Acta Sci. Math. 46 (1983), 311—316

On zeros of analytic multivalued functions

BERNARD AUPETIT*) and JAROSLAV ZEMÁNEK

It had been observed by F. V. ATKINSON [1] and B. SZ.-NAGY [13] that if $f(\lambda)=$ $=I+\lambda V_1+\ldots+\lambda^n V_n$, where $V_1, \ldots, V_n$ are compact operators on a Banach space, then the set of λ in $\mathbf{C}$ for which $0\in \operatorname{Sp} f(\lambda)$ is discrete and closed in the complex plane. For $n=1$ it is exactly the classical result of F. Riesz. For $n>1$ B. SZ.-NAGY [13] believed that this result is deeper than the classical one. The problem was also studied by JU. L. ŠMUL'JAN [12]. Here we show in Theorem 1, by a completely different method, that it comes from Riesz's theorem using only complex function theory. Moreover, we give a generalization of this result when $f(\lambda)$ is any analytic function from a domain Ω of $\mathbf{C}$ into a Banach algebra such that $\operatorname{Sp} f(\lambda)$ is countable for every λ in Ω.

It is known that $\lambda\to\operatorname{Sp} f(\lambda)$ is an analytic multivalued function [3] and that analytic multivalued functions have properties very similar to this special case. So, it is better to formulate all the theorems of this paper in the more general situation (for more details see [3], [5], [8]). However, the reader not familiar with this theory can adapt immediately all the proofs to the spectral case.

Theorem 1. *Let $\lambda\to K(\lambda)$ be an analytic multivalued function defined on a domain Ω in $\mathbf{C}$. Suppose that $K(\lambda)$ has at most 0 as a limit point for every λ in Ω. Let $z\neq 0$ be a fixed complex number. Then the set of those λ in Ω for which $z\in K(\lambda)$ is either closed and discrete in Ω or it is all Ω.*

Proof. Suppose that $z\in K(\lambda_0)$ for some $\lambda_0\in\Omega$. We shall show that the point λ_0 is either isolated or interior in the set $E=\{\lambda\in\Omega\colon z\in K(\lambda)\}$. Because $z\neq 0$ there exists an open disk Δ centred at z and not containing 0 such that $\Delta^-\cap K(\lambda_0)=$ $=\{z\}$. By upper semi-continuity of the function K there exists $r>0$ such that $|\lambda-\lambda_0|<r$ implies $K(\lambda)\cap \operatorname{bdry}\Delta=\emptyset$. Moreover, by Newburgh's property we can also suppose that $K(\lambda)\cap\Delta\neq\emptyset$ for these λ, and in this situation $\lambda\to K(\lambda)\cap\Delta$ is an analytic multivalued function on the disk $B(\lambda_0, r)$, see [5], Theorem 3.14. Because

Received February 9, 1982, and in revised form, October 1, 1982.

*) This work has been supported by Natural Sciences and Engineering Research Council of Canada Grant A 7668.

Δ does not contain 0 the set $K(\lambda)\cap\Delta$ is finite for $|\lambda-\lambda_0|<r$. We apply the scarcity theorem for analytic multivalued functions [3], [5] (we can also use the subharmonicity of $\lambda\to\operatorname{Log}\delta_n(K(\lambda))$, where δ_n denotes the n-th diameter; in the case when $K(\lambda)=\operatorname{Sp}f(\lambda)$ we can use the scarcity theorem ([2], p. 67), or the subharmonicity of $\lambda\to\operatorname{Log}\delta_n(\operatorname{Sp}f(\lambda))$ [11]). So there exist an integer $n\geqq 1$, a closed discrete subset F of the disk $B(\lambda_0, r)$ and n functions $\alpha_1, \ldots, \alpha_n$ which are holomorphic on $B(\lambda_0, r)\setminus F$ such that

$$K(\lambda)\cap\Delta = \{\alpha_1(\lambda), \ldots, \alpha_n(\lambda)\} \quad \text{for} \quad \lambda\in B(\lambda_0, r)\setminus F.$$

There exists s such that $0<s\leqq r$ and $B(\lambda_0, s)\cap F\subset\{\lambda_0\}$. The functions $\alpha_1, \ldots, \alpha_n$ are holomorphic on $B(\lambda_0, s)$ except perhaps at λ_0.

Moreover, by the upper semi-continuity of the function $K(\lambda)$ we have $\lim_{\lambda\to\lambda_0}\alpha_i(\lambda)=z$ for every $i=1, 2, \ldots, n$. Therefore the α_i's can be extended holomorphically to the whole disk $B(\lambda_0, s)$. It follows that either $\alpha_{i_0}(\lambda)\equiv z$ for some i_0, or there exists t with $0<t\leqq s$ such that $\alpha_i(\lambda)\neq z$ for all $\lambda\in B(\lambda_0, t)\setminus\{\lambda_0\}$ and $i=1, 2, \ldots, n$. In the first case λ_0 is an interior point of E, while in the second case λ_0 is isolated in E.

To finish the proof we consider the set E' of all limit points of E in Ω. Because of the upper semi-continuity of the function K the set E is closed in Ω, so $E'\subset E$. Let $\mu\in E'$. Since μ is not isolated in E it is an interior point of E, hence an interior point of E'. So E' is both closed and open in Ω. Consequently we have either $E'=\emptyset$ or $E'=\Omega$. This completes the proof.

Corollary 1. *Let $\lambda\to f(\lambda)$ be an analytic function from a domain Ω into the compact operators on a Banach space. Suppose that $z\notin\operatorname{Sp}f(0)$. Then the set of all λ for which $z\in\operatorname{Sp}f(\lambda)$ is closed and discrete in Ω.*

Remark 1. F. V. Atkinson [1] and B. Sz.-Nagy [13] consider the situation when $\Omega=\mathbf{C}$ and $f(\lambda)=\lambda V_1+\ldots+\lambda^p V_p$ with compact operators $V_1, \ldots, V_p$. Ju. L. Šmul'jan [12] studies the case when $f(\lambda)$ is an analytic family of compact operators, defined on a domain Ω.

We intend to generalize Theorem 1 to the situation when $K(\lambda)$ are general countable sets. Of course, in this situation it is impossible to conclude that the set $\{\lambda: z\in K(\lambda)\}$ is discrete. To see this take, for example, $K(\lambda)=\operatorname{Sp}(\lambda I+C)$ where C is a compact operator with infinite spectrum. In this case the preceding set has z as a limit point.

The situations studied in Theorem 1 and in the last example suggest to introduce the notion of *good isolated point*. Given an analytic multivalued function $\lambda\to K(\lambda)$ on a domain Ω, for $\lambda_0\in\Omega$ we say that $\mu\in K(\lambda_0)$ is a good isolated point of $K(\lambda_0)$ if there exist a disk Δ centred at μ such that $\Delta^-\cap K(\lambda_0)=\{\mu\}$ and an $r>0$ such that the set $K(\lambda)\cap\Delta$ is finite for $|\lambda-\lambda_0|<r$. By the scarcity theorem for analytic

multivalued functions (see [3], Theorem 7) there exists an integer $n\geqq 1$ such that $K(\lambda)\cap\Delta$ has exactly n points for all $|\lambda-\lambda_0|<r$ except perhaps on a closed discrete subset. By definition we put $DK(\lambda)$ to be the set of points of $K(\lambda)$ which are not good isolated points. By transfinite induction we can define $D^\alpha K(\lambda)$ for every ordinal α by

$$D^\alpha K(\lambda)=D(D^{\alpha-1}K(\lambda)) \quad \text{if } \alpha \text{ is not a limit ordinal,}$$

$$D^\alpha K(\lambda)=\bigcap_{\beta<\alpha} D^\beta K(\lambda) \quad \text{if } \alpha \text{ is a limit ordinal.}$$

It is a remarkable fact that if $D^\alpha K(\lambda)$ is not identically void then $\lambda\to D^\alpha K(\lambda)$ is an analytic multivalued function on Ω (see [8] and [5]).

In the situation of Theorem 1 we have $DK(\lambda)$ constant (either empty or equal to $\{0\}$) while in the previous example we have $DK(\lambda)=\{\lambda\}$.

Theorem 2. *Let $\lambda\to K(\lambda)$ be an analytic multivalued function defined on a domain Ω in $\mathbf{C}$. Let z be a fixed complex number. Then every point of the set $\{\lambda\in\Omega: z\in K(\lambda)\setminus DK(\lambda)\}$ is either isolated or interior.*

Proof. We omit the proof because it is similar to the proof of Theorem 1.

We shall need two lemmas the proofs of which are similar to some arguments given in [5].

Lemma 1. *Let $\lambda\to K(\lambda)$ be an analytic multivalued function defined on a domain Ω in $\mathbf{C}$, with $K(\lambda)$ countable for every λ in Ω. Then there exists a point μ in Ω such that $K(\mu)\neq DK(\mu)$.*

Proof. Suppose that $DK(\lambda)=K(\lambda)$ for every λ in Ω. From this we conclude that there exists some $\lambda_0\in\Omega$ for which $K(\lambda_0)$ has an infinite number of points. Because $K(\lambda_0)$ is countable and compact we can assume that there exist two isolated points in $K(\lambda_0)$ (see [9], Theorem 2.43). We denote them by α_0 and α_1. We choose two open disks Δ_0 and Δ_1 centred respectively at α_0 and α_1, having disjoint closures and such that $\Delta_0^-\cap K(\lambda_0)=\{\alpha_0\}$ and $\Delta_1^-\cap K(\lambda_0)=\{\alpha_1\}$. Then we choose $r>0$ such that $B^-(\lambda_0, r)\subset\Omega$ and such that $|\lambda-\lambda_0|<r$ implies $K(\lambda)\cap \operatorname{bdry}\Delta_i=\emptyset$ for $i=0, 1$.

Because $K(\lambda_0)=DK(\lambda_0)$ the isolated point α_i is not a good isolated point of $K(\lambda_0)$, for $i=0, 1$. By applying the scarcity theorem for the two functions $\lambda\to K(\lambda)\cap\cap\Delta_i$ we conclude that the two sets $E_i=\{\lambda\in B(\lambda_0, r): K(\lambda)\cap\Delta_i \text{ is finite}\}$ are of outer capacity zero. Consequently, $E_0\cup E_1$ is of outer capacity zero, therefore there exists some λ_1 in $B(\lambda_0, r/2)$ such that the intersection of $K(\lambda_1)$ on both Δ_0 and Δ_1 is infinite.

As before we find four distinct isolated points in $K(\lambda_1)$, say α_{00}, α_{01} in Δ_0 and α_{10}, α_{11} in Δ_1. We take four open disks Δ_{ij} centred respectively at α_{ij}, having disjoint closures, such that $\Delta_{00}\cup\Delta_{01}\subset\Delta_0$, $\Delta_{10}\cup\Delta_{11}\subset\Delta_1$ and $\Delta_{ij}^-\cap K(\lambda_1)=$ $=\{\alpha_{ij}\}$. By induction we can construct a sequence (λ_n) such that:

(i) $|\lambda_{n+1}-\lambda_n|\leqq r/2^{n+1}$ for $n=0, 1, 2, \ldots$,

(ii) $K(\lambda_n)$ contains at least 2^{n+1} distinct isolated points $\alpha_{i_1\ldots i_{n+1}}$ where i_k takes the values 0, 1,

(iii) each $\alpha_{i_1\ldots i_{n+1}}$ is the centre of an open disk $\Delta_{i_1\ldots i_{n+1}}$, all these 2^{n+1} disks have disjoint closures, and moreover we have $\Delta_{i_1\ldots i_n i_{n+1}}\subset\Delta_{i_1\ldots i_n}$.

Then (λ_n) is a Cauchy sequence converging to some $\mu\in B^-(\lambda_0, r)\subset\Omega$. To obtain a contradiction we shall show that $K(\mu)$ is uncountable.

Let $I=\{i_1, i_2, \ldots, i_n, \ldots\}$ be an arbitrary sequence of 0's and 1's. A subsequence of $\alpha_{i_1}, \alpha_{i_1i_2}, \alpha_{i_1i_2i_3}, \ldots$ converges to an α_I which is in $K(\mu)$ by upper semi-continuity. If $I\neq J$ then for some index k we have $i_k\neq j_k$ with $i_l=j_l$ for $1\leqq l<k$. We have $\alpha_I\in\bar{\Delta}_{i_1i_2\ldots i_k}$ while $\alpha_J\in\bar{\Delta}_{i_1i_2\ldots i_{k-1}j_k}$ and these two disks are disjoint by construction, so $\alpha_I\neq\alpha_J$. But the set of sequences I is uncountable so $K(\mu)$ is uncountable.

Remark 2. For any analytic multivalued function $K(\lambda)$ on Ω it is easy to see that the set of $\lambda\in\Omega$ for which $K(\lambda)\neq DK(\lambda)$ is open. If in addition the set $K(\lambda)$ are countable for $\lambda\in\Omega$, then this set is dense in Ω.

Lemma 2. *Let $\lambda\to K(\lambda)$ be an analytic multivalued function defined on a domain Ω, with $K(\lambda)$ countable for every λ in Ω. Then there exists a first or second class ordinal β such that $D^\beta K(\lambda)=\emptyset$ for every λ in Ω.*

Proof. Let $\mathcal{O}$ denote the set of ordinals in the first and second classes (see [10], p. 369). For every λ in Ω the family of $D^\alpha K(\lambda)$, for α in $\mathcal{O}$, is decreasing, consequently it stabilizes at some ordinal $\alpha(\lambda)$, i.e. we have $D^\gamma K(\lambda)=D^{\alpha(\lambda)}K(\lambda)$ for every $\gamma\geqq\alpha(\lambda)$, γ in $\mathcal{O}$ (see [7], p. 146). For every α in $\mathcal{O}$ we define

$$F_\alpha = \{\lambda\in\Omega\colon D^\gamma K(\lambda) = D^\alpha K(\lambda) \quad \text{for} \quad \gamma\geqq\alpha,\ \gamma\in\mathcal{O}\}.$$

Obviously this family is increasing and exhausts all Ω. Also the sets F_α are closed in Ω (even if the sets $K(\lambda)$ are not countable). Indeed, taking λ_0 in $\Omega\setminus F_\alpha$, we have $D^\gamma K(\lambda_0)\neq D^{\gamma+1}K(\lambda_0)$ for some $\gamma\geqq\alpha, \gamma\in\mathcal{O}$. Since $D^\gamma K(\lambda)\not\equiv\emptyset$, it follows by the Oka—Nishino theorem (see [5], Lemma 3.16) that $\lambda\to D^\gamma K(\lambda)$ is an analytic multivalued function. By the first part of Remark 2 we have $D^\gamma K(\lambda)\neq D^{\gamma+1}K(\lambda)$ in a neighbourhood of λ_0, so $\Omega\setminus F_\alpha$ is open. Using again the results in [7], p. 146, and [10], p. 370, we obtain that for some β in $\mathcal{O}$ we have $F_\beta=\Omega$.

Suppose that $D^\beta K(\lambda)\not\equiv\emptyset$ on Ω. By Oka—Nishino theorem $\lambda\to D^\beta K(\lambda)$ is analytic multivalued on Ω. By hypothesis $D^\beta K(\lambda)$ is countable for every λ in Ω hence by Lemma 1 we have $D^{\beta+1}K(\mu)\neq D^\beta K(\mu)$ for some $\mu\in\Omega$, that is $F_\beta\neq\Omega$, which is a contradiction.

Theorem 3. *Let $\lambda\to K(\lambda)$ be an analytic multivalued function on a domain Ω in $\mathbf{C}$. Suppose that $K(\lambda)$ is countable for every λ in Ω. Let z be a fixed complex number. Then the set of those λ in Ω for which $z\in K(\lambda)$ is either countable or it is all Ω.*

Proof. By Lemma 2 there exists a smallest ordinal β in the first or second class such that $D^{\beta}K(\lambda)\equiv\emptyset$ for λ in Ω. We have $E=\{\lambda\in\Omega: z\in K(\lambda)\}=\bigcup_{0\leqq\gamma<\beta} E_\gamma$ where $E_\gamma=\{\lambda\in\Omega: z\in D^\gamma K(\lambda)\setminus D^{\gamma+1}K(\lambda)\}$. By Theorem 2 applied to the analytic multivalued function $\lambda\to D^\gamma K(\lambda)$ we conclude that E_γ has only isolated or interior points. Therefore E_γ is the disjoint union of an open set and a countable set. Because the set of ordinals less than β is countable the set E is also the disjoint union of an open set and a countable set. If the interior of E is empty then E is countable and we have finished. If not, we shall show that $E=\Omega$. First we note that E is closed in Ω by upper semi-continuity and so the boundary of E in Ω is countable. Let F be the closure of the interior of E in Ω. It is enough to prove that $F=\Omega$. Because F is closed in Ω and Ω is a domain we have only to show that F is open. Let a be a point of F, and let $r>0$ be such that $B(a,r)\in\Omega$. There exists b in the interior of E such that $|a-b|<r$. The set of half-lines Γ with origin at b such that $\Gamma\cap B(a,r)$ contains a boundary point of E is at most countable. So the interior of E is dense in $B(a,r)$ and hence $F\supset B(a,r)$.

Now we give an application of Theorem 3 concerning the problem of spectral classification of projections. In [6] we obtained such result for finite-dimensional algebras. Here we extend it to algebras with countable spectrum.

We say that two idempotents e and f in a Banach algebra A are equivalent if they belong to the same connected component of the set of all idempotents in A. It is possible to prove that e and f are equivalent if and only if there exist elements $a_1,\dots,a_n$ in A such that $f=\exp(-a_n)\dots\exp(-a_1)\cdot e\cdot\exp(a_1)\dots\exp(a_n)$, see [4].

Corollary 2. *Let A be a (real or complex) Banach algebra. Suppose that every element in A has countable spectrum. Let e and f be given idempotents in A. Then e is not equivalent to f if and only if $1\in\mathrm{Sp}\,(e'+f')$ for all idempotents e',f' in neighbourhoods of e and f, respectively.*

Proof. As noted in [6] it is enough to prove that $1\in\mathrm{Sp}\,(e'+f')$ implies e not equivalent to f. Suppose on the contrary that e and f are equivalent. So there are elements $a_1,\dots,a_n$ in A such that

$$f=\exp(-a_n)\dots\exp(-a_1)\cdot e\cdot\exp(a_1)\dots\exp(a_n).$$

Consider the analytic function

$$g(\lambda)=\exp(-\lambda a_n)\dots\exp(-\lambda a_1)\cdot e\cdot\exp(\lambda a_1)\dots\exp(\lambda a_n)$$

defined for all complex λ and with values in the complexification of A. The values of this function are idempotents and for λ real they belong to A. Moreover we have $g(0)=e$, $g(1)=f$. We consider the analytic multivalued function defined on $\mathbf{C}$ by

$$\lambda\to K(\lambda)=\mathrm{Sp}\,(g(\lambda)+g(1-\lambda))$$

which has countable values for λ real. (We recall that for real Banach algebras the spectrum is defined with respect to the complexification.) Hence by Oka—Nishino theorem on scarcity of elements with countable values (see [3], [5], [8]) we conclude that $K(\lambda)$ is countable for every λ in $\mathbf{C}$. But we know that $1 \in K(\lambda)$ if λ is in a small real segment containing zero. So by Theorem 3 we have $1 \in K(\lambda)$ for every λ. In particular, taking $\lambda = 1/2$ we get $1 \in \operatorname{Sp}(2g(1/2))$ which is impossible because $g(1/2)$ is an idempotent.

We thank very much the referee of this paper for his many comments and simplifications in the proofs.

Added in proof. Some related new results are given in [14], [15], [16].

References

[1] F. V. Atkinson, A spectral problem for completely continuous operators, *Acta Math. Acad. Sci. Hungar.*, **3** (1952), 53—60.
[2] B. Aupetit, *Propriétés spectrales des algèbres de Banach,* Lecture Notes in Mathematics 735, Springer-Verlag (Berlin, 1979).
[3] B. Aupetit, Some applications of analytic multivalued functions to Banach algebras, *Proc. Roy. Irish Acad. Sect. A*, **81** (1981), 37—42.
[4] B. Aupetit, Projections in real Banach algebras, *Bull. London Math. Soc.*, **13** (1981), 412—414.
[5] B. Aupetit, Analytic multivalued functions in Banach algebras and uniform algebras, *Adv. in Math.*, **44** (1982), 18—60.
[6] B. Aupetit, T. J. Laffey and J. Zemánek, Spectral classification of projections, *Linear Algebra and Appl.*, **41** (1981), 131—135.
[7] C. Kuratowski, *Topologie I,* Państwowe Wydawnictwo Naukowe (Warszawa, 1958).
[8] T. Nishino, Sur les ensembles pseudoconcaves, *J. Math. Kyoto Univ.*, **1—2** (1962), 225—245.
[9] W. Rudin, *Principles of mathematical analysis,* 3rd edition, McGraw-Hill Book Company (New York, 1976).
[10] W. Sierpiński, *Cardinal and ordinal numbers,* Polish Scientific Publishers (Warszawa, 1965).
[11] Z. Słodkowski, On subharmonicity of the capacity of the spectrum, *Proc. Amer. Math. Soc.*, **81** (1981), 243—249.
[12] Ju. L. Šmul'jan, Completely continuous perturbations of operators, *Doklady Akad. Nauk SSSR,* **101** (1955), 35—38. (Russian)
[13] B. Sz.-Nagy, On a spectral problem of Atkinson, *Acta Math. Acad. Sci. Hungar.*, **3** (1952), 61—66.
[14] B. Aupetit and A. Zraibi, Distribution des valeurs des fonctions analytiques mitiformes, *Studia Math.*, to appear
[15] T. J. Ransford, Open mapping, inversion and implicit function theorems for analytic multivalued functions, *J. London Math. Soc.*, to appear
[16] T. J. Ransford, *Analytic multivalued functions*, Thesis, University of Cambíidge, 1983.

(B. A.)
DÉPARTEMENT DE MATHÉMATIQUES
UNIVERSITÉ LAVAL
QUÉBEC, G1K 7P4, CANADA

(J. Z.)
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
00-950 WARSZAWA, POLAND P. O. BOX 137