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On the equiconvergence of different kinds of partial sums of orthogonal series

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Let N^d $(d \geqq 1)$ be the set of d-tuples $\mathbf{i}=(i_1, \dots, i_d)$ with non-negative integral coordinates. Let $\varphi=\{\varphi_{\mathbf{i}} \mid \mathbf{i}\in N^d\}$ be an orthonormal system (ONS) on $[0, 1]$. Consider the d-multiple orthogonal series

(1) $$\sum_{\mathbf{i}\in N^d} a_{\mathbf{i}}\varphi_{\mathbf{i}}(x), \quad \sum_{\mathbf{i}\in N^d} a_{\mathbf{i}}^2 < \infty.$$

Fixing a sequence $Q=\{Q_k \mid k=0, 1, \dots\}$ of finite sets in N^d with properties

(2) $$\emptyset = Q_0 \subset Q_1 \subset Q_2 \subset \dots, \quad \bigcup_{k=0}^{\infty} Q_k = N^d \stackrel{\text{def}}{=} Q_\infty$$

we can define the Q-partial sums of (1) (see e.g. [1]):

$$s_k^Q(x) = \sum_{\mathbf{i}\in Q_k} a_{\mathbf{i}}\varphi_{\mathbf{i}}(x) \quad (k = 1, 2, \dots).$$

If $P=\{P_k\}$ is another sequence satisfying similar conditions to (2) we write $Q\Rightarrow P$ when the a.e. convergence of $\{s_k^Q(x)\}_{k=1}^\infty$ always implies that of $\{s_k^P(x)\}_{k=1}^\infty$. If not $Q\Rightarrow P$ then we write shortly $Q\not\Rightarrow P$.

F. Móricz [1] proved among others that if

$$Q_k' = \{\mathbf{i}\in N^d \mid \max_{1\leqq j\leqq d} i_j \leqq k\}$$

and

$$P_k' = \left\{\mathbf{i}\in N^d \,\middle|\, \left(\sum_{j=1}^d i_j^2\right)^{1/2} \leqq k\right\}$$

then $Q'\not\Rightarrow P'$ and $P'\not\Rightarrow Q'$.

The aim of this note is to give necessary and sufficient conditions for $Q\Rightarrow P$. Our result has several corollaries which are interesting in themselves.

With the notation $\bar{P}_k=N^d\setminus P_k$ we prove

Theorem 1. *We have $Q\Rightarrow P$ if and only if there is a number K such that*

(i) *each $Q_{k+1}\setminus Q_k$ is the union of at most K sets $(Q_{k+1}\setminus Q_k)\cap(P_{m+1}\setminus P_m)$,*

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(ii) *for every* k, P_k *and* $\bar{P}_k$ *are the (not necessarily disjoint) union of at most* K *sets of the form* $Q_{r+s}\setminus Q_r$ $(s=1, 2, \ldots, \infty)$, $P_{m+1}\setminus P_m$, $(Q_{r'+1}\setminus Q_{r'})\cap(P_{m'+1}\setminus P_{m'})$.

Corollary 1. *The systems* Q *and* P *are equivalent (i.e.* $P\Rightarrow Q$ *and* $Q\Rightarrow P$*) if and only if there is a* K *such that*

(i) *each* $(Q_{k+1}\setminus Q_k)\cup(P_{m+1}\setminus P_m)$ *is the union of at most* K *sets* $(Q_{s+1}\setminus Q_s)\cap\cap(P_{r+1}\setminus P_r)$,

(ii) *each* Q_k *and* P_k *is the union of at most* K *sets* $(Q_{s+1}\setminus Q_s)\cap(P_{r+1}\setminus P_r)$ *and* K *sets of the form* $P_{\varrho+\tau}\setminus P_\varrho$ *and* $Q_{\varrho'+\tau'}\setminus Q_{\varrho'}$, *respectively.*

With the notation

$$(k, l) = \{k, k+1, \ldots, l\} \quad (k \leqq l,\ k, l \in N^1) \tag{3}$$
$$(k, \infty) = \{k, k+1, \ldots\}$$

we have

Corollary 2. *Let* $\{p_k\}$ *and* $\{q_k\}$ *be two subsequences of the natural numbers. Then the a.e. convergence of* $\{s_{p_k}(x)\}_{k=1}^{\infty}$ *implies that of* $\{s_{q_k}(x)\}_{k=1}^{\infty}$ *for every orthogonal series*

$$\sum_{k=0}^{\infty} a_k \varphi_k(x), \quad \sum_{k=0}^{\infty} a_k^2 < \infty \tag{4}$$

if and only if the number of the q_k's *in the intervals* (p_m, p_{m+1}) *is bounded* (here s_k is the ordinary k-th partial sum of (4)).

Corollary 3. *With the above notations the a.e. equiconvergence of* $\{s_{p_k}(x)\}_{k=1}^{\infty}$ *and* $\{s_{q_k}(x)\}_{k=1}^{\infty}$ *for every orthogonal series* (4) *is equivalent to the existence of a* K *for which* $p_k<q_l$ *implies* $p_{k+1}<q_{l+K}$ *and* $q_k<p_l$ *implies* $q_{k+1}<p_{l+K}$.

Corollary 1 follows easily from the proof of Theorem 1. Corollaries 2 and 3 were also proved by H. Schwinn [3].

To formulate another consequence of Theorem 1 let $d=1$, $N=N^1$ and $\pi: N\to N$ be a mapping of N onto N for which the inverse image $\pi^{-1}(k)$ of every number k is finite (one can see easily that the following problem becomes trivial if some of the $\pi^{-1}(k)$ are infinite). Our problem is the following: determine which π has the property: if the orthogonal series (4) converges a.e. then the same is true for the rearranged and bracketed series

$$\sum_{k=0}^{\infty}\Big(\sum_{i\in\pi^{-1}(k)} a_i\varphi_i(x)\Big). \tag{5}$$

The answer is given by

Theorem 2. *The a.e. convergence of* (4) *implies that of* (5) *for every orthogonal series* (4) *if and only if there is a* K *such that for every* k, $\pi^{-1}(0, k)$ *and* $\pi^{-1}(k, \infty)$

are the (not necessarily disjoint) union of at most K *sets of the form* (l, m) $(m=1, 2, \dots, \infty)$ *or* $\pi^{-1}(s)$.

For the definition of (l, m) see (3).

Corollary 4. *If* $\pi: N \to N$ *is a permutation of* N *then the a.e. convergence of* (4) *implies the a.e. convergence of*

$$\sum_{k=0}^{\infty} a_{\pi(k)} \varphi_{\pi(k)}(x)$$

for every orthogonal series (4) *if and only if there is a* K *such that for every* k, $\pi(0, k)$ *consists of at most* K *chains of consecutive integers.*

Remarks. 1. Although we formulated Theorem 1 in d dimensions, the problem and the solution is essentially one-dimensional, namely Theorems 1 and 2 are equivalent (see the proof of Theorem 1 below).

2. If $Q \Rightarrow P$ then our proof yields an orthogonal series (1) for which $\{s_k^Q(x)\}_{k=1}^{\infty}$ converges a.e. but $\{s_k^P(x)\}_{k=1}^{\infty}$ diverges on a set of positive measure. By a standard modification of the proof one could achieve also the a.e. divergence of $\{s_k^P(x)\}_{k=1}^{\infty}$.

3. The ONS $\{\varphi_i\}$ above could be defined on any non-atomic measure space instead of $[0, 1]$ (compare to [2]).

4. Our proof shows that if $Q \Rightarrow P$ and $\{s_k^Q(x)\}_{k=1}^{\infty}$ converges on a set E then $\lim_{k\to\infty} s_k^P(x) = \lim_{k\to\infty} s_k^Q(x)$ a.e. on E, i.e. the P-sums and Q-sums are equal a.e. automatically.

5. Finally, let us remark that to the proof of Corollaries 2 and 3 needs only the consideration used in the proof of the necessity of Theorem 1 (i), by which we obtain a very short proof of Schwinn's results (see [3]). The same is true for a part of Móricz's theorem mentioned earlier (see [1, Theorem 3]).

After these we turn to the proofs our theorems. First we prove Theorem 2.

Proof of Theorem 2. *I. Necessity.* Let us suppose on the contrary that e.g. for each n there is a k such that $\pi^{-1}\{0, \dots, k\} = \pi^{-1}(0, k)$ (see (3)) cannot be represented as the union of at most n sets (l, m) and at most n sets $\pi^{-1}(l)$.

We define sequences $\{N_n\}, \{M_n\}, \{m_n\}, \{m_n^*\}, k_1^{(n)} < k_2^{(n)} < \dots < k_n^{(n)}$ and $\{i_1^{(n)}, \dots, i_n^{(n)}\}, \{j_1^{(n)}, \dots, j_n^{(n)}\}$ in the following way: put $N_0 = M_0 = m_0 = m_0^* = 0$ and if all of the above numbers are already defined up to $n-1$, let N_n and m_n^* be so large that

$$N_n > M_{n-1}, \pi^{-1}(0, N_n) \supseteqq (0, m_{n-1}), \quad m_n^* > m_{n-1}, (0, m_n^*) \supseteqq \pi^{-1}(0, N_n)$$

be satisfied. By our assumption there is an $M_n > N_n$ such that $\pi^{-1}(N_n+1, M_n) \setminus (0, m_n^*)$ cannot be represented as the union of at most n sets (l, m) and at most

n sets $\pi^{-1}(l)$. Let $\pi^{-1}(N_n+1, M_n)\setminus(0, m_n^*)=(r_1, s_1)\cup(r_2, s_2)\cup\ldots\cup(r_t, s_t)$ where $s_i\geqq r_i$ and $r_{i+1}>s_i+1$. We claim that there are n numbers $k_1^{(n)}<\ldots<k_n^{(n)}$ belonging to (N_n+1, M_n) and numbers $i_\varrho\in\pi^{-1}(k_\varrho^{(n)})\setminus(0, m_n^*)$ $(\varrho=1, \ldots, n)$ such that neither two of the i_ϱ belong to the same (r_τ, s_τ). In fact, let $i_1^*\in(r_1, s_1)$, $\pi(i_1^*)=k_1^*$ and if i_ϱ^*, k_ϱ^* $(\varrho<n)$ are already defined and $i_u^*\in(r_{\tau_u}, s_{\tau_u})$ $(1\leqq u\leqq\varrho)$, then, since the ϱ intervals (r_{τ_u}, s_{τ_u}) and the ϱ sets $\pi^{-1}(k_u^*)$ $(1\leqq u\leqq\varrho)$ do not cover $\pi^{-1}(N_n+1, M_n)\setminus(0, m_n^*)$, there is an

$$i_{\varrho+1}^*\in(\pi^{-1}(N_n+1, M_n)\setminus(0, m_n^*))\setminus\left(\left(\bigcup_{u=1}^{\varrho}(r_{\tau_u}, s_{\tau_u})\right)\cup\left(\bigcup_{u=1}^{\varrho}\pi^{-1}(k_u^*)\right)\right).$$

Let $k_{\varrho+1}^*=\pi(i_{\varrho+1}^*)$. We can continue this up to $\varrho=n$, and all what we have to do is to rearrange the set $\{k_1^*, \ldots, k_n^*\}$ into an increasing order $k_1^{(n)}<k_2^{(n)}<\ldots<k_n^{(n)}$ and to carry over this rearrangement to $\{i_1^*, \ldots, i_n^*\}$, by which we obtain $\{i_1^{(n)}, \ldots, i_n^{(n)}\}$. Let $i_\varrho^{(n)}$ belong to $(r_{\tau'_\varrho}, s_{\tau'_\varrho})$ and let us put $j_\varrho^{(n)}=s_{\tau'_\varrho}+1$ $(\varrho=1, \ldots, n)$. Finally, let $m_n>m_n^*$ be so large that $(0, m_n)$ contains $\pi^{-1}(0, M_n)$ as well as the numbers $j_1^{(n)}, \ldots, j_n^{(n)}$.

Our definition is complete and let us observe the following:

(6) $$\pi^{-1}(0, M_{n-1})\subseteqq(0, m_{n-1})\subseteqq\pi^{-1}(0, N_n)\subseteqq(0, m_n^*),$$

(7) $$m_n^*<i_\varrho^{(n)}<j_\varrho^{(n)}\leqq m_n \quad (\varrho=1, \ldots, n),$$

(8) $$M_{n-1}<N_n<k_1^{(n)}<\ldots<k_n^{(n)}\leqq M_n,$$

(9) $$i_\varrho^{(n)}\in\pi^{-1}(k_\varrho^{(n)}),\quad j_\varrho^{(n)}\notin\pi^{-1}(0, M_n) \quad (\varrho=1, \ldots, n),$$

(10) $$\max_{1\leqq\varrho\leqq n-1} j_\varrho^{(n-1)}<\min_{1\leqq\varrho\leqq n} i_\varrho^{(n)},$$

(11) every two $i_{\varrho_1}^{(n)}<i_{\varrho_2}^{(n)}$ is separated by $j_{\varrho_1}^{(n)}$: $i_{\varrho_1}^{(n)}<j_{\varrho_1}^{(n)}<i_{\varrho_2}^{(n)}$.

Now we shall use that there is an orthogonal series (4) with partial sums $S_k(x)$ which diverges unboundedly a.e. on $[0, 1]$. This gives that there is a sequence $p_1<p_2<\ldots$ such that with $q_k=\sum_{l=1}^{k-1} p_l$ we have

(12) $$\sup_n \max_{0<l\leqq p_n} |S_{q_n+l}(x)-S_{q_n}(x)|=\infty \quad \text{(a.e.)}.$$

Let now

(13) $$\psi_{i_\varrho^{(p_n)}}(x)=\psi_{j_\varrho^{(p_n)}}(x)=\frac{1}{2}\varphi_{q_n+\varrho}(x) \quad (x\in[0, 1]),$$

(14) $$b_{i_\varrho^{(p_n)}}=-b_{j_\varrho^{(p_n)}}=a_{q_n+\varrho}$$

for $n=1, 2, \ldots$ and $\varrho=1, \ldots, p_n$ and let $\psi_k(x)=0$ $(x\in[0, 1])$, $b_k=0$ otherwise. Since each ψ_k is orthogonal to all but at most one ψ_l, $l\neq k$ and since $\int_0^1|\psi_k\psi_l|\leqq 1/4$ $(k, l=0, 1, \ldots)$, a standard argument yields that the system $\{\psi_k\}_{k=0}^\infty$ can be extended onto $[-1, 1]$ in such a way that it constitutes an ONS on $[-1, 1]$, and for every $x\in[-1, 0)$ all but at most two of the numbers $\{\psi_k(x)\}_{k=0}^{\infty}$ are zero.

By (10), (11), (13) and (14) the k-th partial sum $s_k(x)$ of

$$\sum_{l=0}^{\infty} b_l\psi_l(x)$$

is equal either to 0 or to some $a_l\varphi_l(x)/2$ if $x\in[0, 1]$. Here l tends to infinity together with k (take into account that if $k>m_{p_n}$ then necessarily $l>q_n$), and by

$$\sum_{l=0}^{\infty}\int_0^1 (a_l\varphi_l(x))^2\,dx = \sum_{l=0}^{\infty} a_l^2 < \infty,$$

$a_l\varphi_l(x)$ tends to 0 a.e. as $l\to\infty$. Hence, $s_k(x)$ tends to zero a.e. on $[0, 1]$ as $k\to\infty$ and so $\{s_k(x)\}_{k=1}^{\infty}$ is convergent a.e. on $[-1, 1]$ (for $x\in[-1, 0)$, $\{s_k(x)\}_{k=1}^{\infty}$ is constant from a certain point on).

However, by (6), (7), (13) and (14)

$$\sum_{k=0}^{N_n}\sum_{l\in\pi^{-1}(k)} b_l\psi_l(x) = 0 \quad (x\in[0, 1]),$$

hence by (8) and (9)

$$\sum_{k=0}^{k_\varrho^{(p_n)}}\sum_{l\in\pi^{-1}(k)} b_l\psi_l(x) = \sum_{k=N_n+1}^{k_\varrho^{(p_n)}}\sum_{l\in\pi^{-1}(k)} b_l\psi_l(x) = \sum_{s=1}^{\varrho} b_{i_\varrho^{(p_n)}}\psi_{i_\varrho^{(p_n)}}(x) =$$

$$= \sum_{s=1}^{\varrho}\frac{1}{2}a_{q_n+s}\varphi_{q_n+s}(x) = \frac{1}{2}\left(S_{q_n+\varrho}(x) - S_{q_n}(x)\right) \quad (1 \leqq \varrho \leqq p_n)$$

and thus, using (12), we obtain that

$$\sum_{k=0}^{\infty}\sum_{l\in\pi^{-1}(k)} b_l\psi_l(x)$$

diverges a.e. on $[0, 1]$.

The necessity of the assumption concerning $\pi^{-1}(k, \infty)$ can be proved similarly, we omit the details.

The proof of the necessity is thus complete (clearly, it is indifferent that the constructed system $\{\psi_k\}_{k=0}^{\infty}$ is orthonormal on $[-1, 1]$ and not on $[0, 1]$).

II. Sufficiency. 1. First we prove that there are no integers

$$x_1 < y_1 < x_2 < y_2 < \ldots < y_{4K+2} < x_{4K+3}$$

with $\pi(x_j)=\pi(x_l)$ $(0\leqq j, l\leqq 4K+3)$ but $\pi(y_j)\neq\pi(y_l)$ $(1\leqq j, l\leqq 4K+2,\ j\neq l)$. Let us suppose the contrary and let $\pi(x_j)=k$ $(1\leqq j\leqq 4K+3)$. We distinguish two cases.

(a) At least $2K+1$ of the distinct numbers $\pi(y_j)$ $(1 \leqq j \leqq 4K+2)$ are less than k. We may suppose without loss of generality that

$$x_1 < y_1 < x_2 < \ldots < y_{2K+1} < x_{2K+2}, \quad \pi(y_j) < k \ (1 \leqq j \leqq 2K+1).$$

For any n, $\pi^{-1}(0, n)$ is the disjoint union of sets of consecutive integers, i.e., for some τ_n,

$$\pi^{-1}(0, n) = (a_1^{(n)}, b_1^{(n)}) \cup \ldots \cup (a_{\tau_n}^{(n)}, b_{\tau_n}^{(n)}) \tag{15}$$

where $a_{j+1}^{(n)} > b_j^{(n)}$ $(1 \leqq j < \tau_n)$. Let us put $n=k-1$ into (15) and let us determine the numbers i_j $(1 \leqq j \leqq 2K+1)$ by $y_j \in (a_{i_j}^{(k-1)}, b_{i_j}^{(k-1)})$. Since x_j $(1 \leqq j \leqq 2K+2)$ does not belong to $\pi^{-1}(0, k-1)$, we have

$$x_j < a_{i_j}^{(k-1)} \leqq y_j \leqq b_{i_j}^{(k-1)} < x_{j+1} < a_{i_{j+1}}^{(k-1)} \quad (1 \leqq j < 2K+1),$$

hence the numbers $i_1, i_2, \ldots, i_{2K+1}$ are all different from each other.

By the assumption of our theorem there are numbers $1 \leqq l_1 < \ldots < l_K \leqq \tau_{k-1}$ and $0 \leqq n_1 < \ldots < n_k \leqq k-1$ so that

$$\pi^{-1}(0, k-1) = (a_{l_1}^{(k-1)}, b_{l_1}^{(k-1)}) \cup \ldots \cup (a_{l_K}^{(k-1)}, b_{l_K}^{(k-1)}) \cup \pi^{-1}(n_1) \cup \ldots \cup \pi^{-1}(n_K). \tag{16}$$

Now at least $K+1$, say $i_1, i_2, \ldots, i_{K+1}$, of the numbers $i_1, i_2, \ldots, i_{2K+1}$ are different from every l_j $(1 \leqq j \leqq K)$ (i.e., we may suppose without loss of generality that $i_j \neq l_{j'}$ for $1 \leqq j \leqq K+1$, $1 \leqq j' \leqq K$) and at least one, say $\pi(y_1)$, of the $K+1$ distinct numbers $\pi(y_1), \pi(y_2), \ldots, \pi(y_{K+1})$ is different from every n_j $(1 \leqq j \leqq K)$. Thus, y_1 does not belong to

$$(a_{l_1}^{(k-1)}, b_{l_1}^{(k-1)}) \cup \ldots \cup (a_{l_K}^{(k-1)}, b_{l_K}^{(k-1)})$$

since $y_1 \in (a_{i_1}^{(k-1)}, b_{i_1}^{(k-1)})$ and $i_1 \neq l_j$ for $1 \leqq j \leqq K$ and also y_1 does not belong to

$$\pi^{-1}(n_1) \cup \ldots \cup \pi^{-1}(n_k)$$

since $\pi(y_1)$ is different from every n_j $(1 \leqq j \leqq K)$. By (16) this means that $y_1 \notin \pi^{-1}(0, k-1)$ which contradicts the assumed inequality $\pi(y_1) < k$. This contradiction proves our assertion in the case (a).

(b) If at most $2K$ of the numbers $y_1, \ldots, y_{4K+2}$ are less than k then at least $2K+1$ of them are greater than k. Now using $\pi^{-1}(k+1, \infty)$ instead of $\pi^{-1}(0, k-1)$ we arrive at a contradiction exactly as above.

2. Let for $k=0, 1, 2, \ldots$

$$\Pi_k = \{\pi^{-1}(k) \cap (a_j^{(n)}, b_j^{(n)}) | n = 0, 1, 2, \ldots, 1 \leqq j \leqq \tau_n\}$$

(for the definition of $a_j^{(n)}$ and $b_j^{(n)}$ see (15)). Our next claim is that for each k and $x \in \pi^{-1}(k)$ there are at most $8K+3$ distinct sets $A \in \Pi_k$ with $x \in A$. In fact, if there were numbers $n_1 < n_2 < \ldots < n_{8K+4}$ and for each $1 \leqq j \leqq 8K+4$ an $1 \leqq i_j \leqq \tau_{n_j}$

such that the sets $(x\in)(a_{i_j}^{(n_j)}, b_{i_j}^{(n_j)})\cap\pi^{-1}(k)$ are all different then either for at least $4K+2$ of the j's we would have

(17) $$(a_{i_{j+1}}^{(n_{j+1})}, a_{i_j}^{(n_j)}-1)\cap\pi^{-1}(k)\neq\emptyset$$

or for at least $4K+2$ of the j's

$$(b_{i_j}^{(n_j)}+1, b_{i_{j+1}}^{(n_{j+1})})\cap\pi^{-1}(k)\neq\emptyset.$$

We might suppose the first case and also that (17) holds for $j=1,2,\dots,4K+2$, i.e., for $j=1,\dots,4K+2$ there would be numbers

$$x_{j+1}\in(a_{i_{j+1}}^{(n_{j+1})}, a_{i_j}^{(n_j)}-1)\cap\pi^{-1}(k).$$

Putting $x_1=x\in(a_{i_1}^{(n_1)}, b_{i_1}^{(n_1)})\cap\pi^{-1}(k)$ and $y_j=a_{i_j}^{(n_j)}-1$ $(1\leqq j\leqq 4K+2)$ we would have $y_j\in\pi^{-1}(0,n_{j+1})$ but $y_j\notin\pi^{-1}(0,n_j)$, i.e., $\pi(y_j)\leqq n_{j+1}<\pi(y_{j+1})$ $(1\leqq j\leqq 4K+1)$, and also $y_j\notin\pi^{-1}(k)$. Thus, we would get a system of numbers

$$x_{4K+3}<y_{4K+1}<x_{4K+2}<\dots<y_1<x_1$$

with $\pi(x_j)\in k$ $(1\leqq j\leqq 4K+3)$ but $\pi(y_j)\neq\pi(y_{j'})$ $(1\leqq j,j'\leqq 4K+2,\ j\neq j')$ and this would contradict the fact proved in point 1 above.

3. After these preliminary considerations we turn to the proof of the sufficiency part of our theorem. First of all, by point 2 above

$$\sum_{k=0}^{\infty}\sum_{A\in\pi_k}\int_0^1\Big(\sum_{i\in A}a_i\varphi_i(x)\Big)^2dx\leqq(8K+3)\sum_{i=0}^{\infty}a_i^2<\infty$$

and hence

$$\lim_{k\to\infty}\sum_{i\in A_k}a_i\varphi_i(x)=0\quad\text{(a.e.)}$$

independently of the choice of the sets $A_k\in\Pi_k$.

Let us suppose that the series (4) converges a.e. and let x be any point in $[0,1]$ for which

(18) $$\lim_{k\to\infty}\sum_{i\in A_k}a_i\varphi_i(x)=0\quad(A_k\in\Pi_k)$$

(19) $$\lim_{k\to\infty}s_k(x)=s(x)\quad\Big(s_k(x)=\sum_{i=0}^{k}a_i\varphi_i(x)\Big)$$

exist. It is enough to show that (5) converges at this point x.

From (19) we have also

(20) $$\lim_{k\to\infty}\big(s_{k+l_k}(x)-s_k(x)\big)=0$$

whatever $l_k\geqq 1$ be.

For a given p let $p<p_1<p_2<p_3$ be chosen so that $\pi(0,p)\subseteqq(0,p_1)$, $\pi^{-1}(0,p_1)\subseteqq$ $\subseteqq(0,p_2)$, $\pi(0,p_2)\subseteqq(0,p_3)$ be satisfied. For $n\geqq p_3$ we have $\pi^{-1}(0,n)\supseteqq(0,p_3)\supseteqq$

$\supseteqq(0, p_2)$ and by the assumption of the theorem

(21) $$\pi^{-1}(0, n) = (a_{i_1}^{(n)}, b_{i_1}^{(n)})\cup\ldots\cup(a_{i_\varrho}^{(n)}, b_{i_\varrho}^{(n)})\cup\pi^{-1}(k_1)\cup\ldots\cup\pi^{-1}(k_\tau)$$

for some $i_1<\ldots<i_\varrho$ and $k_1<\ldots<k_\tau$, where $\tau+\varrho\leqq K$ (if $\tau=0$ or $\varrho=0$ then the corresponding terms are missing). Since $(0, p_2)\subseteqq\pi^{-1}(0, n)$ we may assume (by increasing K by 1 if necessary) $i_1=1, (0, p_2)\subseteqq(a_1^{(n)}, b_1^{(n)})$ and then, since $\pi^{-1}(0, p_1)\subseteqq\subseteqq(0, p_2)$, we can drop those of the k_j's for which $k_j\leqq p_1$. Thus, we may assume that in (21) each $k_j>p_1$ and so, since $\pi^{-1}(0, p_1)\supseteqq(0, p)$,

$$\pi^{-1}(k_j)\cap(p+1, b_1^{(n)}) = \pi^{-1}(k_j)\cap(a_1^{(n)}, b_1^{(n)}) \overset{\text{def}}{=} A_j^{(1)} \quad (1\leqq j\leqq\tau).$$

For $1\leqq j\leqq\tau$ and $2\leqq l\leqq\varrho$ let $A_j^{(l)}=\pi^{-1}(k_j)\cap(a_{i_l}^{(n)}, b_{i_l}^{(n)})$. Then $A_j^{(l)}\in\Pi_{k_j}$ $(1\leqq j\leqq\tau,$ $1\leqq l\leqq\varrho)$ and for $n\geqq p_3$ we have the representation

$$\pi^{-1}(0, n) = (0, p)\cup(p+1, b_1^{(n)})\cup(a_{i_2}^{(n)}, b_{i_2}^{(n)})\cup\ldots\cup(a_{i_\varrho}^{(n)}, b_{i_\varrho}^{(n)})\cup$$
$$\cup\bigcup_{j=1}^{\tau}\left(\pi^{-1}(k_j)\setminus\bigcup_{l=1}^{\varrho}A_j^{(l)}\right)$$

and here the terms on the right are already disjoint. According to this

$$\left|\sum_{k=0}^{n}\sum_{i\in\pi^{-1}(k)}a_i\varphi_i(x)-\sum_{i=0}^{p}a_i\varphi_i(x)\right| =$$

$$=\left|\left(\sum_{i=0}^{p}+\sum_{i=p+1}^{b_1^{(n)}}+\sum_{j=2}^{\varrho}\sum_{i=a_{i_j}^{(n)}}^{b_{i_j}^{(n)}}+\sum_{j=1}^{\tau}\sum_{i\in\pi^{-1}(k_j)}-\sum_{j=1}^{\tau}\sum_{l=1}^{\varrho}\sum_{i\in A_j^{(l)}}\right)a_i\varphi_i(x)-\sum_{i=0}^{p}a_i\varphi_i(x)\right|\leqq$$

$$\leqq|s_{b_1^{(n)}}(x)-s_p(x)|+\sum_{j=2}^{\varrho}|s_{b_{i_j}^{(n)}}(x)-s_{a_{i_j-1}^{(n)}}(x)|+\sum_{j=1}^{\tau}\Big|\sum_{i\in\pi^{-1}(k_j)}a_i\varphi_i(x)\Big|+$$
$$+\sum_{j=1}^{\tau}\sum_{l=1}^{\varrho}\Big|\sum_{i\in A_j^{(l)}}a_i\varphi_i(x)\Big|$$

and (18) and (20) give that here the right hand side tends to zero as $p\to\infty$ by $b_{i_j}^{(n)}\geqq a_{i_j}^{(n)}>b_1^{(n)}>p$ $(2\leqq j\leqq\varrho)$ and $k_j>p$ (notice that $\pi^{-1}(k_j)\in\Pi_{k_j}$ for $1\leqq j\leqq\tau$ and take into account that $\varrho+\tau\leqq K$). Since $s_p(x)\to s(x)$ as $p\to\infty$ and $n>p_3=$ $=p_3(p)$ was arbitrary, we get the convergence of the series (5) at x and the proof is complete.

Proof of Theorem 1. Let us arrange the non-void sets $(Q_{k+1}\setminus Q_k)\cap$ $\cap(P_{m+1}\setminus P_m)$ into a sequence $A_0, A_1, \ldots, A_n, \ldots$ in such a way that $Q_k=\bigcup_{l=0}^{n_k}A_l$ $(k\geqq1)$ be satisfied for some sequence $n_1<n_2<\ldots$.

I. Sufficiency. Let us suppose (i), (ii) and the a.e. convergence of $\{s_k^Q(x)\}$

where s_k^Q are the Q-partial sums of the series (1). Let for $k=0, 1, 2, \ldots$

(22) $$\Phi_k(x) = \frac{1}{\sqrt{\sum_{\mathbf{i}\in A_k} a_{\mathbf{i}}^2}} \sum_{\mathbf{i}\in A_k} a_{\mathbf{i}}\varphi_{\mathbf{i}}(x), \quad b_k = \sqrt{\sum_{\mathbf{i}\in A_k} a_{\mathbf{i}}^2}$$

if $b_k \neq 0$ and

(23) $$\Phi_k(x) = \frac{1}{\sqrt{\sum_{\mathbf{i}\in A_k} 1}} \sum_{\mathbf{i}\in A_k} \varphi_{\mathbf{i}}(x), \quad b_k = 0$$

in the opposite case. Then $\{\Phi_k\}_{k=0}^{\infty}$ is an ONS on $[0, 1]$ and if S_k denotes the k-th partial sum of the ordinary orthogonal series $\sum_{l=0}^{\infty} b_l \Phi_l(x)$ then

(24) $$s_k^Q(x) = S_{n_k}(x) \quad (k = 1, 2, \ldots).$$

(i) gives $n_{k+1} - n_k \leqq K$ by which

$$\sum_{k=1}^{\infty} \sum_{n_k \leqq l < n_{k+1}} \int_0^1 (S_l(x) - S_{n_k}(x))^2 \, dx \leqq K \sum_{k=0}^{\infty} b_k^2 = K \sum_{\mathbf{i}\in N^d} a_{\mathbf{i}}^2 < \infty,$$

and so

$$\lim_{k\to\infty} S_l(x) - S_{n_k}(x) = 0 \quad (n_k \leqq l < n_{k+1})$$

almost everywhere. This, (24) and the assumed a.e. convergence of $\{s_k^Q(x)\}_{k=1}^{\infty}$ imply the a.e. convergence of $\sum_{l=0}^{\infty} b_l \Phi_l(x)$.

Let now $\pi: N \to N$ be defined by $\pi(l) = k$ iff $A_l \subseteqq P_{k+1} \setminus P_k$ $(l, k = 0, 1, \ldots)$. Clearly, π is "onto", $\pi^{-1}(k)$ is a finite set for each k and $P_{k+1} = \bigcup_{l\in\pi^{-1}(0,k)} A_l$, i.e.

$$s_{k+1}^P(x) = \sum_{l=0}^{k} \sum_{i\in\pi^{-1}(l)} b_i \Phi_i(x).$$

By (ii) $\pi^{-1}(0, k)$ and $\pi^{-1}(k, \infty)$ are the union of at most K sets of the form (l, m), $\pi^{-1}(l)$ or $\{l\} = (l, l)$, hence this π satisfies the assumptions of Theorem 2.

Applying Theorem 2 to π and $\sum_{l=0}^{\infty} b_l \Phi_l(x)$ and taking into account the above proved fact that the a.e. convergence of $\{s_k^Q(x)\}_{k=1}^{\infty}$ implies that of $\sum_{l=0}^{\infty} b_l \Phi_l(x)$, we obtain the sufficiency of conditions (i) and (ii).

II. Necessity. First let us prove the necessity of (i). Let us write shortly $Q_k^* = Q_{k+1} \setminus Q_k$, $P_k^* = P_{k+1} \setminus P_k$. If (i) does not hold then for each n there are a k_n and numbers $k_1^{(n)} < \ldots < k_n^{(n)} < l_1^{(n)} < \ldots < l_n^{(n)}$ such that

$$\emptyset \neq Q_{k_n}^* \cap P_{k_1^{(n)}} \subset Q_{k_n}^* \cap P_{k_2^{(k)}} \subset \ldots \subset Q_{k_n}^* \cap P_{l_n^{(n)}}.$$

We may suppose $l_{n-1}^{(n-1)} < k_1^{(n)}$ $(n = 1, 2, \ldots)$. Let $\mathbf{i}_\varrho^{(n)} \in Q_{k_n}^* \cap P_{k_{\varrho-1}^{(n)}}^*$, $\mathbf{j}_\varrho^{(n)} \in Q_{k_n}^* \cap P_{l_{\varrho-1}^{(n)}}^*$ $(1 \leqq \varrho \leqq n)$. Using the orthogonal series $\sum_{k=0}^{\infty} a_k \varphi_k(x)$ and the sequences p_n, q_n from

(12), putting

$$b_{\mathrm{i}_\varrho^{(p_n)}} = -b_{\mathrm{j}_\varrho^{(p_n)}} = a_{q_n+\varrho}, \quad \psi_{\mathrm{i}_\varrho^{(p_n)}}(x) = \psi_{\mathrm{j}_\varrho^{(p_n)}}(x) = \frac{1}{2}\varphi_{q_n+\varrho}(x) \quad (x\in[0, 1])$$

for $n=1, 2, \ldots, \varrho=1, \ldots, n$ and $b_{\mathrm{i}}=0$, $\psi_{\mathrm{i}}(x)=0$ $(x\in[0, 1])$ otherwise, and extending these ψ_i to an ONS on $[-1, 1]$ exactly as above in the necessity proof of Theorem 2 we get a series $\sum_{\mathrm{i}\in N^d} b_{\mathrm{i}}\psi_{\mathrm{i}}(x)$ for which $\sum_{\mathrm{i}\in Q_k} b_{\mathrm{i}}\psi_{\mathrm{i}}(x)=0$ and

$$\sum_{\mathrm{i}\in P_{k_\varrho^{(p_n)}}} b_{\mathrm{i}}\psi_{\mathrm{i}}(x) = \Big(\sum_{\mathrm{i}\in P_{l_{p_n-1}^{(p_n-1)}}} + \sum_{\mathrm{i}\in P_{k_\varrho^{(p_n)}}\setminus P_{l_{p_n-1}^{(p_n-1)}}}\Big) =$$

$$= 0+\sum_{s=1}^{\varrho} b_{\mathrm{i}_s^{(p_n)}}\psi_{\mathrm{i}_s^{(p_n)}}(x) = \frac{1}{2}\big(S_{q_n+\varrho}(x)-S_{q_n}(x)\big) \quad (x\in[0, 1],\ 1\leqq\varrho\leqq p_n,\ n=1, 2, \ldots).$$

Hence $\{s_k^Q(x)\}_{k=1}^\infty$ converges everywhere on $[-1, 1]$ but $\{s_k^P(x)\}_{k=1}^\infty$ diverges a.e. on $[0, 1]$ (see (12)).

Thus, the necessity of (i) is proved and from now on we assume its validity.

Let us now consider the sequence of the sets A_n introduced at the beginning of the proof and the mapping π used in the sufficiency proof. Using (i), (ii) can be expressed as: there is a K_1 such that for every $k, \pi^{-1}(0, k)$ and $\pi^{-1}(k, \infty)$ are the union of at most K_1 sets (l, m) and $\pi^{-1}(l)$. By (i) the a.e. convergence of $\{s_k^Q(x)\}_{k=1}^\infty$ is equivalent to that of

$$\sum_{l=0}^{\infty}\sum_{\mathrm{i}\in A_l} a_{\mathrm{i}}\varphi_{\mathrm{i}}(x) = \sum_{l=0}^{\infty} b_l\Phi_l(x)$$

(see point I above) where we used the notations of (19) and (20). Since the a.e. convergence of $\{s_k^P(x)\}_{k=1}^\infty$ is the same as the a.e. convergence of

$$\sum_{k=0}^{\infty}\sum_{\mathrm{i}\in P_{k+1}\setminus P_k} a_{\mathrm{i}}\varphi_{\mathrm{i}}(x) = \sum_{k=0}^{\infty}\sum_{l\in\pi^{-1}(k)} b_l\Phi_l(x),$$

the necessity of (ii) easily follows from Theorem 2.

We have completed our proof.

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