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Characterizations and invariant subspaces of composition operators

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1. Preliminaries. Let $(X, \mathcal{S}, \lambda)$ be a σ-finite measure space and let T be a non-singular measurable transformation from X into itself. Then the composition transformation C_T from $L^p(\lambda)$ into the space of all complex-valued functions on X is defined by

$$C_T f = f \circ T \quad \text{for every} \quad f \in L^p(\lambda).$$

If C_T happens to be a bounded operator on $L^p(\lambda)$, then we call it a composition operator induced by T.

Let $X=N$, the set of all non-zero positive integers and $\mathcal{S}=P(N)$, the power set of N. Then we can define the measure λ on $P(N)$ by

$$\lambda(E) = \sum_{n\in E}' w_n \quad \text{for every} \quad E\in P(N),$$

where $w=\{w_n\}$ is a sequence of strictly positive real numbers. If $p=2$, then $L^p(\lambda)$ is a Hilbert space with the inner product defined by

$$\langle f, g\rangle = \sum w_n f(n)\overline{g(n)}$$

for all $f, g\in L^p(\lambda)$. This Hilbert space is denoted by l^2_w, and is called a weighted sequence space. By $B(l^2_w)$ we mean the Banach algebra of all bounded linear operators on l^2_w. Let $\{e_n\}$ be the sequence defined by $e_n(p)=\delta_{np}$, the Kronecker delta. If C_T is a composition operator, then C^*_T, the adjoint of C_T, is given by

$$(C^*_T f)(n) = \frac{1}{w_n} \int_{T^{-1}(\{n\})} f\,d\lambda \quad \text{(cf. [4])}.$$

In the present note certain criteria for a bounded operator to be a composition operator are obtained. It is also shown that every composition operator on l^2_w has an invariant subspace. This generalizes a result of SINGH and KOMAL [5] to the weighted sequence spaces.

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2. Criteria for a bounded operator to be a composition operator. In this section we obtain two different criteria for a bounded operator to be a composition operator.

Theorem 2.1. *Let* $A \in B(l^2_w)$. *Then* A *is a composition operator if and only if for every* $n \in N$, *there exists* $m \in N$ *such that* $A^* e'_n = e'_m$, *where* $e'_n = e_n / w_n$.

Proof. The proof follows from Nordgren [2]. Here e'_n's play the role of kernel functions.

Theorem 2.2. *Let* $A \in B(l^2_w)$. *Then* A *is a composition operator if and only if there exists a partition* $\{E_n\}$ *of* N *such that* $Ae_n = X_{E_n}$, *where* X_E *denotes the characteristic function of a set* E.

Proof. Suppose A is a composition operator. Then $A = C_T$ for some mapping $T: N \to N$. The choice $T^{-1}(\{n\}) = E_n$ clearly suits our requirements.

Conversely, if A satisfies the condition of the theorem, then we may define a mapping $T: N \to N$ by $T(m) = n$ for $m \in E_n$. Now $Ae_n = C_T e_n$ and so $Ae_n / \sqrt{w_n} = C_T e_n / \sqrt{w_n}$ for every $n \in N$. Thus A and C_T agree on the basis vectors of l^2_w. It is easy to show that C_T is a bounded operator. Hence $Af = C_T f$ for every $f \in l^2_w$. This completes the proof.

Theorem 2.3. *Let* $T: N \to N$ *be a surjective mapping such that* $C_T \in B(l^2_w)$ *and let* $A \in B(l^2_w)$. *Then* $C_T A$ *is a composition operator if and only if* A *is a composition operator.*

Proof. The proof is an immediate consequence of Theorem 2.1. Indeed if $C_T A = C_S$ then $A^* C_T^* = C_S^*$, i.e., $A^* e'_{T(N)} = A^* C_T^* e'_k = C_S^* e'_k = e'_{S(K)}$ for every $k \in N$. Since $T(N) = N$, for every $m \in N$ there exists $n \in N$ such that $A^* e'_m = e'_n$.

Theorem 2.4. *Let* $T: N \to N$ *be an injection and let* C_T, $A \in B(l^2_w)$. *Then* AC_T *is a composition operator if and only if* A *is a composition operator.*

Proof. Suppose AC_T is a composition operator. Then there is a mapping $S: N \to N$ such that $AC_T = C_S$. Now $Ae_n = AC_T e_{T(n)} = C_S e_{T(n)} = X_{E_n}$, where $E_n = S^{-1}(\{T(n)\})$. By Theorem 2.2, $\{E_n\}$ is a partition of N. Hence A is a composition operator. The proof of the sufficient part of the theorem is straight forward.

Theorem 2.5. *Let* $A \in B(l^2_w)$. *Then* A *is a unitary composition operator if and only if*

$$\{Ae'_n : n \in N\} = \{e'_n : n \in N\} = \{A^* e'_n : n \in N\}.$$

Proof. Assume A is a unitary composition operator. Then by Theorem 2.1

$$\{A^* e'_n : n \in N\} \subseteqq \{e'_n : n \in N\} = \{AA^* e'_n : n \in N\} \subseteqq \{Ae'_n : n \in N\}.$$

From Theorem 3.1 of [6], A^* is a composition operator and hence also the converse inclusions hold.

If the conditions of the theorem are true, then by Theorem 2.1 both A and A^* are composition operators. Hence by Theorem 3.1 of [6] A is a unitary composition operator.

3. Invariant subspaces. Definition. Let $T: N \to N$ be a mapping. Then two integers m and n are said to be in the same orbit of T if each can be reached from the other by composing T and T^{-1} (T^{-1} means a multivalued function) sufficiently many times.

Definition. A closed subspace M of a Hilbert space is called an invariant subspace of A if $AM \subseteq M$.

One of the most outstanding unsolved problems of operator theory is the Invariant Subspace Problem. The problem is simple to state: Does every operator on an infinite dimensional separable Hilbert space have a non-trivial invariant subspace? The answer is not yet known. Recently SINGH and KOMÁL [5] obtained that every composition operator on l^2 has a non-trivial invariant subspace. In the following theorem we generalize this result to the weighted sequence spaces.

Theorem 3.1. *Let $C_T \in B(l^2_w)$. Then C_T has a non-trivial invariant subspace.*

Proof. Suppose C_T is a composition operator induced by a mapping $T: N \to N$. Then either T is invertible or T is not invertible. First assume that T is invertible. Then take $n \in N$. Now either the orbit of n is equal to N or it is not equal to N. Suppose $o(n) = N$, where $o(n)$ is the orbit of n. Then let

$$E_n = \{(T^m)^{-1}(\{n\}) \colon m \in N\}.$$

If $l^2_{E_n} = \text{span}\,\{e'_m \colon m \in E_n\}$, then clearly $l^2_{E_n}$ is invariant under C_T. Next, if $o(n) \neq N$, then $l^2_{E_n} = \text{span}\,\{e'_m \colon m \in o(n)\}$ is an invariant subspace of C_T.

Further, suppose T is not invertible. Then, either T is not an injection or T is not a surjection. If T is not an injection, then C_T has not dense range and hence $\overline{\operatorname{ran} C_T}$ is invariant under C_T. And, if T is not a surjection, then C_T has a non-trivial kernel and hence $\ker C_T$ is invariant under C_T. This completes the proof.

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