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Restrictions of positive operators

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The aim of this note is to give a necessary and sufficient condition for the existence of a positive linear operator on Hilbert space whose restriction to a subset of this space is given. As an application we have a result on division of Hilbert space operators. Krein's theorem on the extension of a bounded symmetric operator from a subspace to the whole space is also established.

Theorem. *Let H be a (complex) Hilbert space, H_0 its subset, and b a function on H_0 with values in H. There exists a positive operator B on H with restriction to H_0 identical to b if and only if*

$$\|\sum_h c_h b(h)\|^2 \leqq M(\sum_h c_h b(h), \sum_h c_h h) \tag{1}$$

holds with some constant $M \geqq 0$ for any finite sequence $\{c_h\}_{h\in H_0}$ of complex numbers indexed by elements of H_0. In this case, $\|B\| \leqq M$.

Proof. The necessity of condition (1) is a simple consequence of a property of positive operators:

$$\|\sum_h c_h b(h)\|^2 = \|B(\sum_h c_h h)\|^2 \leqq \|B\|(B(\sum_h c_h h), \sum_h c_h h) =$$
$$= \|B\|(\sum_h c_h b(h), \sum_h c_h h).$$

Hence (1) holds with $M=\|B\|$ for any finite sequence $\{c_h\}_{h\in H_0}$ of complex numbers.

Conversely, assume that (1) is valid for arbitrary finite sequences $\{c_h\}_{h\in H_0}$ of complex numbers. Since the linear span of H_0 in H, say X, consists of elements $\sum_h c_h h$ with such coefficients, we can introduce a semi-definite inner product on X by

$$\langle\sum_h c_h h, \sum_k d_k k\rangle := (\sum_h c_h b(h), \sum_k d_k k)$$

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for elements $\sum_h c_h h$ and $\sum_k d_k k$ of X. It is well defined because $\sum_h c_h h=0$ implies $\sum_h c_h b(h)=0$, in view of (1). As usual, we get a Hilbert space K from X by factorizing X with respect to the null space of $\langle\cdot,\cdot\rangle$ and by completing with respect to the norm arising on the factor space. For simplicity we denote the image in K of an element of X by the same symbol. With this convention there is a continuous linear operator V from K into H given by

$$V\Big(\sum_h c_h h\Big)=\sum_h c_h b(h)$$

for any element $\sum_h c_h h$ of X. Indeed, according to (1), V is well defined and has norm $\leqq\sqrt{M}$. We are going to prove that $B=VV^*$ is the desired positive operator on H. To see this it is enough to prove that

(2) $$V^*k=k \quad \text{for any } k \text{ in } H_0,$$

since then $Bk=VV^*k=Vk=b(k)$ $(k\in H_0)$.

To prove (2) we see

$$\Big\langle\sum_h c_h h, V^*k\Big\rangle=\Big(V\Big(\sum_h c_h h\Big), k\Big)=\Big(\sum_h c_h b(h), k\Big)=\Big\langle\sum_h c_h h, k\Big\rangle$$

for any element $\sum_h c_h h$ in K. Since these elements are dense in K, the statement follows. The proof of the Theorem is complete.

Corollary 1. *Let A and C be bounded linear operators on the Hilbert space H. There exists a positive operator B on H such that $A=BC$ if and only if there exists a constant $M\geqq 0$ such that*

(3) $$A^*A\leqq M\cdot C^*A$$

Proof. For $h=Ck$, k in H, let $b(h)=Ak$. Then (1) takes the form

$$\|Ak\|^2\leqq M(Ak, Ck)=M(C^*Ak, k)$$

for any k in H, which is the same as (3).

Remark. If b is a linear map of a subspace $H_0\subset H$ into H such that for some constant $M\geqq 0$

(4) $$\|b(h)\|^2\leqq M\big(b(h), h\big) \quad (h\in H_0),$$

then b has a positive extension B defined on H. This is a consequence of the Theorem. On the other hand, the usual condition for positivity

(5) $$0\leqq\big(b(h), h\big) \quad (h\in H_0)$$

is not enough for the existence of such a positive extension: a simple example is the case when $\big(b(h), h\big)=0\neq b(h)$ for some element h in H_0.

Corollary 2. *Let* b *a function on a subset* H_0 *of the Hilbert space* H *with values in* H. *There exists a self-adjoint operator* B *on* H *such that* $m \cdot I \leqq B \leqq M \cdot I$, *where* $m \leqq M$ *are real constants, if and only if*

$$\Big\|\sum_h c_h\big(b(h)-m\cdot h\big)\Big\|^2 \leqq (M-m)\Big(\sum_h c_h\big(b(h)-m\cdot h\big),\ \sum_h c_h h\Big) \tag{6}$$

holds for any finite sequence $\{c_h\}$ *of complex numbers indexed by elements of* H_0.

Corollary 3 (M. G. Kreĭn, cf. [1]). *Let* b *a symmetric and bounded linear operator from a subspace* H_0 *of a Hilbert space* H *into* H. *Then there exists a self-adjoint extension of* b *to the whole space* H *with the same bound.*

Proof. Let M be the norm of the operator b, that is,

$$M = \sup\{\|b(h)\|:\ h\in H_0,\ \|h\| \leqq 1\}.$$

We have then for any h in H_0

$$\|b(h)+M\cdot h\|^2 = \|b(h)\|^2+2M\cdot\big(b(h), h\big)+M^2\cdot\|h\|^2 \leqq$$
$$\leqq 2M\cdot\big(b(h), h\big)+2M^2\cdot\|h\|^2 = 2M\big(b(h)+Mh, h\big).$$

But this is nothing else than (6) in case $-m=M$ and b is a linear function, an operator. As a consequence, $-M\cdot I\leqq B\leqq M\cdot I$ holds for a self-adjoint extension B of b to the space H. This was to be proved.

Reference

[1] F. Riesz and B. Sz.-Nagy, *Functional Analysis*, Ungar (New York, 1960).

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