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Spectral properties of elementary operators

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1. Introduction. Let $\mathfrak{H}$ denote an infinite dimensional complex Hilbert space and let $\mathfrak{L}(\mathfrak{H})$ denote the algebra of all bounded linear operators on $\mathfrak{H}$. For an integer $N \geqq 1$, let $A=(A_1, \ldots, A_N)$ and $B=(B_1, \ldots, B_N)$ denote N-tuples of mutually commuting operators in $\mathfrak{L}(\mathfrak{H})$. The *elementary operator* $\mathfrak{R} \equiv \mathfrak{R}(A, B)$ acting on $\mathfrak{L}(\mathfrak{H})$ is defined by $\mathfrak{R}(X)=A_1XB_1+\ldots+A_NXB_N$ $(X \in \mathfrak{L}(\mathfrak{H}))$. Spectral, metric, and algebraic properties of elementary operators have been studied from a variety of viewpoints [1], [2], [5], [7], [14], [18], [20], [22]. In particular, the *generalized derivation* $\mathfrak{T}(A, B)$ defined by $\mathfrak{T}(X)=AX-XB$, has been analyzed in considerable detail, and various characterizations have been given for the cases when a generalized derivation has dense range [11], or is surjective, bounded below [6], [8], Fredholm [9], or semi- Fredholm [10]. Analogous results are also known for the restriction of a generalized derivation to a norm ideal in $\mathfrak{L}(\mathfrak{H})$ [8], [12].

In the present note we extend several results concerning generalized derivations to an arbitrary elementary operator $\mathfrak{R}$ and its restriction $\mathfrak{R}_{\mathfrak{J}}$ to a norm ideal $\mathfrak{J}$. Descriptions of the right and left spectra of $\mathfrak{R}$ were determined by R. Harte [16] (cf. [5]) and in section 2 we obtain qualitative refinements of these results; we show that $\mathfrak{R}-\lambda$ is right invertible in $\mathfrak{L}(\mathfrak{L}(\mathfrak{H}))$ (and thus surjective) if its range contains each rank one operator, and is left invertible (hence bounded below) if its restriction to the set of rank one operators is bounded below. These results allow us to relate spectral properties of $\mathfrak{R}$ to those of $\mathfrak{R}_{\mathfrak{J}}$ (Theorem 2.3, Theorem 2.8). We also characterize the case when $\mathfrak{R}-\lambda$ has dense range, extending the characterization given for $\mathfrak{T}$ in [11].

In section 3 we specialize to study the *elementary multiplication operator* $\mathfrak{S} \equiv \mathfrak{S}(A, B)$ defined by $\mathfrak{S}(X)=AXB$. The essential spectrum and index function of $\mathfrak{S}$ was determined in [12] and here we describe the semi-Fredholm domain of

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$\mathfrak{S}$ and conditions for $\mathfrak{S}-\lambda$ to have index equal to $+\infty$ or $-\infty$. Analogous results are given for the semi-Fredholm domain of $\mathfrak{S}_{\mathfrak{J}}$. These results complement (but are independent of) the characterization of the semi-Fredholm domain of $\mathfrak{T}$ given in [9] and [10], and we believe they will prove helpful in studying the semi-Fredholm domain and index function of a general elementary operator.

We conclude this section with some preliminary results and notation. Let $\mathfrak{A}$ denote a complex Banach algebra with identity 1, and let $\mathfrak{A}^{(N)}$ denote an N-fold copy of $\mathfrak{A}$. For $a=(a_1, \ldots, a_N)\in\mathfrak{A}^{(N)}$, the joint left spectrum of a in the sense of R. Harte [15] is defined by $\sigma_l(a)=\{\alpha\equiv(\alpha_1, \ldots, \alpha_N)\in\mathbf{C}^{(N)}$: there exists no N-tuple $(b_1, \ldots, b_N)\in\mathfrak{A}^{(N)}$ such that $b_1(a_1-\alpha_1)+\ldots+b_N(a_N-\alpha_N)=1\}$; the joint right spectrum of a, $\sigma_r(a)$, is defined analogously, and the joint spectrum of a is defined by $\sigma(a)=\sigma_l(a)\cup\sigma_r(a)$ [15]. For $a\in\mathfrak{A}$, L_a and R_a denote, respectively, the left and right multiplication operators on $\mathfrak{A}$ induced by a, i.e. $L_a(x)=ax$ and $R_a(x)=xa\ (x\in\mathfrak{A})$. For $a=(a_1, \ldots, a_N)\in\mathfrak{A}^{(N)}$, we set $L_a=(L_{a_1}, \ldots, L_{a_N})$ and $R_a=(R_{a_1}, \ldots, R_{a_N})$. When $\mathfrak{A}=\mathfrak{L}(\mathfrak{H})$, $A=(A_1, \ldots, A_N)\in\mathfrak{A}^{(N)}$, and $\mathfrak{J}$ is a norm ideal in $\mathfrak{L}(\mathfrak{H})$, we define $L_A\mid\mathfrak{J}=(L_{A_1}\mid\mathfrak{J}, \ldots, L_{A_N}\mid\mathfrak{J})$. In this case the left joint spectrum of A may be described in more detail as follows.

Lemma 1.1. [15, Theorem 2.5] *The following are equivalent.*

i) $\alpha\in\sigma_l(A)$;

ii) $\sum_{i=1}^{N}(A_i-\alpha_i)^*(A_i-\alpha_i)$ *is not invertible;*

iii) *There exists a sequence of unit vectors* $\{x_k\}\subset\mathfrak{H}$ *such that*

$$\lim_{k\to\infty}\sum_{i=1}^{N}\|(A_i-\alpha_i)x_k\|=0.$$

Let $\mathfrak{K}(\mathfrak{H})$ denote the ideal of all compact operators in $\mathfrak{L}(\mathfrak{H})$ and let $\mathfrak{A}(\mathfrak{H})=\mathfrak{L}(\mathfrak{H})/\mathfrak{K}(\mathfrak{H})$ denote the Calkin algebra; for $T\in\mathfrak{L}(\mathfrak{H})$, $\tilde{T}$ denotes the image of T in $\mathfrak{A}(\mathfrak{H})$ under the canonical projection. For an N-tuple of operators $T=(T_1, \ldots, T_N)$, we set $\tilde{T}=(\tilde{T}_1, \ldots, \tilde{T}_N)$ and denote the [left] [right] joint essential spectrum of T by $[\sigma_{le}(T)]$ $[\sigma_{re}(T)]$ $\sigma_e(T)$, i.e. $\sigma_{le}(T)=\sigma_l(\tilde{T})$, $\sigma_{re}(T)=\sigma_r(\tilde{T})$, and $\sigma_e(T)=\sigma(\tilde{T})$. The following result is contained in [24, Corollary 2.5, Theorem 2.6].

Lemma 1.2. *The following are equivalent.*

i) $\alpha\in\sigma_{le}(T)$;

ii) $\sum_{i=1}^{N}(T_i-\alpha_i)^*(T_i-\alpha_i)$ *is not Fredholm;*

iii) *There exists an orthonormal sequence* $\{e_n\}_{n=1}^{\infty}\subset\mathfrak{H}$ *such that*

$$\lim_{n\to\infty}\sum_{i=1}^{N}\|(T_i-\alpha_i)e_n\|=0.$$

For $T\in\mathfrak{L}(\mathfrak{H})^{(N)}$, let $\sigma_p(T)=\{\alpha\in\mathbf{C}^{(N)}$: there exists a unit vector $x\in\mathfrak{H}$ such that $(T_i-\alpha_i)x=0,\ 1\leqq i\leqq N\}$, the joint point spectrum of T. Lemmas 1.1 and 1.2 readily imply that $\sigma_l(T)=\sigma_{le}(T)\cup\sigma_p(T)$. For $T\in\mathfrak{L}(\mathfrak{H})^{(N)}$ and $\alpha\in\mathbf{C}^{(N)}$, let $T^*=(T_1^*, \ldots, T_N^*)$ and $\bar{\alpha}=(\bar{\alpha}_1, \ldots, \bar{\alpha}_N)$. Analogues of the preceding results for right spectra follow from the identity $\sigma_r(T)=[\sigma_l(T^*)]^*\equiv\{\bar{\alpha}\in\mathbf{C}^{(N)}: \alpha\in\sigma_l(T^*)\}$; in particular, $\sigma_r(T)=\sigma_{re}(T)\cup\sigma_p(T^*)^*$.

Let $(\mathfrak{I}, |||\cdot|||)$ denote a norm ideal in $\mathfrak{L}(\mathfrak{H})$ in the sense of [21]. Clearly $\mathfrak{I}$ is $\mathfrak{R}$-invariant and $\mathfrak{R}_{\mathfrak{I}}$, the restriction of $\mathfrak{R}$ to $\mathfrak{I}$, is in $\mathfrak{L}(\mathfrak{I})$. If $\mathfrak{I}=C_p$ $(1\leqq p\leqq\infty)$ (the Schatten p-ideal [21]), we denote $\mathfrak{R}_{\mathfrak{I}}$ by $\mathfrak{R}_p$. For $x, y\in\mathfrak{H}$, $x\otimes y$ denotes the rank one operator defined by $(x\otimes y)h=(h, y)x$. $\mathfrak{I}_1$ denotes the set of all rank one operators in $\mathfrak{L}(\mathfrak{H})$; if $F\in\mathscr{F}_1$, then $|||F|||=\|F\|$ [21].

Let $\mathfrak{X}$ denote a complex Banach space and let $\mathfrak{L}(\mathfrak{X})$ denote the algebra of bounded linear operators on $\mathfrak{X}$. For $T\in\mathfrak{L}(\mathfrak{X})$, let $\ker(T)$ and $R(T)$ denote the kernel and range of T; we set $\operatorname{nul}(T)=\dim(\ker(T))$ and $\operatorname{def}(T)=\dim(\mathfrak{X}/R(T)^-)$ (where $R(T)^-$ denotes the norm closure of $R(T)$). T is *semi-Fredholm* if $R(T)$ is closed and either $\operatorname{nul}(T)<\infty$ or $\operatorname{def}(T)<\infty$; in this case, the index of T is defined by $\operatorname{ind}(T)=\operatorname{nul}(T)-\operatorname{def}(T)$ [17]. T is Fredholm if $R(T)$ is closed and both $\operatorname{nul}(T)$ and $\operatorname{def}(T)$ are finite; $\sigma_e(T)=\{\lambda\in\mathbf{C}: T-\lambda$ is not Fredholm$\}$ is the essential spectrum of T. The semi-Fredholm domain of T is defined by $\varrho_{SF}(T)=\{\lambda\in\mathbf{C}: T-\lambda$ is semi-Fredholm$\}$; we denote the complement $\mathbf{C}\setminus\varrho_{SF}(T)$ by $\sigma_{SF}(T)$. For $\alpha=(\alpha_1, \ldots, \alpha_N)$ and $\beta=(\beta_1, \ldots, \beta_N)$ in $\mathbf{C}^{(N)}$ we set $\alpha\circ\beta=\alpha_1\beta_1+\ldots+\alpha_N\beta_N$, and for $\sigma, \varrho\subset\mathbf{C}^{(N)}$, let $\sigma\circ\varrho=\{\alpha\circ\beta: \alpha\in\sigma, \beta\in\varrho\}$. If $N=1$, we abbreviate $\sigma\circ\varrho$ by $\sigma\varrho$. In [12] it is proved that $\sigma_e(\mathfrak{S}(A, B))=\sigma_e(\mathfrak{S}_{\mathfrak{I}})=\sigma(A, B)\equiv\sigma_e(A)\sigma(B)\cup\sigma(A)\sigma_e(B)$. In the sequel we will prove that $\sigma_{SF}(\mathfrak{S})=\sigma_{SF}(\mathfrak{S}_{\mathfrak{I}})=[\sigma_l(A)\sigma_{re}(B)\cup\sigma_{le}(A)\sigma_r(B)]\cap[\sigma_r(A)\sigma_{le}(B)\cup\sigma_{re}(A)\sigma_l(B)]$ (Corollary 3.12, Theorem 3.14).

2. Spectral properties of elementary operators. In this section we present several equivalent descriptions of the left and right spectra of elementary operators and we describe the elementary operators with dense range. The following result will be used to show that an elementary operator is surjective if its range contains each rank one operator: the proof is motivated by that of [8, Theorem 2.1].

Lemma 2.1. *If* $\lambda\in\sigma_r(A)\circ\sigma_l(B)$, *then the range of* $\mathfrak{R}-\lambda$ *does not contain every rank one operator.*

Proof. Let $\alpha\in\sigma_r(A)$ and $\beta\in\sigma_l(B)$ be such that $\lambda=\alpha\circ\beta$. We consider several cases for the location of α and β.

i) $\alpha\in\sigma_r(A)\setminus\sigma_{re}(A)$, $\beta\in\sigma_l(B)\setminus\sigma_{le}(B)$. In this case there exist unit vectors e and f in $\mathfrak{H}$ such that $(A_i-\alpha_i)^*f=(B_i-\beta_i)e=0$ $(1\leqq i\leqq n)$. Let $Yx=(x, e)f$ $(x\in\mathfrak{H})$. If $X\in\mathfrak{L}(\mathfrak{H})$ satisfies $(\mathfrak{R}-\lambda)(X)=Y$, then $1=(Ye, f)=$

$=\sum_{i=1}^{N}[((A_i-\alpha_i)XB_ie,f)+(\alpha_iX(B-\beta_i)e,f)]=\sum_{i=1}^{N}(XB_ie,(A_i-\alpha_i)^*f)=0$, a contradiction; thus the rank one operator Y is not in the range of $\mathfrak{R}-\lambda$.

ii) $\alpha\in\sigma_{re}(A)$, $\beta\in\sigma_{le}(B)$. (Clearly, we may assume that $\max\{\|B_i\|\}>0$.) The following argument is based on J. G. Stampfli's proof that the range of an inner derivation contains no nontrivial unitarily invariant subset of $\mathfrak{L}(\mathfrak{H})$ [23, Theorem 2]; we prove a similar result for $\mathfrak{R}-\lambda$. Let Y be an operator in $\mathfrak{L}(\mathfrak{H})$ that is not a scalar multiple of the identity. We will construct a unitary operator U such that U^*YU is not in the range of $\mathfrak{R}-\lambda$. Let $\{h_n\}_{n=1}^{\infty}$ denote an orthonormal seqence such that $(Yh_n, h_m)\neq 0$ for $n, m\geqq 1$ [19, Theorem 2]. Let $\delta_n=(Yh_{3n}, h_{3n+1})$ for $n\geqq 1$. Let $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ denote orthonormal sequences in $\mathfrak{H}$ such that the following properties are satisfied: i) $\sum_{i=1}^{N}\|(B_i-\beta_i)f_m\|\leqq|\delta_m|/m$ $(m\geqq 1)$; ii) $\sum_{i=1}^{N}\|(A_i-\alpha_i)^*g_m\|<|\delta_m|/m$; iii) $(f_n, g_m)=0$ for $1\leqq n, m$; iv) the subspace spanned by all of the vectors f_n and g_n $(n\geqq 1)$ has an infinite dimensional orthocomplement in $\mathfrak{H}$. Using iii) and iv) we may define a unitary operator U which satisfies $Uf_n=h_{3n}$ and $Ug_n=h_{3n+1}$ $(n\geqq 1)$. If $X\in\mathfrak{L}(\mathfrak{H})$ satisfies $(\mathfrak{R}-\lambda)(X)=U^*YU$, then

$$0<|\delta_n|=|(Yh_{3n}, h_{3n+1})|=|(YUf_n, Ug_n)|=|(U^*YUf_n, g_n)|=$$

$$=\left|\sum_{i=1}^{N}[((A_i-\alpha_i)XB_if_n, g_n)+(\alpha_iX(B_i-\beta_i)f_n, g_n)]\right|\leqq$$

$$\leqq\|X\|\,[\max\{\|B_i\|\}+\max\{|\alpha_i|\}]\,|\delta_n|/n.$$

Thus $\|X\|\geqq n/(\max\{\|B_i\|\}+\max\{|\alpha_i|\})$ for every $n\geqq 1$, so U^*YU is not in the range of $\mathfrak{R}-\lambda$. The proof is completed by taking Y to be a rank one operator.

iii) $\alpha\in\sigma_r(A)\setminus\sigma_{re}(A)$, $\beta\in\sigma_{le}(B)$. If $\beta\in\sigma_p(B)$ we may use the same proof as in part i). We may thus assume that $\beta\notin\sigma_p(B)$. Let $\{e_n\}_{n=1}^{\infty}$ denote an orthonormal sequence such that $0<\sum_{i=1}^{N}\|(B_i-\beta_i)e_n\|<1/n^2$ $(n\geqq 1)$, and set $\delta_n=n\sum_{i=1}^{N}\|(B_i-\beta_i)e_n\|$. Let f be a unit vector such that $(A_i-\alpha_i)^*f=0$ $(1\leqq i\leqq n)$. Since $0<\delta_n<1/n$, we may define a rank one operator Y by the relations $Ye_n=\delta_nf$ $(n\geqq 1)$ and $Yx=0$ if $(x, e_n)=0$ for each n. If $X\in\mathfrak{L}(\mathfrak{H})$ satisfies $(\mathfrak{R}-\lambda)(X)=Y$, then

$$\delta_n=(Ye_n,f)=\sum_{i=1}^{N}[((A_i-\alpha_i)XB_ie_n,f)+(\alpha_iX(B_i-\beta_i)e_n,f)]=$$

$$=\sum_{i=1}^{N}(\alpha_iX(B_i-\beta_i)e_n,f)\quad(n\geqq 1).$$

Thus $0<\delta_n\leqq(\max|\alpha_i|)\|X\|\left(\sum_{i=1}^{N}\|(B_i-\beta_i)e_n\|\right)$ and so $\|X\|\geqq n/\max\{|\alpha_i|\}$ $(n\geqq 1)$. This contradiction shows that Y is not in the range of $\mathfrak{R}-\lambda$.

iv) $\alpha\in\sigma_{re}(A)$, $\beta\in\sigma_l(B)\setminus\sigma_{le}(B)$. Since $\bar{\alpha}\in\sigma_{le}(A^*)$ and $\bar{\beta}\in\sigma_r(B^*)\setminus\sigma_{re}(B^*)$, then iii) implies that there exists a rank one operator Y such that $\left(\sum_{i=1}^{N} B_i^* X^* A_i^*\right)-$ $-\bar{\alpha}\circ\bar{\beta}X^*=Y^*$ has no solution. Then $(\mathfrak{R}-\lambda)(X)=Y$ has no solution and the proof is complete.

Lemma 2.2. i) $\sigma_r(\mathfrak{R}|\mathfrak{J})\subset\sigma_r(A)\circ\sigma_l(B)$;
ii) $\sigma_r(\mathfrak{R})\subset\sigma_r(A)\circ\sigma_l(B)$.

Proof. Part ii) is contained in [16, Theorem 3.4]. The proof of i) is similar. The argument is essentially that used in the proof of [5, Lemma 3]. We first note that $\sigma_r(L_A\mid\mathfrak{J}, R_B\mid\mathfrak{J})\subset\sigma_r(A)\times\sigma_1(B)$. Indeed, suppose $(\alpha,\beta)\in\mathbf{C}^N\times\mathbf{C}^N$ and $\alpha\notin\sigma_r(A)$. There exists an N-tuple of operators $(R_1, \dots, R_N)$ such that $\sum_{i=1}^{N}(A_i-\alpha_i)R_i=1$, and thus $\sum_{i=1}^{N}(L_{A_i}\mid\mathfrak{J}-\alpha_i)(L_{R_i}\mid\mathfrak{J})=1\in\mathfrak{L}(\mathfrak{J})$, so that $(\alpha,\beta)\notin\sigma_r(L_A\mid\mathfrak{J}, R_B\mid\mathfrak{J})$. The proof for the case when $\beta\notin\sigma_l(B)$ is similar. For $z=(z_1, \dots, z_N)$ and $w=$ $=(w_1, \dots, w_N)$ we define the $2N$-variable polynomial p by $p(z, w)=p(z_1, \dots, z_N,$ $w_1, \dots, w_N)=\sum_{i=1}^{N} z_i w_i$. Since $(L_A\mid\mathfrak{J}, R_B\mid\mathfrak{J})$ is a commutative $2N$-tuple in $\mathfrak{L}(\mathfrak{J})$, the spectral mapping theorem for right spectra [15], [16, Theorem 1.2] implies that

$$\sigma_r(\mathfrak{R}|\mathfrak{J}) = \sigma_r\big(p(L_A|\mathfrak{J}, R_B|\mathfrak{J})\big) = p\big(\sigma_r(L_A|\mathfrak{J}, R_B|\mathfrak{J})\big) \subset p\big(\sigma_r(A)\times\sigma_l(B)\big) =$$

$$= \sigma_r(A)\circ\sigma_l(B).$$

Theorem 2.3. *For* $\lambda\in\mathbf{C}$ *and* $\mathfrak{R}=\mathfrak{R}(A, B)$, *the following are equivalent:*
i) $\mathfrak{R}-\lambda$ *is surjective;*
ii) *The range of* $\mathfrak{R}-\lambda$ *contains each rank one operator;*
iii) $\lambda\notin\sigma_r(A)\circ\sigma_l(B)$;
iv) $\mathfrak{R}-\lambda$ *is right invertible in* $\mathfrak{L}(\mathfrak{L}(\mathfrak{H}))$;
v) $\mathfrak{R}_{\mathfrak{J}}-\lambda$ *is right invertible for some norm ideal* $\mathfrak{J}$;
vi) $\mathfrak{R}_{\mathfrak{J}}-\lambda$ *is surjective for some norm ideal* $\mathfrak{J}$;
vii) $\mathfrak{R}_{\mathfrak{J}}-\lambda$ *is right invertible in* $\mathfrak{L}(\mathfrak{J})$ *for every norm ideal* $\mathfrak{J}$;
viii) $\mathfrak{R}_{\mathfrak{J}}-\lambda$ *is surjective for every norm ideal* $\mathfrak{J}$.

Proof. i)⇒ii) is clear, ii)⇒iii) follows from Lemma 2.1, iii)⇒iv) follows from Lemma 2.2, and iv)⇒i) is clear, so i)—iv) are equivalent. Lemma 2.1 implies that vi)⇒ii)⇒iii) and Lemma 2.2 implies that iii)⇒v)⇒vi), so iii), v) and vi) are equivalent. Similarly, we have vii)⇒viii)⇒ii)⇒iii)⇒vi).

We next begin our consideration of elementary operators with dense range.

Corresponding to $\mathfrak{R}(A, B)$ we define an operator $\tilde{\mathfrak{R}}(\tilde{A}, \tilde{B})$ on the Calkin algebra $\mathfrak{A}(\mathfrak{H})$ by $\tilde{\mathfrak{R}}(\tilde{X})=\widetilde{\mathfrak{R}(X)}=\sum_{i=1}^{N}\tilde{A}_i\tilde{X}\tilde{B}_i$ $(X\in\mathfrak{L}(\mathfrak{H}))$.

Lemma 2.4. $\sigma_r(\tilde{\mathfrak{R}})\subset\sigma_{re}(A)\circ\sigma_{le}(B)$.

Proof. The proof is similar to that of Lemma 2.2 ; as before, $\sigma_r(L_{\tilde{A}}, R_{\tilde{B}})\subset\subset\sigma_{re}(A)\times\sigma_{le}(B)$. Let $p(z, w)=p(z_1, ..., z_N, w_1, ..., w_N)=z_1w_1+...+z_Nw_N$. Since $\tilde{\mathfrak{R}}=p(L_{\tilde{A}}, R_{\tilde{B}})$ and $(L_{\tilde{A}}, R_{\tilde{B}})$ is a commutative $2N$-tuple of elements of $\mathfrak{L}(\mathfrak{A}(\mathfrak{H}))$, the spectral mapping theorem for right spectra [15] implies that

$$\sigma_r(\tilde{\mathfrak{R}}) = \sigma_r(p(L_{\tilde{A}}, R_{\tilde{B}})) = p(\sigma_r(L_{\tilde{A}}, R_{\tilde{B}})) \subset$$
$$\subset p(\sigma_{re}(A)\times\sigma_{le}(B)) = \sigma_{re}(A)\circ\sigma_{le}(B).$$

Recall that $C_\infty^*\approx C_1$; a trace class operator K corresponds to the functional $f_K\in C_\infty^*$ defined by $f_K(X)=\mathrm{tr}\,(KX)$ [21]. Under this identification $\mathfrak{S}_\infty(A, B)^*==\mathfrak{S}_1(B, A)$. Indeed, for $X\in C_\infty$ and $K\in C_1$ we have $\mathfrak{S}_\infty(A, B)^*(f_K)(X)==\mathrm{tr}\,(KAXB)=\mathrm{tr}\,(BKAX)=f_{BKA}(X)$. Recall also that $C_1^*\approx\mathfrak{L}(\mathfrak{H})$, where $T\in\mathfrak{L}(\mathfrak{H})$ corresponds to the functional $f_T\in C_1^*$ defined by $f_T(K)=\mathrm{tr}\,(TK)$. For $K\in C_1$ and $T\in\mathfrak{L}(\mathfrak{H})$, $\mathfrak{S}_1(B, A)^*(f_T)(K)=\mathrm{tr}\,(TBKA)=\mathrm{tr}\,(ATBK)=f_{ATB}(K)$, and therefore $\mathfrak{S}_1(B, A)^* = \mathfrak{S}(A, B)$. By linearity, we see that $\mathfrak{R}_\infty(A, B)^* = \mathfrak{R}_1(B, A)$ and $\mathfrak{R}_1(B, A)^*=\mathfrak{R}(A, B)$.

Theorem 2.5. *The following are equivalent for* $\lambda\in\mathbf{C}$.

i) $\mathfrak{R}(A, B)-\lambda$ *has norm dense range;*

ii) $\lambda\notin\sigma_{re}(A)\circ\sigma_{le}(B)$ *and* $\mathfrak{R}_1(B, A)$ *is injective;*

iii) *For* $\varepsilon>0$ *and* $Y\in\mathfrak{L}(\mathfrak{H})$, *there exists* $X\in\mathfrak{L}(\mathfrak{H})$ *such that* $(\mathfrak{R}-\lambda)(X)-Y$ *is a compact operator with norm less than* ε.

Proof. We first prove ii)$\Rightarrow$iii). Suppose ii) is satisfied, let $\varepsilon>0$, and let Y be in $\mathfrak{L}(\mathfrak{H})$. Lemma 2.4 shows that $\tilde{\mathfrak{R}}-\lambda$ is surjective; thus there exists $X\in\mathfrak{L}(\mathfrak{H})$ and $K\in\mathfrak{K}(\mathfrak{H})$ such that $(\mathfrak{R}-\lambda)(X)-Y=K$. Since $\mathfrak{R}_1(B, A)-\lambda$ is injective, $\mathfrak{R}_\infty(A, B)-\lambda$ has dense range. Thus there exists $\{K_n\}\subset\mathfrak{K}(\mathfrak{H})$ such that $\|(\mathfrak{R}-\lambda)(K_n)--K\|\to 0$. Now $(\mathfrak{R}-\lambda)(X-K_n)-Y=K-(\mathfrak{R}-\lambda)(K_n)\in\mathfrak{K}(\mathfrak{H})$, and for sufficiently large n, $\|K-(\mathfrak{R}-\lambda)(K_n)\|<\varepsilon$.

Clearly iii)$\Rightarrow$i), so it suffices to prove that i)$\Rightarrow$ii). If $\mathfrak{R}(A, B)-\lambda$ has dense range, then duality implies that $\mathfrak{R}_1(B, A)-\lambda$ is injective. Suppose $\lambda\in\sigma_{re}(A)\circ\circ\sigma_{le}(B)$; it suffices to prove that the range of $\mathfrak{R}(A, B)-\lambda$ is not dense. Let $\alpha\in\sigma_{re}(A)$ and $\beta\in\sigma_{le}(B)$ satisfy $\lambda=\alpha\circ\beta$. Let $\{e_n\}$ and $\{f_n\}$ denote orthonormal sequences such that $\sum_{i=1}^{N}\|(A_i-\alpha_i)^*e_n\|\to 0$ $(n\to\infty)$ and $\sum_{i=1}^{N}\|(B_i-\beta_i)f_n\|\to 0$ $(n\to\infty)$.

Let Y denote an operator in $\mathfrak{L}(\mathfrak{H})$ such that $Yf_n=e_n$ $(n\geq 1)$. For $X\in\mathfrak{L}(\mathfrak{H})$,

$$\|(\mathfrak{R}-\lambda)(X)-Y\| \geq \left|\sum_{i=1}^{N}(((A_i-\alpha_i)XB_i+\alpha_i X(B_i-\beta_i))f_n, e_n)-(Yf_n, e_n)\right| =$$

$$= \left|\left[\sum_{i=1}^{N}(XB_i f_n, (A_i-\alpha_i)^* e_n)+(\alpha_i X(B_i-\beta_i)f_n, e_n)\right]-1\right| \geq$$

$$\geq 1-\|X\| \max\{\|B_i\|\} \left(\sum_{i=1}^{N}\|(A_i-\alpha_i)^* e_n\|\right)-\max\{|\alpha_i|\}\|X\|\left(\sum_{i=1}^{N}\|(B_i-\beta_i) f_n\|\right),$$

and thus $\|(\mathfrak{R}-\lambda)(X)-Y\|\geq 1$. The proof is complete.

We conclude this section with an analogue of Theorem 2.3 for left spectra of elementary operators.

Lemma 2.6. *If* $\lambda\in\sigma_l(A)\circ\sigma_r(B)$, *then* $(\mathfrak{R}-\lambda)\mid\mathfrak{F}_1$ *and* $(\mathfrak{R}_3-\lambda)\mid\mathfrak{F}_1$ *are not bounded below.*

Proof. Let $\alpha\in\sigma_l(A)$ and $\beta\in\sigma_r(B)$ be such that $\lambda=\alpha\circ\beta$. There exist sequences of unit vectors $\{x_k\}$, $\{y_k\}\subset\mathfrak{H}$ such that

$$\lim_{k\to\infty}\sum_{i=1}^{N}\|(A_i-\alpha_i)x_k\| = \lim_{k\to\infty}\sum_{i=1}^{N}\|(B_i-\beta_i)^* y_k\| = 0.$$

Now

$$|||(\mathfrak{R}_3-\lambda)(x_k\otimes y_k)||| \leq$$

$$\leq \sum_{i=1}^{N}[|||(A_i-\alpha_i)(x_k\otimes y_k)B_i|||+|\alpha_i|\,|||(x_k\otimes y_k)(B_i-\beta_i)|||] =$$

$$= \sum_{i=1}^{N}[\|(A_i-\alpha_i)(x_k\otimes y_k)B_i\|+|\alpha_i|\,\|\alpha_i(x_k\otimes y_k)(B_i-\beta_i)\|].$$

For $t\in\mathfrak{H}$, $\|t\|=1$, we have

$$\|(A_i-\alpha_i)(x_k\otimes y_k)B_i t\| =$$

$$= \|(A_i-\alpha_i)(B_i t, y_k)x_k\| \leq \|B_i\|\,\|y_k\|\,\|(A_i-\alpha_i)x_k\|.$$

Thus

$$\sum_{i=1}^{N}\|(A_i-\alpha_i)(x_k\otimes y_k)B_i\| \to 0 \quad (k\to\infty);$$

similarly,

$$\sum_{i=1}^{N}\|(x_k\otimes y_k)(B_i-\beta_i)\| \to 0 \quad (k\to\infty).$$

Since $|||x_k\otimes y_k|||=\|x_k\otimes y_k\|=1$, it follows that neither $(\mathfrak{R}_3-\lambda)\mid\mathfrak{F}_1$ nor $(\mathfrak{R}-\lambda)\mid\mathfrak{F}_1$ is bounded below.

Lemma 2.7. i) $\sigma_l(\mathfrak{R})\subset\sigma_l(A)\circ\sigma_r(B)$;
ii) $\sigma_l(\mathfrak{R}_{\mathfrak{J}})\subset\sigma_l(A)\circ\sigma_r(B)$.

Proof. The proof is similar to the proof of Lemma 2.2, but using the spectral mapping theorem for left spectra [16].

Theorem 2.8. *For $\lambda\in\mathbf{C}$ the following are equivalent.*

i) $\mathfrak{R}-\lambda$ *is left invertible;*
ii) $\mathfrak{R}-\lambda$ *is bounded below;*
iii) $(\mathfrak{R}-\lambda)\mid\mathfrak{F}_1$ *is bounded below;*
iv) $\lambda\notin\sigma_l(A)\circ\sigma_r(B)$;
v) $\mathfrak{R}_{\mathfrak{J}}-\lambda$ *is left invertible in* $\mathfrak{L}(\mathfrak{J})$ *for some norm ideal* $\mathfrak{J}$;
vi) $\mathfrak{R}_{\mathfrak{J}}-\lambda$ *is bounded below for some norm ideal* $\mathfrak{J}$;
vii) $\mathfrak{R}_{\mathfrak{J}}-\lambda$ *is left invertible for every norm ideal* $\mathfrak{J}$;
viii) $\mathfrak{R}_{\mathfrak{J}}-\lambda$ *is bounded below for every norm ideal* $\mathfrak{J}$.

Proof. The implications i)⇒ii)⇒iii) are trivial; iii)⇒iv) follows from Lemma 2.6, and Lemma 2.7 implies that iv)⇒i); thus i)—iv) are equivalent. The implications v)⇒vi)⇒iv)⇒v) and vii)⇒viii)⇒iv)⇒vii) also follow by application of Lemmas 2.6 and 2.7.

Corollary 2.9. *For each norm ideal* $\mathfrak{J}$,

$$\sigma(R_{\mathfrak{J}}(A, B)) = \sigma(R(A, B)) = \sigma_r(A)\circ\sigma_l(B)\cup\sigma_l(A)\circ\sigma_r(B).$$

Proof. The result follows from Theorem 2.3 and Theorem 2.8.

Remark. The identity for $\sigma(R)$ given above is due to R. HARTE [16]; our contribution is the identity $\sigma(R_{\mathfrak{J}})=\sigma(R)$. A special case of the latter identity for the Hilbert—Schmidt ideal C_2 was obtained by R. CURTO [5, Lemma 3]. The main result of [5] presents a new description of $\sigma(R)$ in terms of Taylor joint spectra.

3. The semi-Fredholm domain of $\mathfrak{S}_{\mathfrak{J}}$. In the present section we describe the semi-Fredholm domain and index function of $\mathfrak{S}_{\mathfrak{J}}$ and $\mathfrak{S}$. To this end we define the following sets:

$$\sigma_{lr} \equiv \sigma_{lr}(A, B) = \sigma_l(A)\sigma_{re}(B)\cup\sigma_{le}(A)\sigma_r(B);$$
$$\sigma_{rl} \equiv \sigma_{rl}(A, B) = \sigma_r(A)\sigma_{le}(B)\cup\sigma_{re}(A)\sigma_l(B).$$

It follows from [12, Lemma 3.2] that if $\lambda\in\sigma_{lr}$ and $\mathfrak{S}_{\mathfrak{J}}-\lambda$ is semi-Fredholm, then $\operatorname{ind}(\mathfrak{S}_{\mathfrak{J}}-\lambda)=+\infty$; [12, Lemma 3.3] implies that if $\lambda\in\sigma_{rl}$ and $\mathfrak{S}_{\mathfrak{J}}-\lambda$ is semi-Fredholm, then $\operatorname{ind}(\mathfrak{S}_{\mathfrak{J}}-\lambda)=-\infty$. Thus $\sigma_{lr}\cap\sigma_{rl}\subset\sigma_{SF}(\mathfrak{S}_{\mathfrak{J}})$ and in the sequel we prove the reverse inclusion. We begin with the following special case.

Proposition 3.1. *If* $\lambda\in\mathbf{C}\setminus\sigma_{rl}$, *then* $\mathfrak{S}_{\mathfrak{J}}(A, B)-\lambda$ *is semi-Fredholm with* $\operatorname{ind}(\mathfrak{S}_{\mathfrak{J}}(A, B)-\lambda)>-\infty$.

Proof. If $\lambda\in\mathbf{C}\setminus\sigma_r(A)\sigma_l(B)$, then Theorem 2.3 implies that $\mathfrak{S}_3-\lambda$ is surjective, so the result is clear in this case. We may thus assume that $\lambda\in\sigma_r(A)\sigma_l(B)\setminus\sigma_{rl}$. We require the following preliminary lemmas.

Lemma 3.2. *If $\alpha\in\sigma_r(A)$, $\beta\in\sigma_l(B)$ and $\alpha\beta\notin\sigma_{rl}$ then α is isolated in $\sigma_r(A)$ or β is isolated in $\sigma_l(B)$.*

Proof. Since $\alpha\beta\notin\sigma_{rl}$, then $\alpha\in\sigma_r(A)\setminus\sigma_{re}(A)$ and $\beta\in\sigma_l(B)\setminus\sigma_{le}(B)$. Suppose that α is not isolated in $\sigma_r(A)$ and β is not isolated in $\sigma_l(B)$. Since $\alpha\notin\sigma_{re}(A)$, [10, Lemma 3.6 (i)] implies that $\alpha\in\mathfrak{U}\equiv\operatorname{int}(\sigma_r(A))$. Similarly, since $\beta\notin\sigma_{le}(B)$ and β is not isolated in $\sigma_l(B)$, then [10, Lemma 3.6 (ii)] implies that $\beta\in\mathfrak{B}\equiv\operatorname{int}(\sigma_l(B))$. $\mathfrak{U}$ and $\mathfrak{B}$ are nonempty, open, bounded subsets of the plane, so [12, Lemma 2.11] implies that there exists $t>0$ such that

i) $t\alpha\in\operatorname{bdry}(\mathfrak{U})$ and $\beta/t\in\mathfrak{B}^-$, or

ii) $t\alpha\in\mathfrak{U}$ and $\beta/t\in\operatorname{bdry}(\mathfrak{B})$.

It follows from [10, Lemma 3.6] that $\operatorname{bdry}(\mathfrak{U})\subset\sigma_{re}(A)$ and $\operatorname{bdry}(\mathfrak{B})\subset\sigma_{le}(B)$. In case i), $t\alpha\in\operatorname{bdry}(\mathfrak{U})\subset\sigma_{re}(A)$ and $\beta/t\in\mathfrak{B}^-\subset\sigma_l(B)$, so $\lambda=\alpha\beta=(t\alpha)(\beta/t)\in\sigma_{re}(A)\cdot\sigma_l(B)\subset\sigma_{rl}$, which is a contradiction. In case ii), $t\alpha\in\mathfrak{U}\subset\sigma_r(A)$ and $\beta/t\in\operatorname{bdry}(\mathfrak{B})\subset\sigma_{le}(B)$, so $\lambda=(t\alpha)(\beta/t)\in\sigma_r(A)\sigma_{le}(B)\subset\sigma_{rl}$, also a contradiction; the proof is now complete.

Lemma 3.3. *If $\lambda\in\sigma_r(A)\sigma_l(B)\setminus\sigma_{rl}$, then $\lambda\neq0$ and $X\equiv\{(\alpha,\beta)\in\sigma_r(A)\times\sigma_l(B)\colon\alpha\beta=\lambda\}$ is finite.*

Proof. If $0\in\sigma_r(A)\sigma_l(B)$, either $0\in\sigma_r(A)$ or $0\in\sigma_l(B)$, so $0\in\sigma_r(A)\sigma_{le}(B)$ or $0\in\sigma_{re}(A)\sigma_l(B)$, and so $0\in\sigma_{rl}$; thus $\lambda\neq0$.

Assume that X is infinite and let $\{(\alpha_n,\beta_n)\}_{n=1}^{\infty}$ be a sequence of distinct points of X. It follows readily that the α_n's are distinct and the β_n's are distinct. There exists a convergent subsequence $(\alpha_{n_k},\beta_{n_k})\to(\alpha,\beta)$, and clearly $\alpha\in\sigma_r(A)$, $\beta\in\sigma_l(B)$, and $\alpha\beta=\lambda$. Since α is not isolated in $\sigma_r(B)$ and β is not isolated in $\sigma_l(B)$, we have a contradiction to Lemma 3.2.

We return to the proof of Proposition 3.1 and consider $\lambda\in\sigma_r(A)\sigma_l(B)\setminus\sigma_{rl}$. Lemma 3.2 and Lemma 3.3 imply that $\lambda\neq0$ and that there exist integers p and n, $p\geqq n\geqq0$, $p>0$, distinct nonzero points $\alpha_1,\dots,\alpha_p\in\sigma_r(A)\setminus\sigma_{re}(A)$, and distinct nonzero points $\beta_1,\dots,\beta_p\in\sigma_l(B)\setminus\sigma_{le}(B)$ such that the following properties are satisfied:

1) $\{(\alpha,\beta)\in\sigma_r(A)\times\sigma_l(B)\colon\alpha\beta=\lambda\}=\{(\alpha_i,\beta_i)\}_{i=1}^{p}$;

2) if $n>0$, then α_i is isolated in $\sigma_r(A)$, $1\leqq i\leqq n$;

3) if $p>n$, then β_i is isolated in $\sigma_l(B)$, $n+1\leqq i\leqq p$.

If each β_i isolated in $\sigma_l(B)$ we may take $n=0$ and delete $\{(\alpha_1,\beta_1),\dots,(\alpha_n,\beta_n)\}$; likewise, if each α_i is isolated in $\sigma_r(A)$, we may take $p=n$ and delete $\{(\alpha_{n+1},\beta_{n+1}),$

$\dots, (\alpha_p, \beta_p)\}$. We assume in the sequel that $1 \leqq n < p$, for the other cases require only obvious modifications of the argument for this case.

Let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ denote copies of $\mathfrak{H}$ with $A \in \mathfrak{L}(\mathfrak{H}_1)$ and $B \in \mathfrak{L}(\mathfrak{H}_2)$. We identify $\mathfrak{L}(\mathfrak{H})$ with $\mathfrak{L}(\mathfrak{H}_2, \mathfrak{H}_1)$ and consider $\mathfrak{S}(A, B)$ as an operator on $\mathfrak{L}(\mathfrak{H}_2, \mathfrak{H}_1)$. [10, Corollary 2.4] implies that there exists an orthogonal decomposition $\mathfrak{H}_1 = \mathfrak{M}_0 \oplus \dots \oplus \mathfrak{M}_n$ and operators $A_i \in \mathfrak{L}(\mathfrak{M}_i)$ $(0 \leqq i \leqq n)$ such that:

4) $\mathfrak{M}_i$ is finite dimensional $(1 \leqq i \leqq n)$;

5) $\sigma(A_i) = \{\alpha_i\}$ $(1 \leqq i \leqq n)$;

6) $\sigma_r(A_0) \cap \{\alpha_1, \dots, \alpha_n\} = \emptyset$;

7) A is similar to $A' \equiv A_0 \oplus A_1 \oplus \dots \oplus A_n$.

An application of [10, Corollary 2.3] implies that there is an orthogonal decomposition $\mathfrak{H}_2 = \mathfrak{K}_{n+1} \oplus \dots \oplus \mathfrak{K}_{p+1}$ and operators $B_i \in \mathfrak{L}(\mathfrak{K}_i)$ $(n+1 \leqq i \leqq p+1)$ such that:

8) $\mathfrak{K}_i$ is finite dimensional, $n+1 \leqq i \leqq p$;

9) $\sigma(B_i) = \{\beta_i\}$, $n+1 \leqq i \leqq p$;

10) $\sigma_l(B_{p+1}) \cap \{\beta_{n+1}, \dots, \beta_p\} = \emptyset$;

11) B is similar to $B' \equiv B_{n+1} \oplus \dots \oplus B_{p+1}$.

[12, Proposition 2.5] implies that to complete the proof it suffices to prove that $\mathfrak{S}_3(A', B') - \lambda$ is semi-Fredholm with $\operatorname{ind}(\mathfrak{S}_3(A', B') - \lambda) > -\infty$. The argument is formally similar to that in the proof of [12, Theorem 3.1] so we give the outline and refer the reader to [12] for certain details.

Relative to the above decompositions of $\mathfrak{H}_1$ and $\mathfrak{H}_2$, let $(X_{ij})_{0 \leqq i \leqq n, n+1 \leqq j \leqq p+1}$ denote the operator matrix of an operator $X \in \mathfrak{L}(\mathfrak{H}) = \mathfrak{L}(\mathfrak{H}_2, \mathfrak{H}_1)$. A calculation (using 7) and 11)) shows that the row i, column j entry of the matrix of $S'(X) \equiv \equiv A'XB' - \lambda X$ is equal to $A_i X_{ij} B_j - \lambda X_{ij}$, $0 \leqq i \leqq n$, $n+1 \leqq j \leqq p+1$. For $X \in \mathfrak{L}(\mathfrak{H})$, let $R(X)$ be defined by the matrix which modifies the first row and last column of $S'(X)$ as follows:

$$\begin{bmatrix} (A_0 - \alpha_{n+1}) X_{0,n+1} \beta_{n+1} \dots (A_0 - \alpha_p) X_{0,p} \beta_p & A_0 X_{0,p+1} B_{p+1} - \lambda X_{0,p+1} \\ & \alpha_1 X_{1,p+1} (B_{p+1} - \beta_1) \\ [A_i X_{ij} B_j - \alpha_j \beta_j X_{ij}] & \vdots \\ & \alpha_n X_{n,p+1} (B_{p+1} - \beta_n) \end{bmatrix}$$

We first prove that $R \mid \mathfrak{J}$ is semi-Fredholm with $\operatorname{ind}(R \mid \mathfrak{J}) > -\infty$. Let R_{ij} be the operator on $\mathfrak{L}(\mathfrak{K}_j, \mathfrak{H}_i)$ defined by the row i, column j entry of $R(X)$, $0 \leqq i \leqq n$, $n+1 \leqq j \leqq p+1$. It follows from 1), 6), 7), 10), and 11) above that $\lambda \notin \sigma_r(A_0) \sigma_l(B_{p+1})$, so Theorem 2.3 implies that $R_{0,p+1} = \mathfrak{S}(A_0, B_{p+1}) - \lambda$ is surjective; in particular, $\operatorname{ind}(R_{0,p+1}) \geqq 0$. Let $1 \leqq i \leqq n$ and $n+1 \leqq j \leqq p$. Since $\sigma(A_i) = \{\alpha_i\}$ $(1 \leqq i \leqq n)$ and $\sigma(B_j) = \{\beta_j\}$ $(n+1 \leqq j \leqq p)$, it follows that $\lambda = \alpha_j \beta_j \notin \notin \sigma(A_i) \sigma(B_j)$, and thus $R_{ij} = \mathfrak{S}(A_i, B_j) - \lambda$ is invertible for $1 \leqq i \leqq n$ and $n+1 \leqq \leqq j \leqq p$ [4].

We next consider the operators $R_{0,j}$ $(n+1\leqq j\leqq p)$ defined by $R_{0,j}(X)=$ $=(A_0-\alpha_j)X\beta_j$ $(X\in\mathfrak{L}(\mathfrak{K}_j,\mathfrak{H}_0))$. Since $\alpha_j\in\sigma_r(A)\setminus\sigma_{re}(A)$, 7) implies that $\alpha_j\notin\sigma_{re}(A_0)$, and thus $A_0-\alpha_j$ is semi-Fredholm and $\operatorname{ind}(A_0-\alpha_j)>-\infty$. Since $\dim(\mathfrak{K}_j)<\infty$, [10, Lemma 3.5] implies that $R_{0,j}$ is semi-Fredholm with $\operatorname{ind}(R_{0,j})=$ $=\operatorname{ind}(A_0-\alpha_j)\dim(\mathfrak{K}_j)>-\infty$. Similarly, since $\mathfrak{H}_i$ is finite dimensional and $\beta_i\notin\sigma_{le}(B_{p+1})$ $(1\leqq i\leqq n)$, then [10, Lemma 3.5] and [12, Lemma 2.6] imply that $R_{i,p+1}$ is semi-Fredholm with $\operatorname{ind}(R_{i,p+1})=\operatorname{ind}((B_{p+1}-\beta_i)^*)\dim(\mathfrak{H}_i)>-\infty$.

It now follows exactly as in the proof of [12, Theorem 3.1] that $R\mid\mathfrak{J}$ is semi-Fredholm with

$$\operatorname{ind}(R|\mathfrak{J})=\sum_{j=n+1}^{p}\operatorname{ind}(R_{0,j})+\sum_{i=1}^{n}\operatorname{ind}(R_{i,p+1})+\operatorname{ind}(R_{0,p+1})>-\infty.$$

Let $K_j\in\mathfrak{L}(\mathfrak{K}_j)$ be invertible $(n+1\leqq j\leqq p)$ and let $M_i\in\mathfrak{L}(\mathfrak{M}_i)$ be invertible $(1\leqq i\leqq n)$. For $X\in\mathfrak{J}$, $X=(X_{ij})$, define $T(X)$ by the matrix

$$\begin{bmatrix} A_0X_{0,n+1}K_{n+1}^{-1}(B_{n+1}-\beta_{n+1})K_{n+1} & \dots & A_0X_{0,p}K_p^{-1}(B_p-\beta_p)K_p & 0\\ 0 & \dots & 0 & M_1^{-1}(A_1-\alpha_1)M_1X_{1,p+1}B_{p+1}\\ \vdots & & \vdots & \vdots\\ 0 & \dots & 0 & M_n^{-1}(A_n-\alpha_n)M_nX_{n,p+1}B_{p+1}\end{bmatrix}.$$

Since $B_j-\beta_j$ $(n+1\leqq j\leqq p)$ and $A_i-\alpha_i$ $(1\leqq i\leqq n)$ are nilpotent, appropriate choices of the K_j's and M_i's insure that $Q\equiv R\mid\mathfrak{J}+T$ is semi-Fredholm with $\operatorname{ind}(Q)=\operatorname{ind}(R\mid\mathfrak{J})>-\infty$. The matrix of $Q(X)$ $(X\in\mathfrak{J})$ is of the form

$$\begin{bmatrix} A_0X_{0,n+1}K_{n+1}^{-1}B_{n+1}K_{n+1}-\lambda X_{0,n+1}\dots A_0X_{0,p}K_p^{-1}B_pK_p-\lambda X_{0,p} & A_0X_{0,p+1}B_{p+1}-\lambda X_{0,p+1}\\ & M_1^{-1}A_1M_1X_{1,p+1}B_{p+1}-\lambda X_{1,p+1}\\ [A_iX_{ij}B_j-\lambda X_{ij}] & \vdots\\ & M_n^{-1}A_nM_nX_{n,p+1}B_{p+1}-\lambda X_{n,p+1}\end{bmatrix}.$$

It now follows as in [12, Theorem 3.1] that $\mathfrak{S}_{\mathfrak{J}}(A',B')-\lambda$ is semi-Fredholm with $\operatorname{ind}(\mathfrak{S}_{\mathfrak{J}}(A',B')-\lambda)=\operatorname{ind}(Q)>-\infty$, so the proof is complete.

Corollary 3.4. *$\mathfrak{S}_{\mathfrak{J}}(A,B)-\lambda$ is semi-Fredholm with* $\operatorname{ind}(\mathfrak{S}_{\mathfrak{J}}-\lambda)>-\infty$ *if and only if $\lambda\in\mathbf{C}\setminus\sigma_{rl}(A,B)$.*

Proof. The result follows from [12, Lemma 3.3] and Proposition 3.1.

Corollary 3.5. *$\mathfrak{S}_{\mathfrak{J}}(A,B)-\lambda$ is semi-Fredholm with* $\operatorname{ind}(\mathfrak{S}_{\mathfrak{J}}-\lambda)=+\infty$ *if and only if $\lambda\in\sigma(A,B)\setminus\sigma_{rl}(A,B)$.*

Proof. Apply [12, Theorem 3.1] and Corollary 3.4.

We now consider the case when $\lambda\in\mathbf{C}\setminus\sigma_{lr}(A,B)$.

Proposition 3.6. *If $\lambda\in\mathbf{C}\setminus\sigma_{lr}$, then $\mathfrak{S}_{\mathfrak{J}}-\lambda$ is semi-Fredholm with* $\operatorname{ind}(\mathfrak{S}_{\mathfrak{J}}-\lambda)<+\infty$.

The proof is completely analogous to that of Proposition 3.1; for this reason we omit the details and merely mention the necessary preliminary results.

Lemma 3.7. *If* $\alpha\in\sigma_l(A)$, $\beta\in\sigma_r(B)$ *and* $\alpha\beta\notin\sigma_{lr}$, *then* α *is isolated in* $\sigma_l(A)$ *or* β *is isolated in* $\sigma_r(B)$.

Proof. The proof is similar to that of Lemma 3.2.

Using Lemma 3.7, the proof of the next result is based on that of Lemma 3.3.

Lemma 3.8. *If* $\lambda\in\sigma_l(A)\sigma_r(B)\setminus\sigma_{lr}(A, B)$, *then* $\lambda\neq 0$ *and* $\{(\alpha, \beta)\in\sigma_l(A)\times\times\sigma_r(B)\colon \alpha\beta=\lambda\}$ *is finite.*

Using the preceding two lemmas, the proof of Proposition 3.6 follows the argument in the proof of Proposition 3.1, except that instead of using Theorem 2.3, we now use Theorem 2.8 to show that $\mathfrak{S}(A_0, B_{p+1})-\lambda$ is bounded below.

Corollary 3.9. $\mathfrak{S}_{\mathfrak{J}}(A, B)-\lambda$ *is semi-Fredholm with* $\operatorname{ind}(\mathfrak{S}_{\mathfrak{J}}-\lambda)<+\infty$ *if and only if* $\lambda\in\mathbf{C}\setminus\sigma_{lr}(A, B)$.

Proof. The result follows from [12, Lemma 3.2] and Proposition 3.6.

Corollary 3.10. $\mathfrak{S}_{\mathfrak{J}}-\lambda$ *is semi-Fredholm with* $\operatorname{ind}(\mathfrak{S}_{\mathfrak{J}}-\lambda)=-\infty$ *if and only if* $\lambda\in\sigma(A, B)\setminus\sigma_{lr}(A, B)$.

Proof. The result follows from Corollary 3.9 and [12, Theorem 3.1].

An immediate consequence of Corollary 3.4 and Corollary 3.9 is the following description of the semi-Fredholm domain of $\mathfrak{S}_{\mathfrak{J}}$.

Theorem 3.11. $\mathfrak{S}_{\mathfrak{J}}-\lambda$ *is semi-Fredholm if and only if* $\lambda\in\mathbf{C}\setminus(\sigma_{rl}\cap\sigma_{lr})$.

Corollary 3.12. $\sigma_{SF}(\mathfrak{S}_{\mathfrak{J}})=\sigma_{rl}\cap\sigma_{lr}$.

For the case when $\mathfrak{S}_{\mathfrak{J}}-\lambda$ is Fredholm, a formula for $\operatorname{ind}(\mathfrak{S}_{\mathfrak{J}}-\lambda)$ is given in [12, Theorem 3.8]. The latter result, when combined with Corollary 3.5 and Corollary 3.10, thus gives a complete description of $\operatorname{ind}(\mathfrak{S}_{\mathfrak{J}}-\lambda)$ for $\lambda\in\varrho_{SF}(\mathfrak{S}_{\mathfrak{J}})$.

Example 3.13. Consider the case when $\mathfrak{J}$ is the ideal of all Hilbert—Schmidt operators endowed with its (separable) Hilbert space structure [4]. In this case $\mathfrak{S}_{\mathfrak{J}}(A, B)$ is again a Hilbert space operator; we will show that if A and B^* are quasitriangular, then so is $\mathfrak{S}_{\mathfrak{J}}$. By a theorem of C. Apostol, C. Foiaş, and D. Voiculescu [3], an operator T on a separable Hilbert space is quasitriangular if and only if $\operatorname{ind}(T-\lambda)\geqq 0$ for every $\lambda\in\varrho_{SF}(T)$.

Suppose A and B^* are quasitriangular; thus $\operatorname{ind}(A-\lambda)\geqq 0$ $(\lambda\in\varrho_{SF}(A))$ and $\operatorname{ind}(B-\lambda)\leqq 0$ $(\lambda\in\varrho_{SF}(B))$. It follows directly from the index formula of [12, Theorem 3.8] that $\operatorname{ind}(\mathfrak{S}_{\mathfrak{J}}-\lambda)\geqq 0$ for every $\lambda\in\mathbf{C}\setminus\sigma_e(\mathfrak{S}_{\mathfrak{J}})$. To complete the proof it thus suffices to verify that the case $\operatorname{ind}(\mathfrak{S}_{\mathfrak{J}}-\lambda)=-\infty$ cannot occur.

Suppose to the contrary that $\operatorname{ind}(\mathfrak{S}_3-\lambda)=-\infty$; from Corollary 3.10 we have

$$\lambda\in(\sigma_e(A)\sigma(B)\cup\sigma(A)\sigma_e(B))\setminus(\sigma_l(A)\sigma_{re}(B)\cup\sigma_{le}(A)\sigma_r(B)).$$

We consider the case $\lambda\in\sigma_e(A)\sigma(B)$ and let $\alpha\in\sigma_e(A)$ and $\beta\in\sigma(B)$ satisfy $\alpha\beta=\lambda$. If $\beta\in\sigma_r(B)$, then $\alpha\notin\sigma_{le}(A)$ and thus $\operatorname{ind}(A-\alpha)=-\infty$, a contradiction. Therefore $\beta\in\sigma(B)\setminus\sigma_r(B)$, so $\operatorname{ind}(B-\beta)>0$, which is also a contradiction. The case when $\lambda\in\sigma(A)\sigma_e(B)$ can be handled similarly, so we omit the details.

We note that the converse of this example is false. [12] contains an example of operators A and B such that A, A^*, B, and B^* are non-quasitriangular but $\mathfrak{S}_3(A, B)$ is biquasitriangular, i.e. $\mathfrak{S}_3$ and $\mathfrak{S}_3{}^*$ are both quasitriangular.

Systematic revision of the proofs of this section (replacing the norm ideal $\mathfrak{J}$ by $\mathfrak{L}(\mathfrak{H})$) yields a description of the semi-Fredholm domain of $\mathfrak{S}(A, B)$.

Theorem 3.14. i) $\sigma_{SF}(\mathfrak{S})=\sigma_{lr}\cap\sigma_{rl}$;

ii) $\mathfrak{S}-\lambda$ *is semi-Fredholm with* $\operatorname{ind}(\mathfrak{S}-\lambda)<+\infty$ *if and only if* $\lambda\in\mathbf{C}\setminus\sigma_{lr}$;

iii) $\mathfrak{S}-\lambda$ *is semi-Fredholm with* $\operatorname{ind}(\mathfrak{S}-\lambda)>-\infty$ *if and only if* $\lambda\in\mathbf{C}\setminus\sigma_{rl}$.

This result, together with [12, Theorem 3.9], completes the description of $\operatorname{ind}(\mathfrak{S}-\lambda)$ $(\lambda\in\varrho_{SF}(\mathfrak{S}))$. More generally, the present results, together with those of [9], [10] and [12], completely describe the semi-Fredholm domain and index function of the operators $\mathfrak{T}$, $\mathfrak{T}_3$, $\mathfrak{S}$, and $\mathfrak{S}_3$. Corresponding results for arbitrary elementary operators $\mathfrak{R}$, or the operators $\mathfrak{R}_3$, appear to be unknown at present. Some partial results are known for the general case. In [12, Theorem 3.14] it is proved that $\sigma_e(\mathfrak{R}_3)\subset\sum_{i=1}^{N}(\sigma(A_i)\sigma_e(B_i)\cup\sigma_e(A_i)\sigma(B_i))$ (and similarly for the operator $\mathfrak{R}$). By combining the techniques of [9], [10], [12] with the multi-variate techniques used in section 2, it is possible to prove the following result for the general case. The proof, and applications, will appear elsewhere. For n-tuples of operators A and B, let $\sigma_{lr}(A, B)=\sigma_{le}(A)\circ\sigma_r(B)\cup\sigma_l(A)\circ\sigma_{re}(B)$ and let $\sigma_{rl}(A, B)=\sigma_{re}(A)\circ\sigma_l(B)\cup\sigma_r(A)\circ\sigma_{le}(B)$. Let $\mathfrak{J}$ be an arbitrary norm ideal.

Theorem 3.15. i) $\sigma_e(A)\circ\sigma(B)\cup\sigma(A)\circ\sigma_e(B)\subset\sigma_e(\mathfrak{R}_3)$;

ii) *If* $\lambda\in\sigma_{lr}(A, B)$ *and* $\mathfrak{R}_3-\lambda$ *is semi-Fredholm, then* $\operatorname{ind}(\mathfrak{R}_3-\lambda)=+\infty$;

iii) *If* $\lambda\in\sigma_{rl}(A, B)$ *and* $\mathfrak{R}_3-\lambda$ *is semi-Fredholm, then* $\operatorname{ind}(\mathfrak{R}_3-\lambda)=-\infty$;

iv) $\sigma_{lr}(A, B)\cap\sigma_{rl}(A, B)\subset\sigma_{SF}(\mathfrak{R}_3)$.

We note that parts ii)—iv) are valid for elementary operators with arbitrary (non-commutative) coefficient sequences. A similar result holds for the operator $\mathfrak{R}$.

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