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Compact and Hilbert—Schmidt composition operators on Hardy spaces of the upper half-plane

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Introduction. Let $H^p(\pi^+)$ denote the Banach space of functions f holomorphic in π^+ (the upper half-plane) for which

$$\|f\|_p = \operatorname*{Sup}_{y>0}\left\{\left|\int_{-\infty}^{\infty} |f(x+iy)|^p\,dx\right|^{1/p}\right\} < \infty.$$

Let $T:\pi^+\to\pi^+$ be analytic. Then the composition mapping C_T, defined by

$$C_T f = f\circ T,$$

maps $H^p(\pi^+)$ into the vector space of all analytic functions on π^+. This mapping C_T is a linear transformation. If the range of C_T is a subspace of $H^p(\pi^+)$ and C_T happens to be bounded, we call it the composition operator induced by T. We are interested in the case when $p=2$. In this case $H^2(\pi^+)$ becomes a Hilbert space. For the sake of simplicity we will denote $\|\ \|_2$ simply by $\|\ \|$. Composition operators on $H^2(\pi^+)$ have been studied by SINGH [6] and SINGH and SHARMA [7]. In [7], we have proved that if T is an analytic function from π^+ into itself and the only singularity that T can have is a pole at infinity, then C_T is a bounded operator on $H^2(\pi^+)$ if and only if the point at infinity is a pole of T. In Section 2, we give a characterization of compact composition operators on $H^2(\pi^+)$. A sufficient condition for a composition operator to be compact is also provided. In Section 3, Hilbert—Schmidt composition operators are characterized.

2. Compact composition operators on $H^2(\pi^+)$. A linear operator A on a Hilbert space H is called compact if A takes bounded sets into sets with compact closures. This definition is equivalent to the statement that the image of every bounded sequence under A has a convergent subsequence [2]. This is further equivalent to saying that if $f_n\to f$ weakly in H, then $Af_n\to Af$ strongly in H. In this section we give a characterization of compact composition operators on $H^2(\pi^+)$.

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Theorem 2.1. *Let C_T be a composition operator on $H^2(\pi^+)$. Then C_T is compact if and only if for every sequence $f_n \to f$ uniformly on compact subsets of π^+ and bounded in $H^2(\pi^+)$ norm, the image sequence $C_T f_n \to C_T f$ strongly.*

We need the following lemmas to prove the theorem.

Lemma 2.1. *Let $f \in H^2(\pi^+)$. Then $|f(x+iy)|^2 \leqq \|f\|^2/2\pi y$ for $x+iy \in \pi^+$.*

Proof. First suppose $f \in H^1(\pi^+)$. Then by Cauchy-formula [1, p. 195],

$$f(w) = (2\pi i)^{-1} \int_{-\infty}^{\infty} \frac{f(r)}{r-w}\, dr.$$

Writing $w = x+iy$ and taking absolute values we get

$$|f(x+iy)| \leqq (2\pi y)^{-1} \int_{-\infty}^{\infty} |f(r)|\, dr = \|f\|_1/2\pi y.$$

(Here $\|f\|_1$ is the $H^1(\pi^+)$-norm.)

Let $f \in H^2(\pi^+)$. Then we can write $f = B \cdot g$, where B is a Blaschke product and g is an analytic function in π^+ and does not have any zero in π^+ [3, pp. 132—133]. It is obvious that $\|f\| = \|g\|$. Let $h_1 = g^2$. Then $h_1 \in H^1(\pi^+)$. Hence $f = B \cdot h_1^{1/2}$ and

$$|f(x+iy)| = |B(x+iy)|\, |h_1(x+iy)|^{1/2} \leqq \|h_1\|_1^{1/2}/\sqrt{2\pi y},$$

which implies that $|f(x+iy)|^2 \leqq \|f\|^2/2\pi y$ for every $x+iy \in \pi^+$.

Lemma 2.2. *Let $\{f_n\}$ be a sequence in $H^2(\pi^+)$. Then $f_n \to f$ in norm implies that $f_n \to f$ uniformly on compact subsets of π^+.*

Proof. Suppose $f_n \to f$ strongly. Let K be a compact subset of π^+. Then by Lemma 2.1

$$|f_n(x+iy) - f(x+iy)| \leqq (2\pi)^{-1/2} M_K \|f_n - f\|,$$

where $M_K = \sup_{x+iy \in K} \{y^{-1/2}\}$. The right hand side tends to zero as $n \to \infty$ for every point $x+iy \in K$. Since K is an arbitrary compact subset of π^+ and $f_n \to f$ uniformly on the compact subset K of π^+, the proof follows.

Lemma 2.3. *If $\{f_n\}$ is a bounded sequence in $H^2(\pi^+)$, then there exists a subsequence $\{f_{n_k}\}$ which converges uniformly on compact subsets.*

Proof. In the light of Theorem 14.6 of [4] it is enough to show that the sequence $\{f_n\}$ is uniformly bounded on each compact subset of π^+. If K is a compact subset of π^+, then again by Lemma 2.1 we have for $x+iy \in K$ that

$$|f_n(x+iy)| \leqq (2\pi)^{-1/2} M_K M,$$

where M_K is as in Lemma 2.2 and $M \geqq 0$ is such that $\|f_n\| \leqq M$ for all n. This finishes the proof.

Proof of Theorem 2.1. Suppose C_T is compact. Let $\{f_n\}$ be a sequence in $H^2(\pi^+)$ and $f \in H^2(\pi^+)$ such that $f_n \to f$ uniformly on compact subsets of π^+ and let $M \geqq 0$ be such that $\|f_n\| \leqq M$ for all n. Then we want to show that

$$\|C_T f_n - C_T f\| \to 0 \quad \text{as} \quad n \to \infty.$$

Suppose this is not true. Then there exists a subsequence $C_T f_{n_k}$ and an $\varepsilon > 0$ such that $\|C_T f_{n_k} - C_T f\| \geqq \varepsilon > 0$. Since $\{f_{n_k}\}$ is norm bounded and C_T is compact, there exists a subsequence $\{f_{n_{k_l}}\}$ such that $C_T f_{n_{k_l}} \to g$ strongly for some $g \in H^2(\pi^+)$. Hence, by Lemma 2.2, $C_T f_{n_{k_l}} \to g$ uniformly on compact subsets. Also, the subsequence $\{f_{n_{k_l}}\}$ converges to f uniformly on compact subsets, implying that $\{C_T f_{n_{k_l}}\}$ converges to $C_T f$ uniformly on compact subsets. This shows that $C_T f = g$, which is a contradiction. This proves that $\|C_T f_n - C_T f\| \to 0$ as $n \to \infty$.

In order to prove the converse, let F be a bounded set in $H^2(\pi^+)$. We want to show that the closure of $\{C_T f : f \in F\}$ is compact. Let $\{C_T f_n\}$ be a sequence in this closure. Then, since $\{f_n\}$ is norm bounded, by Lemma 2.3 there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ converging uniformly on compact subsets to some function f. Hence, by our hypothesis, $C_T f_{n_k} \to C_T f$ strongly, which shows that $\{C_T f_n\}$ has an accumulation point. Thus the closure of $\{C_T f : f \in F\}$ is countably compact and hence compact. This completes the proof.

In the next theorem the above result is used to give a sufficient condition for a composition operator to be compact.

Theorem 2.2. *Let $T : \pi^+ \to \pi^+$ be an analytic function such that C_T is a bounded operator on $H^2(\pi^+)$. Suppose $T_*(x) = \lim_{y \to 0} T(x+iy)$ exists a.e. and $T_*(x) \in \pi_+$ for almost all $x \in R$ (the set of reals). If $\int_{-\infty}^{\infty} [\operatorname{im} T_*(x)]^{-1} dx < \infty$, then C_T is a compact composition operator on $H^2(\pi^+)$.*

The following lemma is required to prove the theorem.

Lemma 2.4. *If $T_*(x) = \lim_{y \to 0} T(x+iy)$ exists a.e. and $T_*(x) \in \pi^+$ for almost all $x \in R$, then for every $f \in H^2(\pi^+)$*

$$(f \circ T_*)(x) = (f \circ T)_*(x) \quad a.e. \text{ on } R.$$

Proof. Let $E_1 = \{x \in R : T_*(x) \text{ does not exist}\}$, $E_2 = \{x \in R : T_*(x) \notin \pi^+\}$ and $E = E_1 \cup E_2$. Then for $x \in R \setminus E$, $T_*(x) = \lim_{y \to 0} T(x+iy)$ belongs to π^+. Since f is analytic at $T_*(x)$, it follows by the continuity of f that

$$(f\circ T_*)(x) = f(\lim_{y\to 0} T(x+iy)) = \lim_{y\to 0} (f\circ T)(x+iy) = (f\circ T)_*(x)$$

for every $x\in R\setminus E$. Since the set E has Lebesgue measure zero, the result follows.

Proof of Theorem 2.2. Let $\{f_n\}$ be a bounded sequence in $H^2(\pi^+)$ such that $f_n\to f$ uniformly on compact subsets. If we show that $C_Tf_n\to C_Tf$ strongly, then we are done. Using Lemmas 2.4 and 2.1 we have

$$|(C_Tf_n-C_Tf)_*(x)|^2 = |(f_n\circ T)_*-(f\circ T)_*(x)|^2 =$$
$$= |(f_n\circ T_*)(x)-(f\circ T_*)(x)|^2 = |(f_n-f)(T_*(x))|^2 \leqq M/im\, T_*(x),$$

where $M\geqq 0$ is such that $\|f_n-f\|/2\pi\leqq M$ for all n. Since $T_*(x)\in\pi^+$ for $x\in R\setminus E$ and the convergence is uniform on compact subsets, we have

$$(C_Tf_n-C_Tf)_*(x) = f_n(T_*(x))-f(T_*(x)) \to 0 \quad \text{as} \quad n\to\infty \quad \text{for all} \quad x\in R\setminus E,$$

where E is the set as described in Lemma 2.4. This shows that $|(C_Tf_n-C_Tf)_*|^2\to 0$ as $n\to\infty$ pointwise on $R\setminus E$ and the functions $|C_Tf_n-C_Tf)|^2$ are bounded by an integrable function g defined by $g(x)=1/\text{im}\, T_*(x)$ for $x\in R$. Hence, by Lebesgue's dominated convergence theorem and by the equality

$$\|f\|^2 = \int_{-\infty}^{\infty} |f_*(x)|^2\,dx \quad \text{for every} \quad f\in H^2(\pi^+)$$

(see [1, p. 190]), it follows that $\|C_Tf_n-C_Tf\|^2\to 0$ as $n\to\infty$. This completes the proof.

3. Hilbert—Schmidt composition operators on $H^2(\pi^+)$. A linear operator A on an infinite dimensional separable Hilbert space is said to be Hilbert—Schmidt if there exists an orthonormal basis $\{e_n: n\in \mathrm{N}\}$ in H such that

$$\sum_{n\in \mathrm{N}} \|Ae_n\|^2 < \infty. \tag{3.1}$$

It is easy to see that the sum on the right side of (3.1) does not depend upon the particular choice of the orthonormal basis $\{e_n: n\in\mathrm{N}\}$ [5].

In Theorem 2.2 it has been analysed that if an analytic function T maps the upper half-plane into the upper half-plane and C_T is a composition operator on $H^2(\pi^+)$, then the following condition

$$\int_{-\infty}^{\infty} 1/im\, T_*(x)\,dx < \infty \tag{3.2}$$

is sufficient for C_T to be a compact composition operator on $H^2(\pi^+)$. In fact, the condition (3.2) turns out to be a necessary and sufficient condition for C_T to be a Hilbert—Schmidt composition operator on $H^2(\pi^+)$. This we demonstrate in the following theorem.

Theorem 3.1. *Let $T:\pi^+\to\pi^+$ be an analytic function such that C_T is a composition operator on $H^2(\pi^+)$. Suppose $T_*(x)=\lim_{y\to 0} T(x+iy)$ exists a.e. and $T_*(x)\in\pi^+$ for almost all $x\in R$. Then the condition (3.2) is necessary and sufficient for C_T to be Hilbert—Schmidt.*

Proof. We know that the family of functions S_n defined by

$$S_n(w)=\frac{(w-i)^n}{\sqrt{\pi}(w+i)^{n+1}}\quad (n=0,1,\dots)$$

forms an orthonormal basis for $H^2(\pi^+)$. Therefore, C_T is Hilbert—Schmidt if and only if

$$\infty>\sum_{n=0}^{\infty}\|C_TS_n\|^2=\sum_{n=0}^{\infty}\int_{-\infty}^{\infty}|(S_n\circ T)_*(x)|^2\,dx=\sum_{n=0}^{\infty}\int_{-\infty}^{\infty}|S_n(T_*(x))|^2\,dx$$

(the equalities above follow from [1, p. 190] and Lemma 2.4, respectively). A simple computation yields that C_T is Hilbert—Schmidt if and only if

$$\infty>\pi^{-1}\int_{-\infty}^{\infty}[4\,im\,T_*(x)]^{-1}\,dx.$$

Hence the theorem.

Remark. It is worthwhile to remark here that Theorem 2.2 follows as an easy consequence of Theorem 3.1. In spite of this we have presented an independent proof to Theorem 2.2 because of the following reason: With a little modification Theorem 2.1 and consequently Theorem 2.2 can easily be developed for the Banach spaces $H^p(\pi^+)\,(1\leq p<\infty)$. Hence if we consider a composition operator on $H^p(\pi^+)$, the condition (3.2) turns out to be sufficient for a composition operator C_T to be compact on $H^p(\pi^+)$. Whereas, in case of $H^2(\pi^+)$, the condition (3.2) is necessary as well as sufficient for a composition operator C_T to be a Hilbert—Schmidt operator.

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