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Approximation in L^1 by Kantorovich polynomials

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1.

This paper is a continuation of two earlier ones [11, 12]. Let

$$K_n(f;x)=\sum_{k=0}^{n}\Big((n+1)\int\limits_{k/(n+1)}^{(k+1)/(n+1)} f(u)\,du\Big)\, b_{n,k}(x),\quad b_{n,k}(x)=\binom{n}{k}x^k(1-x)^{n-k}$$

be the Kantorovich-variant of the Bernstein operator. A series of papers contains results for the approximation properties of $K_n(f)$ in integral metrics (for references see the survey article [3]). However, the analogue of the well-known equivalence theorem of BERENS and LORENTZ [5] or that of LORENTZ and SCHUMAKER [7] and DITZIAN [6] is not known for them. The problem is the characterization of $\|K_n(f)-f\|_{L^1(0,1)}=O(n^{-\alpha})$ $(0<\alpha<1)$ in terms of a certain modulus of smoothness, and the aim of this paper is to give this characterization.

For $f\in L^p(0,1)$, $p>1$ we proved in [12]

Theorem A. *If* $1<p<\infty$, $0<\alpha<1$ *and* $f\in L^p(0,1)$ *then*

(i) $\|K_n(f)-f\|_{L^p}=O(n^{-\alpha})$

and

(ii) (α) $\|\Delta^*_{h\sqrt{x(1-x)}}(f;x)\|_{L^p(h^2,1-h^2)}=O(h^{2\alpha})$,

(β) $\|f(\cdot+h)-f(\cdot)\|_{L^p(0,1-h)}=O(h^\alpha)$

are equivalent.

Here

$$\Delta^*_h(f;x)=f(x-h)-2f(x)+f(x+h)$$

(we deviate from the custom and write $\|f(x)\|_{L^p}$ instead of $\|f(\cdot)\|_{L^p}$ if the former is more suggestive).

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For the saturation case $\alpha=1$ we have (see [9, 10, 4, 12])

Theorem B. *If* $1<p<\infty$ *and* $f\in L^p(0,1)$ *then the following are equivalent:*

(i) $\|K_n(f)-f\|_{L^p}=O(n^{-1})$,
(ii) f *has an absolutely continuous derivative with* $x(1-x)f''(x)\in L^p(0,1)$
(iii) $\|(x(1-x)\Delta_h^*(F;x))'\|_{L^p(h,1-h)}=O(h^2)$,
(iv) $\|x(1-x)\Delta_h^*(f;x)\|_{L^p(h,1-h)}=O(h^2)$,
(v) $\|\Delta^*_{h\sqrt{x(1-x)}}(f;x)\|_{L^p(h^2,1-h^2)}=O(h^2)$.

Here $F(x)=\int_0^x f(u)\,du$ and naturally (ii) means that "f coincides a.e. with a function which has absolutely continuous derivative".

Turning to L^1 let us mention the saturation result (see [8, 2]):

Theorem C. *For* $f\in L^1(0,1)$ *the following conditions are equivalent:*
(i) $\|K_n(f)-f\|_{L^1}=O(n^{-1})$,
(ii) f *is absolutely continuous and* $x(1-x)f'(x)$ *is of bounded variation,*
(iii) $\|x(1-x)\Delta_h^*(F,x)\|_{BV+L^\infty(h,1-h)}=O(h^2)$

Here $BV+L^\infty$ denotes the sum of the two norms: total variation and ess. supremum. Examples show that Theorem B does not hold for L^1, i.e., the BV-norm in Theorem C seems to be the appropriate one and we cannot hope in replacing it by an L^1-norm. The difference between Theorems B and C suggests also that we should exchange the L^p-norm in Theorem A for a BV-norm or something like that to obtain a correct result in L^1 (see also the conjecture in [3]). Thus, it is rather surprising that Theorem A holds word for word when $p=1$:

Theorem 1. *If* $0<\alpha<1$ *and* $f\in L^1(0,1)$ *then*

(i) $\|K_n(f)-f\|_{L^1}=O(n^{-\alpha})$

and

(ii) (α) $\|\Delta^*_{h\sqrt{x(1-x)}}(f;x)\|_{L^1(h^2,1-h^2)}=O(h^2)$,
(β) $\|f(\cdot+h)-f(\cdot)\|_{L^1(0,1-h)}=O(h^\alpha)$

are equivalent.

Let us mention that although (ii)⇒(i) holds also for $\alpha=1$, neither (ii) (α) nor (ii) (β) is necessary for (i) in the case $\alpha=1$. This is shown by the function $f(x)=\log x$ $(x\in(0,1))$.

The first result with the modulus of smootheness $\sup_{0<h\le\delta}\|\Delta^*_{h\sqrt{x(1-x)}}(f,x)\|_{L^p(h^2,1-h^2)}$ (more precisely with its analogue) was proved in [11] for the Szász—Kantorovich operators:

$$M_n(f;x)=\sum_{k=0}^{\infty}\Big(n\int_{k/n}^{(k+1)/n}f(u)\,du\Big)p_{n,k}(x),\quad p_{n,k}(x)=e^{-nx}\frac{(nx)^k}{k!},\quad x\ge 0.$$

Theorem D. *For* $1<p<\infty$, $0<\alpha<1$ *and* $f\in L^p(0,\infty)$ *the following conditions are equivalent:*

(i) $\|M_n(f)-f\|_{L^p(0,\infty)} = O(n^{-\alpha})$,
(ii) (α) $\|\Delta^*_{h\sqrt{x}}(f;\,x)\|_{L^p(h^2,\infty)} = O(h^{2\alpha})$
(β) $\|f(\cdot+h)-f(\cdot)\|_{L^p(0,\infty)} = O(h^\alpha)$.

This is true just as well for $p=1$:

Theorem 2. *Theorem* D *holds also when* $p=1$.

We shall prove only Theorem 2, but our method works also for K_n (the technical details are somewhat easier for M_n); we refer to [12] for the necessary changes in the proof (observe that [12] relates to [11] about as Theorem 1 relates to Theorem 2). The only point in our proof which might not be obvious for K_n is the delicate formula (2.5) but the analogue of this was given in [12, (4.5)].

Although Theorems A and 1 (D and 2) have the same form, here we have to use a different method since in the case $p>1$ the proof rested heavily on the maximal inequality. Nevertheless, the roots of the proofs of the inverse parts are the same: the so called elementary method of inverse results developed by Berens and Lorentz [5], and Becker and Nessel [1].

2. Proof of Theorem 2

I. Proof of (ii)$\Rightarrow$(i). First we derive from (ii) three further inequalities.
Inequality 1.

$$\int_0^h\!\!\int |f(x)-f(y)|\,dx\,dy = 2\int_0^h d\varepsilon \int_0^{h-\varepsilon} |f(x+\varepsilon)-f(x)|\,dx \leqq K\int_0^h \varepsilon^\alpha\,d\varepsilon \leqq Kh^{\alpha+1}.$$

Inequality 2.

$$A(f,h) \overset{\text{def}}{=} \left\| \frac{1}{x}\int_0^{h\sqrt{x}} |f(x\pm\tau)-f(x)|\,d\tau \right\|_{L^1(h^2,\infty)} \leqq Kh^{2\alpha} \quad (h\geqq 0).$$

Proof. For any $f\in L^1(0,\infty)$,

$$\int_{h^2}^\infty \frac{1}{x}\int_0^{h\sqrt{x}} |f(x\pm\tau)|\,d\tau\,dx \leqq Kh^{-2}\int_0^{2h^2}\!\!\int |f(x+u)|\,du\,dx +$$
$$+K\int_{2h^2}^\infty |f(u)|\frac{h\sqrt{u}}{u}\,du \leqq K\|f\|_{L^1}$$

and if f is absolutely continuous with $f' \in L_1$ then

$$A(f, h) \leqq \int_{h^2}^{\infty} \frac{1}{x} \int_0^{h\sqrt{x}} d\tau \left| \int_0^{\pm\tau} |f'(x+u)|\, du \right| dx \leqq$$

$$\leqq \int_{h^2}^{\infty} \frac{1}{x} \int_0^{h\sqrt{x}} (h\sqrt{x}-|u|)\, |f'(x\pm u)|\, du\, dx \leqq Kh^2 \|f'\|_{L^1}$$

Let now $f \in L^1$ be arbitrary for which (ii) (β) holds, and let

$$g_h(x) = \frac{1}{h^2} \int_0^{h^2} f(x+\tau)\, d\tau.$$

For this

$$\|f-g_h\|_{L^1} \leqq h^{-2} \int_0^{h^2} \|f(\cdot+\tau)-f(\cdot)\|_{L^1}\, d\tau \leqq Kh^{-2} \int_0^{h^2} \tau^{\alpha}\, d\tau \leqq Kh^{2\alpha}$$

and

$$\|g_h'\|_{L^1} = h^{-2} \|f(\cdot+h^2)-f(\cdot)\|_{L^1} \leqq Kh^{2\alpha-2}$$

by which

$$A(f, h) \leqq A(f-g_h, h)+A(g_h, h) \leqq K(\|f-g_h\|_{L^1}+h^2\|g_h'\|_{L^1}) \leqq Kh^{2\alpha}.$$

Inequality 3.

$$\left\| \frac{1}{h\sqrt{x}} \int_0^{h\sqrt{x}} |\Delta_\tau^*(f;\, x)|\, d\tau \right\|_{L^1(h^2, \infty)} = \left\| \frac{1}{h} \int_0^{h} |\Delta_{u\sqrt{x}}^*(f;\, x)|\, du \right\|_{L^1(h^2, \infty)} \leqq$$

$$\leqq \frac{1}{h} \int_0^h \|\Delta_{u\sqrt{x}}^*(f;\, x)\|_{L^1(h^2, \infty)}\, du \leqq K\frac{1}{h} \int_0^h u^{2\alpha}\, du \leqq Kh^{2\alpha}.$$

Now the analogous inequalities for L^p were the only tools used at the proof of (ii)$\Rightarrow$(i) in [11, Theorem 1], and this proof equally holds, using Inequalities 1—3, for $p=1$. For the details see [11].

II. Proof of (i)$\Rightarrow$(ii) (β). Let

$$v(f;\, \delta) = v(f) = \sup_{0\leqq h\leqq \delta} \|f(\cdot+h)-f(\cdot)\|_{L^1(0, \infty)}.$$

It is sufficient to prove that for $0<h\leqq 1$, $n\geqq 1$,

$$v(h) \leqq K\left(n^{-\alpha}+nhv\left(\frac{1}{n}\right)\right);$$

see [1, Lemma 2.1].

But

$$v(f;\, h) \leqq v(f-M_n(f);\, h)+v(M_n(f);\, h)$$

and here, by (i),

$$v(f-M_n(f);\, h) \leqq 2\|f-M_n(f)\|_{L^1} \leqq Kn^{-\alpha}.$$

By

$$M_n'(f; x) = n \sum_{k=0}^{\infty} \left(n \int_0^{1/n} \left(f\left(\frac{k}{n}+u\right) - f\left(\frac{k+1}{n}+u\right)\right) du \right) p_{n,k}(x)$$

we have

$$v(M_n(f);\, h) \leqq \int_0^{\infty} dx \int_0^{h} |M_n'(f;\, x+u)|\, du \leqq \int_0^{h} \|M_n'(f)\|_{L^1}\, du \leqq$$

$$\leqq hn \sum_{k=0}^{\infty} \int_0^{1/n} \left| f\left(\frac{k}{n}+u\right) - f\left(\frac{k+1}{n}+u\right)\right| du \int_0^{\infty} np_{n,k}(x)\, dx =$$

$$= hn \sum_{k=n}^{\infty} \int_0^{1/n} \left| f\left(\frac{k}{n}+u\right) - f\left(\frac{k+1}{n}+u\right)\right| du = hn \left\| f\left(\cdot + \frac{1}{n}\right) - f(\cdot)\right\|_{L^1} \leqq hnv\left(\frac{1}{n}\right),$$

and the proof is complete.

For later application let us prove also the inequality

$$I(f;\, \delta) = \left\| \frac{\delta}{\sqrt{x}} (f(x+\delta\sqrt{x}) - f(x-\delta\sqrt{x})) \right\|_{L^1(\delta^2,\, \infty)} \leqq K\delta^{2\alpha}. \tag{2.1}$$

In fact, for the function

$$g_\delta(x) = \frac{1}{\delta^2} \int_0^{\delta^2} f(x+u)\, du$$

we have proved above

$$\|f-g_\delta\|_{L^1} \leqq \delta^{-2} \int_0^{\delta^2} \|f(\cdot+u) - f(\cdot)\|_{L^1}\, du \leqq K\delta^{2\alpha}$$

and

$$\|g_\delta'\|_{L^1} \leqq \delta^{-2} \|f(\cdot+\delta^2) - f(\cdot)\|_{L^1} \leqq K\delta^{2\alpha-2},$$

by which

$$I(f;\, \delta) \leqq I(f-g_\delta;\, \delta) + I(g_\delta;\, \delta) \leqq$$

$$\leqq \|(f-g_\delta)(x+\delta\sqrt{x})\|_{L^1(\delta^2,\, \infty)} + \|(f-g_\delta)(x-\delta\sqrt{x})\|_{L^1(\delta^2,\, \infty)} +$$

$$+ \int_{\delta^2}^{\infty} \frac{\delta}{\sqrt{x}} \left(\int_{-\delta\sqrt{x}}^{\delta\sqrt{x}} |g_\delta'(x+u)|\, du \right) dx \leqq K\delta^{2\alpha} + \int_{\delta^2}^{\infty} \delta \int_{-\delta}^{\delta} |g_\delta'(x+u\sqrt{x})|\, du\, dx \leqq$$

$$\leqq K\delta^{2\alpha} + K\delta \int_{-\delta}^{\delta} \|g_\delta'\|_{L^1} \leqq K(\delta^{2\alpha} + \delta^2 \|g_\delta'\|_{L^1}) \leqq K\delta^{2\alpha}.$$

III. Proof of (i)$\Rightarrow$(ii) (α). First let us prove the following

Lemma. *Let* $0<h\leqq 1$, $h^2\leqq n^{-1}\leqq h$, $k=0,1,2,\ldots$. *Then there is an absolute constant* K *for which*

$$(1)\quad \int_{h^2}^{\infty} dx \iint_{-h\sqrt{x}/2}^{h\sqrt{x}/2} p_{n,k}(x+u+v)\,du\,dv \leqq K\frac{h^2(k+1)}{n^2},$$

$$(2)\quad \int_{h^2}^{\infty} dx \iint_{-h\sqrt{x}/2}^{h\sqrt{x}/2} \frac{k}{(x+u+v)^2}\,p_{n,k}(x+u+v)\,du\,dv \leqq Kh^2,$$

$$(3)\quad \int_{h^2}^{\infty} dx \iint_{-h\sqrt{x}/2}^{h\sqrt{2}/x} \frac{\left(\frac{k}{n}-(x+u+v)\right)^2}{(x+u+v)^2}\,p_{n,k}(x+u+v)\,du\,dv \leqq K\frac{h^2}{n^2}.$$

Proof. $p_{n,k}(x)$ increases on $(0, k/n)$ and decreases on $(k/n, \infty)$, hence

$$\iint_{-h\sqrt{x}/2}^{h\sqrt{x}/2} p_{n,k}(x+u+v)\,du\,dv \leqq$$

$$\leqq \begin{cases} h^2xp_{n,k}(x+h\sqrt{x}) & \text{for } x\in(0, k/n-h\sqrt{k/n}),\\ h^2x\max\limits_{y} p_{n,k}(y) & \text{for } x\in(k/n-h\sqrt{k/n},\ k/n+2h\sqrt{k/n}),\\ h^2xp_{n,k}(x-h\sqrt{x}) & \text{for } x\in(k/n+2h\sqrt{k/n},\ \infty).\end{cases}$$

Since

$$\int_{h^2}^{\infty} |g(x\pm h\sqrt{x})|\,dx \leqq 2\int_{0}^{\infty} g(x)\,dx, \quad xp_{n,k}(x)=\frac{k+1}{n}p_{n,k+1}(x),$$

$$\int_{0}^{\infty} p_{n,k}(x)\,dx=\frac{1}{n}$$

and $\max\limits_{y} p_{n,k}(y)=p_{n,k}(k/n)\leqq K/\sqrt{k+1}$ (use Stirling's formula), we obtain easily

$$\int_{h^2}^{\infty} dx \iint_{-h\sqrt{x}/2}^{h\sqrt{x}/2} p_{n,k}(x+u+v)\,du\,dv \leqq$$

$$\leqq K\left(\frac{(k+1)h^2}{n}\|p_{n,k+1}\|_{L^1}+h^2\frac{k}{n}\frac{1}{\sqrt{k+1}}h\sqrt{\frac{k}{n}}\right) \leqq K\left(\frac{h^2(k+1)}{n^2}\right) \quad (k=0,1,2,\ldots).$$

For $k\geqq 2$ inequality (2) follows from (1), since $kx^{-2}p_{n,k}(x)=(n^2/(k-1))p_{n,k-2}(x)$. For $k=1$ we have

$$\int_{h^2}^{\infty} dx \iint_{-h\sqrt{x}/2}^{h\sqrt{x}/2} n(x+u+v)^{-1}e^{-n(x+u+v)}\,du\,dv = n\int_{h^2}^{\infty} dx \int_{-h\sqrt{x}}^{h\sqrt{x}} \frac{h\sqrt{x}-|\tau|}{x+\tau}e^{-n(x+\tau)}\,d\tau \leqq$$

$$\leqq 2n\int_{h^2}^{\infty}\left(\frac{h}{\sqrt{x}}\int_{-h\sqrt{x}}^{h\sqrt{x}} e^{-n(x-\tau)}\,d\tau\right)dx \leqq Kh^2.$$

Finally, (3) follows from (1) for $k=0$, and for $k\geqq 1$ we have

$$\int_{h^2}^{\infty} dx \iint_{-h\sqrt{x}/2}^{h\sqrt{x}/2} \frac{(k/n-(x+u+v))^2}{(x+u+v)^2}\, p_{n,k}(x+u+v)\,du\,dv \leqq$$

$$\leqq K\int_{h^2}^{2h^2} dx \iint_{-h\sqrt{x}/2}^{h\sqrt{x}/2} \left(\frac{k+1}{n(x+u+v)}\right)^2 p_{n,k}(x+u+v)\,du\,dv+$$

$$+K\int_{2h^2}^{\infty} \frac{dx}{x^2} \iint_{-h\sqrt{x}/2}^{h\sqrt{x}/2} \left(\frac{k}{n}-(x+u+v)\right)^2 p_{n,k}(x+u+v)\,du\,dv.$$

Here the first term is at most Kh^2/n^2 for $k=1$ (see (2)) and

$$K\int_{h^2}^{2h^2} dx \iint_{-h\sqrt{x}/2}^{h\sqrt{x}/2} p_{n,k-2}(x+u+v)\,du\,dv \leqq Kh^6 \leqq Kh^2/n^2$$

for $k\geqq 2$.

The second term can be estimated as we have done in inequality (1) (use that $(k/n-x)^2 p_{n,k}(x)$ increases on $(0, (k+1)/n-\sqrt{2k+1}/n)$ and decreases on $((k+1)/n+ +\sqrt{2k+1}/n, \infty)$ together with the facts

$$\int_0^{\infty} \frac{n}{x}\left(\frac{k}{n}-x\right)^2 p_{n,k}(x)\,dx = \frac{1}{n},$$

$$\int_{\max\left(\frac{k+1}{n}-\frac{\sqrt{2k+1}}{n}-h\sqrt{\frac{k+1}{n}};\,h^2\right)}^{\frac{k+1}{n}+\frac{\sqrt{2k+1}}{n}+4h\sqrt{\frac{k+1}{n}}} \frac{dx}{x^2} \iint_{-h\sqrt{x}/2}^{h\sqrt{x}/2} \left(\frac{k}{n}-(x+u+v)\right)^2 p_{n,k}(x+u+v)\,du\,dv \leqq$$

$$\leqq K\frac{\sqrt{k}}{n}\left(\frac{k}{n}\right)^{-2} h^2\frac{k}{n}\left(\frac{\sqrt{k}}{n}\right)^2 \frac{1}{\sqrt{k}} \leqq K\frac{h^2}{n^2}.$$

Let us turn back to (ii) (α), and let

$$\omega(f;\delta)=\omega(\delta)=\sup_{0<h\leqq\delta}\int_{h^2}^{\infty} |\Delta^*_{h\sqrt{x}}(f;x)|\,dx.$$

It is sufficient to prove that for $0<h^2\leqq 1/n\leqq h\leqq 1$ we have

$$\omega(h)\leqq K\left(n^{-\alpha}+h^2 n\left(n^{-\alpha}+\omega\left(\frac{1}{n}\right)\right)\right),$$

see [1, Lemma 2.1]. Since (i) yields

$$\left\|\Delta^*_{h\sqrt{x}}(f-M_n(f);x)\right\|_{L^1(h^2,\infty)} \leqq K\,\|f-M_n(f)\|_{L^1} \leqq Kn^{-\alpha},$$

an easy consideration shows that it is enough to prove

$$(2.2)\qquad \left\|\Delta^*_{h\sqrt{x}}(M_n(f);\,x)\right\|_{L^1(h^2,\infty)} \leqq Kh^2n\left(n^{-\alpha}+\omega\left(\frac{1}{\sqrt{n}}\right)\right)\quad \left(h^2 \leqq \frac{1}{n} \leqq h\right).$$

Let

$$(2.3)\quad \mathcal{M}_n(f;\,x) = \sum_{k=1}^{\infty}\left(n\int_{k/n}^{(k+1)/n} f(u)\,du\right)p_{n,k}(x) = M_n(f;\,x)-ne^{-nx}\int_0^{1/n} f(u)\,du.$$

a) (ii) (β) (which we have proved above) gives

$$\left\|\left(n\int_0^{1/n} f(u)\,du\right)\Delta^*_{h\sqrt{x}}(e^{-nt};\,x)\right\|_{L^1(h^2,\infty)} \leqq Knh^2\left|\int_0^{1/n} f(u)\,du\right| \leqq$$

$$\leqq Knh^2\left\|f\left(\cdot+\frac{1}{n}\right)-f(\cdot)\right\|_{L^1} \leqq Knh^2n^{-\alpha}.$$

b) Let $F_1(x)=\int_0^x f(t)\,dt$, $F_2(x)=\int_0^x F_1(t)\,dt$ and

$$f_\delta(x) = \frac{1}{\delta^2}\int_0^\delta du\int_{-u}^{u} f(x+v\sqrt{x})\,dv = \frac{1}{\delta^2}\int_0^\delta du\int_0^u \left(f(x+v\sqrt{x})+f(x-v\sqrt{x})\right)dv =$$

$$= \frac{1}{\delta^2 x}\,\Delta^*_{\delta\sqrt{x}}(F_2;\,x).$$

We have

$$(2.4)\qquad \|f-f_\delta\|_{L^1(\delta^2,\infty)} \leqq \frac{1}{\delta^2}\int_0^\delta du\int_0^u \|\Delta^*_{v\sqrt{x}}(f;\,x)\|_{L^1(\delta^2,\infty)}\,dv \leqq \omega(\delta)$$

and

$$f_\delta''(x) = \frac{2}{\delta^2x^3}\,\Delta^*_{\delta\sqrt{x}}(F_2;\,x)-\frac{2}{\delta^2x^2}\,\Delta^*_{h\sqrt{x}}(F_1;\,x)-$$

$$-\frac{5}{4\delta x^{5/2}}\left(F_1(x+\delta\sqrt{x})-F_1(x-\delta\sqrt{x})\right)+\frac{1}{\delta^2x}\,\Delta^*_{\delta\sqrt{x}}(f;\,x)+$$

$$+\frac{1}{\delta x^{3/2}}\left(f(x+\delta\sqrt{x})-f(x-\delta\sqrt{x})\right)+\frac{1}{4x^2}\left(f(x+\delta\sqrt{x})+f(x-\delta\sqrt{x})\right),$$

and the key point in our theorem is that the latter is equal to

$$f_\delta''(x) = \frac{2}{x^2}\big(f_\delta(x)-f(x)\big)-\frac{5}{4x^2}\frac{1}{\delta}\int_0^\delta \Delta_{h\sqrt{x}}^*(f;\,x)\,dt+$$

$$(2.5)\quad +\frac{1}{4x^2}\Delta_{\delta\sqrt{x}}^*(f;\,x)+\frac{1}{\delta^2 x}\Delta_{\delta\sqrt{x}}^*(f;\,x)+\frac{1}{\delta x^{3/2}}\big(f(x+\delta\sqrt{x})-f(x-\delta\sqrt{x})\big)-$$

$$-\frac{2}{\delta x^{3/2}}\frac{1}{\delta}\int_0^\delta \big(f(x+t\sqrt{x})-f(x-t\sqrt{x})\big)\,dt.$$

Now

$$\|\Delta_{h\sqrt{x}}^*(\mathcal{M}_n(f);\,x)\|_{L^1(h^2,\infty)} \leqq$$

$$\leqq \|\Delta_{h\sqrt{x}}^*(\mathcal{M}_n(f-f_{\frac{1}{\sqrt{n}}});\,x)\|_{L^1(h^2,\infty)}+\|\Delta_{h\sqrt{x}}^*(\mathcal{M}_n(f_{\frac{1}{\sqrt{n}}});\,x)\|_{L^1(h^2,\infty)}$$

and below we estimate the two terms on the right side separately.

c) Since

$$(p_{n,k}(x))'' = \frac{n^2}{x^2}\left[\left(\frac{k}{n}-x\right)^2-\frac{k}{n^2}\right]p_{n,k}(x),$$

we obtain by (2) and (3) from the Lemma, and by (2.4) that

$$\int_{h^2}^\infty |\Delta_{h\sqrt{x}}^*(\mathcal{M}_n(f-f_{\frac{1}{\sqrt{n}}});\,x)|\,dx = \int_{h^2}^\infty dx\Big|\iint_{-h\sqrt{x}/2}^{h\sqrt{x}/2} \mathcal{M}_n''(f-f_{\frac{1}{\sqrt{n}}};\,x+u+v)\,du\,dv\Big| \leqq$$

$$\leqq \sum_{k=1}^\infty\Big(n\int_{k/n}^{(k+1)/n}|f-f_{\frac{1}{\sqrt{n}}}|(u)\,du\Big)\Big\{\int_{h^2}^\infty dx\iint_{-h\sqrt{x}/2}^{h\sqrt{x}/2}\frac{n^2}{(x+u+v)^2}\Big(\frac{k}{n}-(x+u+v)\Big)^2\times$$

$$\times p_{n,k}(x+u+v)\,du\,dv+\int_{h^2}^\infty dx\iint_{-h\sqrt{x}/2}^{h\sqrt{x}/2}\frac{k}{(x+u+v)^2}p_{n,k}(x+u+v)\,du\Big\} \leqq$$

$$\leqq Kh^2 n\sum_{k=1}^\infty\int_{k/n}^{(k+1)/n}|f-f_{\frac{1}{\sqrt{n}}}|(u)\,du = Kh^2 n\|f-f_{\frac{1}{\sqrt{n}}}\|_{L^1\left(\frac{1}{n},\infty\right)} \leqq Kh^2 n\omega\left(\frac{1}{\sqrt{n}}\right).$$

d) We have also

$$(p_{n,k}(x))'' = n^2\big(p_{n,k-2}(x)-2p_{n,k-1}(x)+p_{n,k}(x)\big)\quad (k=1,2,\ldots,\ p_{n-1}(x)\equiv 0),$$

thus

$$\int_{h^2}^{\infty} |\Delta^*_{h\sqrt{x}}(\mathcal{M}_n(f_{\frac{1}{\sqrt{n}}}); x)|\, dx =$$

$$= n^2 \int_{h^2}^{\infty} dx \left| \sum_{k=1}^{\infty} \left(n \iiint_0^{1/n} f''_{\frac{1}{\sqrt{n}}}\left(\frac{k}{n}+u+v+w\right) du\, dv\, dw \right) \iint_{-h\sqrt{x}/2}^{h\sqrt{x}/2} p_{n,k}(x+s+t)\, ds\, dt + \right.$$

$$\left. + \left(-2n \int_{1/n}^{3/n} f_{\frac{1}{\sqrt{n}}}(u)\, du + n \int_{2/n}^{3/n} f_{\frac{1}{\sqrt{n}}}(u)\, du \right) \iint_{-h\sqrt{x}/2}^{h\sqrt{x}/2} p_{n,0}(x+s+t)\, ds\, dt \right| \leqq$$

$$\leqq K \sum_{k=1}^{\infty} \left(n \iiint_0^{1/n} \left| f''_{\frac{1}{\sqrt{n}}}\left(\frac{k}{n}+u+v+w\right) \right| du\, dv\, dw \right) kh^2 +$$

$$+ Knh^2 \left(\left| \int_{1/n}^{2/n} f_{\frac{1}{\sqrt{n}}}(u)\, du \right| + \left| \int_{2/n}^{3/n} f_{\frac{1}{\sqrt{n}}}(u)\, du \right| \right) \leqq$$

$$\leqq Knh^2 \sum_{k=1}^{\infty} k \iiint_0^{1/n} \left| f''_{\frac{1}{\sqrt{n}}}\left(\frac{k}{n}+u+v+w\right) \right| du\, dv\, dw +$$

$$+ Knh^2 \left(n^{-\alpha} + \omega\left(\frac{1}{\sqrt{n}}\right) \right) = Knh^2 A + Knh^2 \left(n^{-\alpha} + \omega\left(\frac{1}{\sqrt{n}}\right) \right),$$

where

$$A = \sum_{k=1}^{\infty} k \iiint_0^{1/n} \left| f''_{\frac{1}{\sqrt{n}}}\left(\frac{k}{n}+u+v+w\right) \right| du\, dv\, dw,$$

and where we used that

$$\left| \int_{1/n}^{2/n} f_{\frac{1}{\sqrt{n}}}(u)\, du \right| + \left| \int_{2/n}^{3/n} f_{\frac{1}{\sqrt{n}}}(u)\, du \right| \leqq \|f - f_{\frac{1}{\sqrt{n}}}\|_{L^1\left(\frac{1}{n}, \infty\right)} +$$

$$+ \left| \int_{1/n}^{\infty} f(u)\, du - \int_{2/n}^{\infty} f(u)\, du \right| + \left| \int_{2/n}^{\infty} f(u)\, du - \int_{3/n}^{\infty} f(u)\, du \right| \leqq K \left(\omega\left(\frac{1}{\sqrt{n}}\right) + n^{-\alpha} \right).$$

To estimate A we apply (2.5). Taking absolute value in (2.5) term by term we increase $|f''_{\frac{1}{\sqrt{n}}}(x)|$. Now the first term on the right of (2.5) contributes to A at most by

$$\sum_{k=1}^{\infty} k \iiint_0^{1/n} \frac{2}{\left(\frac{k}{n}+u+v+w\right)^2} \left| n \int_0^{1/\sqrt{n}} ds \int_0^s \Delta^*_{t\sqrt{\frac{k}{n}+u+v+w}} \left(f; \frac{k}{n}+u+v+w\right) dt \right| du\,dv\,dw \leqq$$

$$\leqq 2n \int_0^{1/\sqrt{n}} ds \int_0^s dt \sum_{k=1}^{\infty} \frac{n^2}{k} \iiint_0^{1/n} \left| \Delta^*_{t\sqrt{\frac{k}{n}+u+v+w}} \left(f; \frac{k}{n}+u+v+w\right) \right| du\,dv\,dw \leqq$$

$$\leqq Kn \int_0^{1/\sqrt{n}} ds \int_0^s \|\Delta^*_{t\sqrt{x}}(f; x)\|_{L^1\left(\frac{1}{n},\infty\right)} dt \leqq K\omega\left(\frac{1}{\sqrt{n}}\right).$$

Quite similarly the contribution of the second, third and fourth terms to A is at most $K\omega\left(\frac{1}{\sqrt{n}}\right)$.

Using inequality (2.1), the fifth term contributes to A at most by

$$\sum_{k=1}^{\infty} k \iiint_0^{1/n} \frac{\sqrt{n}}{\left(\frac{k}{n}+u+v+w\right)^{3/2}} \left| f\left(\frac{k}{n}+u+v+w+\frac{1}{\sqrt{n}}\sqrt{\frac{k}{n}+u+v+w}\right) - \right.$$

$$\left. -f\left(\frac{k}{n}+u+v+w-\frac{1}{\sqrt{n}}\sqrt{\frac{k}{n}+u+v+w}\right) \right| du\,dv\,dw \leqq$$

$$\leqq K \int_{1/n}^{\infty} \frac{1}{\sqrt{nx}} \left| f\left(x+\frac{1}{\sqrt{n}}\sqrt{x}\right) - f\left(x-\frac{1}{\sqrt{n}}\sqrt{x}\right) \right| dx \leqq Kn^{-\alpha},$$

and a similar estimate can be given for the contribution of the sixth term:

$$\sum_{k=1}^{\infty} k \iiint_0^{1/n} \frac{2\sqrt{n}}{\left(\frac{k}{n}+u+v+w\right)^{3/2}} \left(\sqrt{n} \int_0^{1/\sqrt{n}} \left| f\left(\frac{k}{n}+u+v+w+t\sqrt{\frac{k}{n}+u+v+w}\right) - \right.\right.$$

$$\left.\left. -f\left(\frac{k}{n}+u+v+w-t\sqrt{\frac{k}{n}+u+v+w}\right) \right| dt \right) du\,dv\,dw \leqq$$

$$\leqq \int_0^{1/\sqrt{n}} \frac{1}{t} \left(\int_{1/n}^{\infty} \frac{t}{\sqrt{x}} |f(x+t\sqrt{x})-f(x-t\sqrt{x})|\,dx \right) dt \leqq K \int_0^{1/\sqrt{n}} t^{2\alpha-1}\,dt \leqq Kn^{-\alpha}.$$

Collecting our estimates from a) to d) we obtain (2.2) by which the proof is complete.

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