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Conditions for hermiticity and for existence of an equivalent C*-norm

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The author has found a sufficient condition for a self-adjoint element in a Banach *-algebra to have purely real spectrum. This is contained in Theorem 1 below. Using this result it becomes possible to prove that a fairly weak condition provides for the existence of an equivalent C^*-norm (see Theorem 2).

The problem discussed here is a version of the Araki—Elliott problem. ARAKI and ELLIOTT [3] proved in 1973 that if the B^*-condition

$$\|a^*a\| = \|a^*\| \cdot \|a\|$$

holds for a linear norm and the * is continuous, then it is a C^*-norm. They conjectured that the continuity of the involution is also a consequence of the B^*-condition. Z. SEBESTYÉN and the author [4] verified this conjecture, and gave a condition for a norm to be a C^*-norm which can hardly be weakened.

We shall use [1] without further reference.

Theorem 1. *Let $\mathscr{A}$ be a Banach *-algebra, and let r be the spectral radius in it. Consider a self-adjoint element $h(\in\mathscr{A})$. Let $\langle h\rangle$ be the algebra generated by h. Assume there are a seminorm p on $\langle h\rangle$ and constants $0<M_1\leqq M_2$ such that*

(i) $M_1^2\cdot r(a^*a)\leqq p(a^*)\cdot p(a)\leqq M_2^2\cdot r(a^*a)$ *for all* $a\in\langle h\rangle$.

Then $\mathrm{Sp}\,(h)\subset\mathbf{R}$ *or* $\mathrm{Sp}\,(h)\subset\{0, w, \bar{w}\}$ *with a suitable* $w\in\mathbf{C}$. *Further, if p is a norm then* $\mathrm{Sp}\,(h)\subset\mathbf{R}$. ("Sp" denotes the spectrum in $\mathscr{A}$.)

The proof will consist of two parts. Part I contains independent propositions with independent notations. Then we shall prove Theorem 1 in Part II utilizing the results of the previous part.

Part I. We start with an easy lemma.

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Lemma 1.1. *Let $\mathcal{A}$ be a *-algebra, p, r be seminorms on it such that $r(a^2)=r(a)^2$, $r(a^*)=r(a)$ and*

$$(1)\qquad M_1^2\cdot r(a^*a)\leqq p(a^*)\cdot p(a)\leqq M_2^2\cdot r(a^*a)\quad \textit{for all}\quad a\in\mathcal{A}.$$

Then the following also hold:

$$(2)\qquad M_1\cdot r(h)\leqq p(h)\leqq M_2\cdot r(h)\quad \textit{if}\quad h=h^*\in\mathcal{A},$$

$$(3)\qquad p(a)\leqq 2M_2\cdot r(a)\quad \textit{for all}\quad a\in\mathcal{A}.$$

Proof. Writing $a=h$, $a^*=h$, (2) is immediate from the properties of r. For an arbitrary element a consider the real and imaginary part of a, that is, $h=2^{-1}(a+a^*)$, $k=(2i)^{-1}(a-a^*)$. Then $r(a^*)=r(a)$ implies $r(k)\leqq r(a)$, $r(h)\leqq r(a)$, and so (3) follows from (2).

We call a set $K\subset\mathbf{C}$ symmetric if it is stable under conjugation, i.e. $\bar{z}\in K$ if $z\in K$. In the remainder of this part let K be a fixed symmetric non-void compact subset of the complex plain. Denote by $C(K)$ the algebra of continuous functions on K, and by r the customary sup-norm in $C(K)$. Define an involution in $C(K)$ setting $f^*(z)=\overline{f(\bar{z})}$. This definition is correct and this involution is norm-preserving, since K is symmetric.

Let $A\subset C(K)$ be the polynomials without constant terms. This is a *-subalgebra. Consider the following condition: there are a seminorm p on A and constants $0<M_1\leqq M_2$ such that

$$(\mathrm{P1})\qquad M_1^2\cdot r(f^*f)\leqq p(f^*)\cdot p(f)\leqq M_2^2\cdot r(f^*f)\quad \text{for all}\quad f\in A.$$

Our goal is to prove that this condition implies that the shape of K is very special (see Propositions 1.2 and 1.5 below).

First we list some immediate consequences of (P1). We see from Lemma 1.1 that

$$(\mathrm{P2})\qquad M_1\cdot r(h)\leqq p(h)\leqq M_2\cdot r(h)\quad \text{if}\quad h=h^*\in A,$$

$$(\mathrm{P3})\qquad p(f)\leqq 2M_2\cdot r(f)\quad \text{for all}\quad f\in A.$$

Let B be the norm-closure of A in $C(K)$. Because of (P3) p has a unique continuous extension to B, which will also be denoted by p. Then this extended p will also be a seminorm and (P1), (P2), (P3) remain valid on B.

Notation. We say that a set $T\subset\mathbf{C}$ is a *cross* if there is a real number s such that $T\subset\mathbf{R}\cup\{s+it;\ t\in\mathbf{R}\}$.

Proposition 1.2. (P1) *implies that K is a cross.*

Proof. Suppose the contrary. Then we shall find f, g in B with $p(f)+p(g)<p(f+g)$, which is a contradiction. We need two lemmas for this.

Denote by C (resp. β) the maximum of $|z|$ (resp. $\operatorname{Im} z$) on K. Note that $C, \beta>0$ because K is symmetric and not a cross. Let $\alpha\in\mathbf{R}$ be such that $\alpha+i\beta\in K$. Write $w_1=\alpha+i\beta$, $w_2=\bar{w}_1$, $m=|w_1|$.

Lemma 1.3. *For any* $n\in\mathbf{R}$ *there are* a, b *in* B *such that*
(4) $r(a^*a), r(b^*b)\leqq C^2$, (5) $r(a)=r(b)>n$, (6) $|b(w_1)|=|b(w_2)|=m$,
(7) $|a(w_1)|\geqq mC^{-1}\cdot r(a)$, (8) $|a(w_2)|<2^{-1}m$.

Proof. Let $a_t(z)=z\cdot\exp(-it(z-\alpha))$, $b_t(z)=z\cdot\exp(-it(z-\alpha)^2)$ where t is real and $z\in K$. Then $a_t, b_t\in B$ for all t. Since K is not a cross, there is a $u=\gamma i\delta\in K$ such that $\gamma\neq\alpha$ and $\delta\neq 0$ ($\gamma, \delta\in\mathbf{R}$). Thus $|b_t(u)|=|u|\cdot\exp(2t(\gamma-\alpha)\delta)$ and hence there is a t for which $|b_t(u)|>n$. Let $b=b_t$ with such a t.

Since $|a_t(w_1)|=m\cdot\exp(t\beta)$, $|a_t(w_2)|=m\cdot\exp(-t\beta)$, there is a $t>0$ with $|a_t(w_2)|<2^{-1}m$, $r(a_t)>r(b)$. With such a t let $a=r(b)r(a_t)^{-1}a_t$. It is easy to check that (4)—(8) hold for this a, b (for (7) use that β is the maximum of $\operatorname{Im} z$ on K).

Lemma 1.4. *Assume that for an* $a\in B$ *the condition*

(9) $$r(a^*a)^{1/2}\leqq C\leqq 2^{-1}\cdot r(a)$$

holds. Then there is a constant L *(e.g.* $L=4M_2^2C^2M_1^{-1}$ *is appropriate) such that*

(10) $$\min\left(p(a), p(a^*)\right)\leqq L\cdot r(a)^{-1}.$$

Proof. Choosing z in K with $r(a)=a(z)$ we have by (9)

$$|a^*(z)|\leqq C^2\cdot r(a)^{-1}\leqq 2^{-1}C\leqq 4^{-1}\cdot r(a),$$

and thus

$$r(a+a^*)\geqq |(a+a^*)(z)|\geqq |a(z)|-|a^*(z)|\geqq r(a)-4^{-1}\cdot r(a)\geqq 2^{-1}\cdot r(a).$$

Then we get from (P1), (P2), (9) and the subadditivity of p that

$$p(a)+p(a^*)\geqq 2^{-1}M_1\cdot r(a) \quad\text{and}\quad p(a)\cdot p(a^*)\leqq M_2^2C^2.$$

Writing $c=\min\left(p(a), p(a^*)\right)$, $d=\max\left(p(a), p(a^*)\right)$, we then have $2d\geqq c+d\geqq$ $\geqq 2^{-1}M_1\cdot r(a)$, $c\cdot d\leqq M_2^2C^2$, and hence $c\leqq 4M_2^2C^2M_1^{-1}r(a)^{-1}$.

We turn to the proof of Proposition 1.2. Let $a, b\in B$ be such that (4)—(8) hold with "large enough" n. Let further f (resp. g) be the one from a and a^* (resp. b and b^*) for which p is less. Since $r(g)=r(f)=r(a)>n$ and n is large ($>2C$), we can apply Lemma 1.4 and have

(11) $$p(f)+p(h)<2Ln^{-1}.$$

On the other hand, (P1) and (5)—(8) give us

$$M_1^{-2}\cdot p(f^*+g^*)\cdot p(f+g)\geqq r((f^*+g^*)(f+g))\geqq |[(f^*+g^*)(f+g)](w_1)|\geqq$$
$$\geqq\left(mC^{-1}\cdot r(a)-m\right)\cdot(m-2^{-1}m)\geqq (4C)^{-1}m^2\cdot r(a)$$

if n is large (since $n>2C$ implies $m \leqq (2C)^{-1} m \cdot r(a)$). Further, by (P3)

$$p(f^*+g^*) \leqq 2M_2 \cdot r(f^*+g^*) \leqq 4M_2 \cdot r(a)$$

and thus

$$p(f+g) \geqq M_1^2 m^2 (16 M_2 C)^{-1} \geqq 2Ln^{-1}$$

if n is large. This and (11) show the desired contradiction. Proposition 1.2 is proved.

Proposition 1.5. *If* $\mathrm{card}(K-\mathbf{R})=2$ *and* (P1) *holds then* $K \cap \mathbf{R} \subset \{0\}$.

Proof. Suppose $K-\mathbf{R}=\{w, \bar{w}\}$. Since $\mathbf{C}-K$ is connected now, by Runge's theorem there are polynomials P_k converging to $w^{-1} \cdot 1_{\{w\}}$ in $C(K)$, where $1_{\{w\}}$ denotes the characteristic function of the one point set $\{w\}$. Hence $z \cdot P_k(z)$ converges to $1_{\{w\}}$ in $C(K)$, consequently $1_{\{w\}} \in B$.

Since $1_{\{w\}}^* \cdot 1_{\{w\}}=0$, thus by (P1) we infer that one of the functions $1_{\{w\}}$ and $1_{\{w\}}^*$, say f, is such that $p(f)=0$. This implies

$$p(f+g) = p(g) \quad \text{for all} \quad g \in B. \tag{12}$$

Applying this to $g=f^*$ we get from (P2) that

$$p(f^*) \geqq M_1. \tag{13}$$

Let $h(z)=z$ on K and let $h_0=h-w \cdot 1_{\{w\}}-\bar{w} \cdot 1_{\{w\}}^*$; thus $h_0 \in B$. We will show that $h_0=0$, i.e. $K \cap \mathbf{R} \subset \{0\}$. Write $g=\alpha \cdot h_0$, where α is a real number, and let $k=f+g$. Since g is self-adjoint, further $g \cdot f=0=g \cdot f^*$, therefore $k^*k=g^2$ and so (P1) implies

$$p(k^*) \cdot p(k) \leqq M_2^2 \cdot r(g)^2. \tag{14}$$

On the other hand, we can see from (12), (13) and (P2) that $p(k) \geqq M_1 \cdot r(g)$, $p(k^*) \geqq$ $\geqq M_1 - M_2 \cdot r(g)$. This contradicts (14), if $r(g)$ is a small positive number. But if $h_0 \neq 0$, then $r(g)$ runs over all of $\mathbf{R}_+$ when α does. Thus $h_0=0$ and the proof of Proposition 1.5 is complete.

Part II. If $P=\sum_{k=1}^{n} a_k X^k$ is a complex polynomial without constant term then we write $P^*=\sum_{k=1}^{n} \bar{a}_k X^k$. It is clear that $P^*(h)=P(h)^*$, where h is the self-adjoint element considered in Theorem 1.

Let $K=\mathrm{Sp}(h)$. Then K is symmetric, because in each *-algebra $\mathrm{Sp}(a^*)=$ $=\overline{\mathrm{Sp}(a)}$ for any a. We will show that this K satisfies (P1). Consider the following relation between A and $\langle h \rangle$: $f \sim a$ if there is a polynomial P such that $P(h)=a$ and $P(z)=f(z)$ for all $z \in K$. Denote by r' the sup-norm in $C(K)$. Then $r'(f)=r(a)$ if $f \sim a$, because $P(\mathrm{Sp}(h))=\mathrm{Sp}(P(h))$. Further, $f \sim a$, $g \sim b$ ensure $f+\lambda g \sim a+\lambda b$, $f^* \sim a^*$, since $P^*(z)=\overline{P(\bar{z})}$. Finally we see from (i) and Lemma 1.1 that $p \leqq 2M_2 \cdot r$.

Hence the following definition is correct: let $p'(f)=p(a)$ if $f\sim a$. Moreover, this p' shows that K satisfies (P1). Thus we know that

(15) $\mathrm{Sp}(h)$ is a cross,

(16) if $\mathrm{card}(\mathrm{Sp}(h)-\mathbf{R})=2$ then $\mathrm{Sp}(h)\cap\mathbf{R}\subset\{0\}$.

Suppose that $K=\mathrm{Sp}(h)\not\subset\mathbf{R}$ and $K\not\subset\{0,w,\bar{w}\}$ for any $w\in\mathbf{C}$. Then by (15) and (16) we can find w_1, w_2 in $K-\mathbf{R}$ such that $\mathrm{Re}\,w_1=\mathrm{Re}\,w_2$, $\mathrm{Im}\,w_1\neq\pm\mathrm{Im}\,w_2$. Thus $\mathrm{Re}(w_1+sw_1^2)\neq\mathrm{Re}(w_2+sw_2^2)$ for any $s\in\mathbf{R}-\{0\}$, and if $|s|$ is small then $w_1+sw_1^2$, $w_2+sw_2^2$ are not real. Therefore $\mathrm{Sp}(h+sh^2)$ is not a cross. But this is impossible, since $g=h+sh^2$ is self-adjoint and $\langle g\rangle\subset\langle h\rangle$.

It remains to prove the last statement of the theorem. Assume the contrary, that is, $K\not\subset\mathbf{R}$ and p is a norm. We know already that $K\cup\{0\}=\{0,w,\bar{w}\}$ where $w\in\mathbf{C}-\mathbf{R}$. Let $y=h^2-wh$. Then $y^*y=h^4-wh^3-\bar{w}h^3+w\bar{w}h^2$ and hence $\mathrm{Sp}(y)\neq\{0\}$, $\mathrm{Sp}(y^*y)=\{0\}$. Thus, on the one hand, $r(y^*y)=0$; on the other hand, $p(y^*)\cdot p(y)\neq 0$, since $y\in\langle h\rangle-\{0\}$ and p is a norm on $\langle h\rangle$. This contradicts (i). Theorem 1 is proved.

Theorem 2. *Let $\mathscr{A}$ be a *-algebra. Let p be a norm on it, and assume that the following hold with suitable positive constants C, D:*

(i) $p(a^*a)\leqq C\cdot p(a^*)\cdot p(a)$ *for all* $a\in\mathscr{A}$,

(ii) $p(b^*b)\geqq D\cdot p(b^*)\cdot p(b)$ *if* $b\in\langle h\rangle$, $h=h^*\in\mathscr{A}$.

Then $(\mathscr{A},p)$ is an equivalent pre-C^*-algebra (that is, there is a norm on the completion of $(\mathscr{A},p)$, equivalent to p and such that the completion with this norm is a C^*-algebra).

Proof. This identity holds in each *-algebra:

$$4xy=(x^*+y)^*(x^*+y)-(-x^*+y)^*(-x^*+y)+ \tag{1}$$
$$+i(ix^*+y)^*(ix^*+y)-i(-ix^*+y)^*(-ix^*+y).$$

From this and (i) we get

$$4p(xy)\leqq 4C\cdot\big(p(x)+p(y^*)\big)\cdot\big(p(x^*)+p(y)\big). \tag{2}$$

Writing $x=\big(p(v^*)^{1/2}+\varepsilon\big)\big(p(v)^{1/2}+\varepsilon\big)u$, $y=\big(p(u^*)^{1/2}+\varepsilon\big)\big(p(u)^{1/2}+\varepsilon\big)v$ in (2) (where $\varepsilon>0$) and letting ε tend to 0, we infer

$$p(uv)\leqq C\cdot\big(p(u^*)^{1/2}p(v^*)^{1/2}+p(u)^{1/2}p(v)^{1/2}\big)^2. \tag{3}$$

Define a new norm on $\mathscr{A}$ by setting

$$\|a\|=4C\cdot\max\big(p(a^*),p(a)\big)\quad\text{for all}\quad a\in\mathscr{A}. \tag{4}$$

Then we have

$$\|ab\|\leqq\|a\|\cdot\|b\|,\ \|a^*\|=\|a\|,\ p(a)\leqq(4C)^{-1}\|a\|\quad\text{for all}\quad a,b\in\mathscr{A}. \tag{5}$$

Let $\mathscr{B}$ be the completion of $(\mathscr{A}, \|\cdot\|)$. Because of (5) the operations and p have unique continuous extensions to $\mathscr{B}$ and (i), (ii), (4), (5) remain valid in $\mathscr{B}$.

Let r be the spectral radius in $\mathscr{B}$. Since $\mathscr{B}$ is a Banach-algebra, thus

$$r(a) = \lim \|a^n\|^{1/n} \quad \text{for all} \quad a \in \mathscr{B}. \tag{6}$$

If h is a self-adjoint element in $\mathscr{B}$, then $D \cdot p(h)^2 \leqq p(h^2)$, and hence $p(h) \leqq \leqq D^{-1/2} p(h^2)^{1/2} \leqq D^{-1/2} D^{-1/4} p(h^4)^{1/4} \leqq \dots$. Therefore $p(h) \leqq D^{-1} \cdot \limsup p(h^n)^{1/n}$. Thus we see from (5) and (6) that $p(h) \leqq D^{-1} \cdot r(h)$. On the other hand, $r(h) \leqq \leqq \|h\| = 4C \cdot p(h)$ and we have

$$(4C)^{-1} \cdot r(h) \leqq p(h) \leqq D^{-1} \cdot r(h) \quad \text{if} \quad h^* = h \in \mathscr{B}. \tag{7}$$

From this and (i), (ii) we can see that

$$(4C^2)^{-1} \cdot r(a^*a) \leqq p(a^*) \cdot p(a) \leqq D^{-2} \cdot r(a^*a) \quad \text{if} \quad a \in \langle h \rangle,\ h^* = h \in \mathscr{A}; \tag{8}$$

furthermore, p is a norm on $\langle h \rangle$. Thus Theorem 1 shows that $\mathrm{Sp}\,(h) \subset \mathbf{R}$ if $h^* = h \in \mathscr{A}$. Then $r(\sin h) \leqq 1$, $r(\cos h - 1) \leqq 2$ via functional calculus. Since * is continuous in $\mathscr{B}$, hence $\sin h$, $\cos h - 1$ are self-adjoint. Therefore (7) and (4) imply $\|\sin h\| \leqq 4CD^{-1}$, $\|\cos h - 1\| \leqq 8CD^{-1}$, and so

$$\|\exp(ih) - 1\| \leqq 12CD^{-1} \quad \text{if} \quad h^* = h \in \mathscr{A}. \tag{9}$$

The self-adjoint part of $\mathscr{A}$ is dense in that of $\mathscr{B}$, and hence (9) remains valid for $h = h^* \in \mathscr{B}$, too. But this ensures that $\|a\|_c = r(a^*a)^{1/2}$ is a C^*-norm on $\mathscr{B}$, which is equivalent to $\|\cdot\|$ (see [2]). Thus p is continuous with respect to $\|\cdot\|_c$; let $E > 0$ be such that

$$p(a) \leqq E \cdot \|a\|_c \quad \text{for all} \quad a \in \mathscr{B}.$$

Comparing this with (i) and (7) we see that for any $a \in \mathscr{B}$

$$E \cdot \|a\|_c \cdot p(a) = E \cdot \|a^*\|_c \cdot p(a) \geqq p(a^*) \cdot p(a) \geqq (4C^2)^{-1} r(a^*a) = (4C^2)^{-1} \|a\|_c^2,$$

that is, $p(a) \geqq (4EC^2)^{-1} \|a\|_c$. Therefore p is equivalent to $\|\cdot\|_c$. Theorem 2 is proved.

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