty$. Hence we can write

$$\sigma_T(A)-\Phi(A) = [\sigma_T(A)-\sigma_{n\delta}(A)]+[\sigma_{n\delta}\sigma_k(A)-\sigma_T\sigma_k(A)]+ \tag{20}$$

$$+[s_n^\delta\sigma_\delta(A)-s_n^\delta s_l^\delta\sigma_\delta(A)]+\sigma_{l\delta}\left(\frac{1}{n}\sum_{i=0}^{n-1}\alpha_{i\delta}(A)-\sigma_{n\delta}(A)\right)+\sigma_T(A_k)$$

where $s_m^\delta(B)=\frac{1}{m}\sum_{i=1}^{m}\alpha_{i\delta}(B)$ and we assume that l, n are integers, $k=l\delta$, $[T+1]=n\delta$.

For the sake of notational simplicity we denote by $D_j(T, k, \delta)$ the jth term on the right hand side in (20) ($j\leqq 3$). Then

$$\|D_j(T, k, \delta)\| \leqq 2T^{-1}(n\delta-T)\|A\| \leqq 2T^{-1}\|A\| \quad (j=1, 2),$$

and by Lemma 4

$$\|D_3(T, k, \delta)\| \leqq 2ln^{-1}\|\sigma_\delta(A)\| \leqq 2kT^{-1}\|A\|.$$

On the other hand, taking

$$D_4(T, k, \delta) = \frac{1}{n}\sum_{i=1}^{n-1}\alpha_{i\delta}(A)-\sigma_{n\delta}(A)$$

we have

$$\|D_4(T, k, \delta)\|_2 = \left\|\frac{1}{n}\sum_{i=0}^{n-1}V_{i\delta}(\xi)-\zeta_{n\delta}(\xi)\right\| \to 0$$

if $[T+1]=n\delta$ is fixed and $\delta\to 0$. For every integer $[T]$ we choose $\delta>0$ such that $\|D_4(T, k, \delta)\|_2\leqq T^{-1}2^{-T}\varepsilon$. Splitting A into selfadjoint and skewadjoint part, taking a subsequence (δ_{m_k}) of (δ_k) with the requirement $\delta_{m_k}<k^{-1}2^{-k}\varepsilon$, we obtain

$$\sigma_T(A)-\Phi(A) = \sigma_T(B^1(k))+i\sigma_T(B^2(k))+\sigma_{m_k}(B^3([T]))+$$

$$+i\sigma_{m_k}(B^4([T]))+\sum_{j=1}^{3}D_j(T, m_k).$$

Here $B^i(l)$ is selfadjoint and $\|B^i(l)\|_2 \leqq l^{-1}2^{-l}\varepsilon$. Now split all the $B^j(l)$'s into 3 summands by Lemma 3. So $B^j(l) = C^j(l) + D^j(l) - iE^j(l)$ and $\|C^j(l)\|$, $\|D^j(l)\|_1$, $\|E^j(l)\|_1 \leqq l^{-1}2^{-l}\varepsilon$. Apply Theorem 5 to $D^j(l)$ and $E^j(l)$ with the constant l^{-1} ($l \in \mathrm{N}, j \leqq 3$) and get a projection E. Then on the one hand,

$$\varphi(I-E) \leqq 2\cdot 8 \sum_{l=1}^{\infty} l l^{-1}2^{-l}\varepsilon = 16\varepsilon,$$

and on the other hand, we estimate in the following fashion:

$$\left\|E\big(\sigma_T(A)-\Phi(A)\big)E\right\| \leqq 2k^{-1}\cdot 2^{-k}\varepsilon + 8k^{-1} + 2[T]^{-1}2^{-[T]} + 8[T]^{-1} + \\ + 4T^{-1}\|A\| + 2m_k T^{-1}\|A\|.$$

Therefore $\|E\big(\sigma_T(A)-\Phi(A)\big)E\| \to 0$ as $T \to +\infty$ and the proof is complete.

Finally we formulate a continuous form of Theorem 4, which is a Dunford—Schwartz—Zygmund type theorem (cf. [17]). Let $\mathfrak{A}$ be a von Neumann algebra and φ a semifinite faithful normal weight on it and for $i \leqq k$ let (α_t^i) be a one-parameter semigroup of kernels possessing the continuity requirement (iii). Define

$$\sigma_T^i(A) = \frac{1}{T}\int_0^T \alpha_t^i(A)\,dt$$

and we know that $\sigma_T^i(A) \to \Phi^i(A)$ under the conditions and in the sense of Theorem 6, under the hypotheses of Theorem 6.

Theorem 7. *Let* $\mathfrak{A}$, φ, (α_t^i), Φ^i, σ^i *be as above and* $A \in \mathfrak{A}$ *such that* $\varphi(A^*A)$, $\varphi(AA^*)$ *are finite. Then for* $\varepsilon > 0$ *there is a projection* $E \in \mathfrak{A}$ *such that*

$$\left\|E\big(\sigma_{T_k}^k \ldots \sigma_{T_1}^1(A) - \Phi^k \ldots \Phi^1(A)E\right\| \to 0$$

if $T_1 \to +\infty, \ldots, T_k \to +\infty$ *independently and* $\varphi(I-E) < \varepsilon$.

Since the proof is very similar to that of Theorem 4, we omit it. We only note that instead of Theorem 3, one has to use the continuous form of it.

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