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An exact description of Lorentz spaces

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1. Introduction

We assume that f is a measurable complex-valued function on a measure space (Ω, μ), where μ is a σ-finite positive measure. The function f can be rearranged to a non-increasing function, denoted f^*, on $[0, \infty[$. The function f^* is continuous from the right and equidistributed with f (see e.g. [13, p. 131]).

We suppose that p and q are real numbers satisfying $0<p<\infty$, $0<q<\infty$. The Lorentz space $L(p, q)$ consists of all functions f satisfying

$$\|f\|^*_{p,q} = \Big(\int\limits_0^\infty (f^*)^q t^{q/p-1}\, dt\Big)^{1/q} < \infty.$$

See [7], [9] or [13, p. 132]. The $L(p, q)$-spaces are of great interest in pure and applied mathematics. In particular, they appear as intermediate spaces in the theory of interpolation (see e.g. [6, p. 264] or [13, p. 134]).

Obviously $L(p, p)=L^p$. It is well known that if $q_2 \leqq q_1$, then $\|f\|^*_{p,q_1} \leqq \|f\|^*_{p,q_2}$ (see [6, p. 253]). In particular, $L(p, q) \supset L^p$ when $p<q$ and $L(p, q) \subset L^p$ when $p>q$. Moreover, in a sense, every $L(p, q)$-space is "close to" the corresponding L^p-space. In particular, by generalizing the definition of the $L(p, q)$-norm in the natural way we obtain the usual weak L^p-space when $q=\infty$. However, it is not possible to identify an $L(p, q)$-space by some Orlicz space of the type $L^p(\log L)^a$. One aim of this paper is to give an exact description of the $L(p, q)$-spaces at least in similar terms.

Throughout this paper we let the letter h stand for a strictly positive and continuous function on $[0, \infty[$ which is constant on $[0, 1]$.

The following theorem by the present author can be found in [12, p. 270].

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Theorem A. *Let* $p>q$. *Then*

$$\int_0^1 (f^*)^q t^{q/p-1}\,dt<\infty$$

if and only if

$$\int_0^1 (f^* h(\log^+ f^*))^p\,dt<\infty \tag{1.1}$$

for some function h *such that, for some* $a>0$,

(1.2) $h(x)a^x$ *is a decreasing or an increasing function of* x

and

$$\int_1^\infty (h(x))^{pq/(q-p)}\,dx<\infty. \tag{1.3}$$

We may assume, without loss of generality, that $\log=\log_2$.

In Section 2 of this paper we shall state a theorem (Theorem 2.1) which generalizes Theorem A in two directions. On the one hand, by also studying conditions of the type $\int_1^\infty (f^*)^q t^{q/p-1}dt<\infty$ and, on the other hand, by also considering the case $p<q$. In this way we obtain an exact characterization of the $L(p,q)$-spaces not only for the special case when $\mu(\Omega)<\infty$ and $p>q$. Some applications to the theory of Fourier series (and transforms) are also given in Section 2. In particular, we shall see that the conclusion we usually extract from Hausdorff—Young's inequality (see e.g. [14, vol II, p. 101]) is, in a sense, far from being the sharpest possible. Some useful lemmas can be found in Section 3. The proof of the main theorem in Section 2 is carried out in Sections 4 (the case $p>q$) and 5 (the case $p<q$).

We say that the function f belongs to the Lorentz—Zygmund space $L^{p,q}(\log L)^\alpha$, $0<p<\infty$, $0<q<\infty$, $-\infty<\alpha<\infty$ if the quasi-norm

$$\|f\|^*_{p,q;\alpha}=\Big(\int_0^\infty (f^*(t)t^{1/p}(|\log t|+1)^\alpha)^q\,dt/t\Big)^{1/q}$$

is finite (see [2, p. 7]). In particular, we have $L^{p,q}(\log L)^0=L(p,q)$ and $L^{p,p}(\log L)^\alpha$ can be identified with the Zygmund space $L^p(\log L)^\alpha$ (see [2, p. 35]).

In Section 6 we shall generalize our main theorem so that we obtain an exact characterization of the spaces $L^{p,q}(\log L)^\alpha$. We shall also point out the fact that a recent embedding result by BENNETT and RUDNICK [2, p. 31] is a consequence of this characterization.

In Section 7 we shall give some concluding remarks. In particular, we shall compare the functional spaces introduced in this paper with the similarly defined Beurling—Herz spaces (see [1, p. 2] and [5, pp. 298—300]).

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2. A description of the $L(p, q)$-spaces

We make the following definition.

Definition. Let $p>q$. Then

a) $f\in E_0(p, q)$ if

$$\int_0^1 (f^* h(\log^+ f^*))^p \, dt < \infty \tag{2.1}$$

for some function h such that, for some $a>0$,

(2.2) $h(x)a^x$ is a decreasing or an increasing function of x

and

$$\int_1^\infty (h(x))^{pq/(q-p)} \, dx < \infty. \tag{2.3}$$

b) $f\in E_\infty(p, q)$ *if* $f^*(t)>0$ *and*

$$\int_1^\infty \left(f^* h\left(\log^+ \frac{1}{f^*}\right)\right)^p dt < \infty \tag{2.4}$$

for some function h satisfying (2.2) and (2.3).

Let $p<q$. Then

c) $f\in E_0(p, q)$ if (2.1) holds for every function h satisfying (2.2) and (2.3).

d) $f\in E_\infty(p, q)$ if (2.4) holds for every function h satisfying (2.2) and (2.3).

Let $p\neq q$. Then

e) $f\in E(p, q)$ if $f\in E_0(p, q)$ and $f\in E_\infty(p, q)$.

The main theorem in this section can now be formulated in the following way.

Theorem 2.1. *Let* $0<p<\infty$ *and* $0<q<\infty$. *Then*

a) $\int_0^1 (f^*)^q t^{q/p-1} dt < \infty$ *if and only if* $f\in E_0(p, q)$

and

b) $\int_1^\infty (f^*)^q t^{q/p-1} dt < \infty$ *if and only if* $f\in E_\infty(p, q)$.

We see that part a) of this theorem gives an exact description of the desired type for the case when the μ-measure of Ω is finite. By combining the equivalences in Theorem 2.1 we obtain a characterization of the $L(p, q)$-spaces in the general case, namely that

$$f\in L(p, q) \text{ if and only if } f\in E(p, q). \tag{2.5}$$

It can be somewhat difficult to see what this equivalence really means so we shall formulate it in another way. Therefore we let D be a subset of Ω such that $|f|\geqq 1$ on D and $|f|\leqq 1$ on $\Omega\setminus D$. Then we can make some elementary calculations to find that $f\in E(p, q)$ if and only if

$$\int_D (|f|\, h(\log|f|))^p\, d\mu + \int_{\Omega\setminus D}\left(|f|\, h\left(\log\frac{1}{|f|}\right)\right)^p d\mu < \infty \tag{2.6}$$

for *some* (the case $p>q$) or *every* (the case $p<q$) *function* h *satisfying* (2.2) and (2.3). In the sequel we say that $f\in L^p h(\log L)$ when (2.6) holds. For the special case $h(x)=x^\alpha$ we get the Zygmund space $L^p(\log L)^\alpha$. We can now formulate the equivalence (2.5) in the following way.

Theorem 2.2. *Let* $0<p<\infty$ *and* $0<q<\infty$.

a) *Let* $p>q$. *Then* $f\in L(p, q)$ *if and only if* $f\in L^p h(\log L)$ *for some function* h *satisfying* (2.2) *and* (2.3).

b) *Let* $p<q$. *Then* $f\in L(p, q)$ *if and only if* $f\in L^p h(\log L)$ *for every function* h *satisfying* (2.2) *and* (2.3).

We apply Theorem 2.2 with $h(x)=x^{(1+\delta)(1/q-1/p)}$, $\delta>0$, and find that if $p>q$, then, for every $\varepsilon>0$,

$$L(p, q) \supset L^p(\log L)^{1/q-1/p+\varepsilon} \tag{2.7}$$

and if $p<q$, then, for every $\varepsilon>0$,

$$L(p,q) \subset L^p(\log L)^{1/q-1/p-\varepsilon}. \tag{2.8}$$

The inclusions (2.7) and (2.8) are the sharpest possible in the sense that they are in general false if we permit $\varepsilon=0$. In order to verify this fact we set $(\Omega, \mu)= =([0, 1], dx)$ and study the function

$$f(x) = \frac{1}{x^{1/p}(\log 1/x)^{1/q}(\log(\log 1/x+2))^\alpha}.$$

Then, as $t\to 0$,

$$(f^*)^q t^{q/p-1} \simeq \frac{1}{t\log 1/t\,(\log(\log 1/t+2))^{\alpha q}},$$

and

$$(f^*)^p(\log^+ f^*+1)^{p/q-1} \simeq \frac{1}{t\log 1/t\,(\log(\log 1/t+2))^{\alpha p}}.$$

We obtain suitable counterexamples by choosing α satisfying $1/p<\alpha<1/q$ for the case $p>q$ and $1/q<\alpha\leqq 1/p$ for the case $p<q$.

We shall now consider a function f on $[0, 1]$. Let c_n, $n\in Z$, be the complex Fourier coefficients of f (with respect to a uniformly bounded system of orthonormal functions). The sequence $(c_n^*)_0^\infty$ is the sequence $(|c_n|)_{-\infty}^\infty$ rearranged in non-increasing order. Hausdorff—Young's inequality (see e.g. [14, vol. II, p. 101]) can be used to obtain the following implication:

$$\text{(2.9)} \qquad \text{if } f\in L^p,\ 1<p<2,\ p'=p/(p-1), \text{ then } \sum_{-\infty}^{\infty} |c_n|^{p'}<\infty.$$

By an estimate of Paley it is also well known that if $f\in L^p$, $1<p<2$, then $\sum_1^\infty (c_n^*)^p n^{p-2}<$ $<\infty$ (see e.g. [14, vol II, p. 123]).

Therefore we can use Theorem 2.1 b) and make some straightforward calculations to obtain the following more precise implication than that in (2.9).

Corollary 2.3. *If $f\in L^p$, $1<p<2$, $p'=p/(p-1)$, then*

$$\sum_{-\infty}^{\infty} |c_n|^{p'} \left(h\left(\log^+ \frac{1}{|c_n|}\right)\right)^{(2-p)/(p-1)} <\infty$$

for some function h, $h\geqq 1$, satisfying (2.2) and

$$\text{(2.10)} \qquad \int_1^\infty \frac{1}{h(x)}\,dx<\infty.$$

Remark. The result in Corollary 2.3 cannot be improved. In fact, by using the results obtained in [12, p. 268] we find that the implication in Corollary 2.3 can be replaced by an equivalence in a relatively large class of functions. This class consists at least of all non-negative functions f satisfying the condition that

$$\int_0^t f^*(u)\,du \leqq K \int_0^t f(x)\,dx$$

for some constant K. Of course it is impossible to replace the implication in (2.9) by an equivalence in some similar relatively large class of functions.

In Corollary 2.3 we have seen that the condition $f\in L^p$ is an unnecessarily restricted condition to ensure the convergence of the series $\sum_{-\infty}^{\infty} |c_n|^{p'}$. However, it is well known that also the condition $\int_0^1 (f^*)^{p'} t^{p'-2}\,dt$ (that is $f\in L(p, p')$) implies that $\sum_{-\infty}^{\infty} |c_n|^{p'}<\infty$ (see [14, vol II, p. 124]). Therefore we can use Theorem 2.1 a) and obtain the following more precise criterion.

Corollary 2.4. *Let* $1<p<2$ *and* $p'=p/(p-1)$. *If*

$$\int_0^1 |f|^p (h(\log^+ |f|))^{p-2}\, dx < \infty$$

for every function $h, h \geqq 1$, *satisfying* (2.2) *and* (2.10), *then*

$$\sum_{-\infty}^{\infty} |c_n|^{p'} < \infty.$$

Remark. We can use the estimates obtained in [12, p. 268] to see that the implication in Corollary 2.4 can be replaced by an equivalence in the same class of functions as that in the remark after Corollary 2.3.

Finally we note that we can use Theorem 2.1 and similar arguments as before to obtain the corresponding results for a function $f \in R^n$ and its Fourier transform $\hat{f} \in R^n$. For example the corollary corresponding to Corollary 2.3 can be formulated in the following way.

Corollary 2.5. *If* $f \in L^p(R^n)$, $1<p<2$, $p'=p/(p-1)$, *then*

$$\int_{R^n} |\hat{f}|^{p'} (h(|\log |\hat{f}||))^{(2-p)/(p-1)}\, d\xi < \infty$$

for some function h *satisfying* (2.2) *and* (2.10).

Remark. It may be tempting to try to find some function h_0, *not* depending on f, such that

$$\|f\|_p \leqq 1 \Rightarrow \int_{R^n} |\hat{f}|^{p'} h_0(|\log |\hat{f}||)\, d\xi \leqq K_0 < \infty. \tag{2.11}$$

However, this is not possible for any positive function h_0 such that $h_0(x) \to \infty$ as $x \to \infty$. This fact follows when using the following homogeneity argument: Let f be a function on R^n such that $\hat{f}(\xi) \geqq a_0 > 0$ on a set E of positive measure. If $f_a(x) = a^{1/p} f(ax_1, x_2, \dots, x_n)$, then

$$\|f_a(x)\|_p = \|f\|_p \leqq 1, \quad \hat{f}_a(\xi) = a^{1/p-1} \hat{f}\left(\frac{\xi_1}{a}, \xi_2, \dots, \xi_n\right)$$

and

$$I_a = \int_{R^n} |\hat{f}_a(\xi)|^{p'} h_0(|\log \hat{f}_a(\xi)|)\, d\xi = \int_{R^n} |\hat{f}(\eta)|^{p'} h_0(|\log (a^{-1/p'} \hat{f}(\eta))|)\, d\eta.$$

Since $h_0(x) \to \infty$ as $x \to \infty$ we can choose a small enough to obtain that

$$h_0(|\log (a^{-1/p'} \hat{f}(\eta))|) \geqq 2K_0/(m(E) a_0^{p'}) \quad \text{on} \quad E.$$

Therefore $I_a \geqq m(E) a_0^{p'} 2K_0/(m(E) a_0^{p'}) = 2K_0$. We conclude that (2.11) does not hold for any of the functions h_0 considered.

3. Some lemmas

Lemma 3.1. *Let* $\sum_1^\infty c_k$ *be a non-negative and divergent series. If* $S_k=\sum_1^k c_n$, *then the series* $\sum_1^\infty c_k/S_k$ *is divergent and, for every* $a>0$, *the series* $\sum_1^\infty c_k/S_k^{1+a}$ *is convergent.*

A proof of this lemma by Abel can be found for example in [4, p. 121]. We shall now state two useful regularization lemmas.

Lemma 3.2. *Let* $\sum_{k=0}^\infty a_k$ *be a positive and convergent series and let* $c>1$. *Then there exists a sequence* $(b_k)_{k=0}^\infty$ *such that, for* $k=0, 1, 2, \ldots$, *we have* $a_k\leqq b_k$, $c^{-1}\leqq \leqq b_{k+1}/b_k\leqq c$ *and*

$$\sum_{k=0}^\infty b_k \leqq \frac{c+1}{c-1}\sum_{k=0}^\infty a_k.$$

Lemma 3.3. *Let* δ *be a positive number and let* g *be a positive, integrable function on* $[1, \infty[$ *such that, for some* $b>0$, $g(x)x^b$ *is a decreasing or an increasing function of* x. *Then there exists a constant* K *(depending only on* b *and* δ*) and a function* $g_1(x)$, *such that* $g_1(x)\geqq g(x)$,

(3.1) $$g_1(x)x^{1+\delta} \quad \textit{is increasing},$$

(3.2) $$g_1(x)x^{1-\delta} \quad \textit{is decreasing},$$

and

$$\int_1^\infty g_1(x)\,dx \leqq K\int_1^\infty g(x)\,dx.$$

Somewhat less precise versions of Lemmas 3.2 and 3.3 have been proved in [11, pp. 292—294]. The proofs we shall give here are elementary and based on convolutions.

Proof of Lemma 3.2. We choose $b_k=\sum_{n=0}^\infty a_n c^{-|k-n|}$. Then

$$\sum_{k=0}^\infty b_k = \sum_{k=0}^\infty\sum_{n=0}^k a_n c^{-(k-n)} + \sum_{k=0}^\infty\sum_{n=k+1}^\infty a_n c^{(k-n)} =$$

$$= \sum_{n=0}^\infty a_n c^n \sum_{k=n}^\infty c^{-k} + \sum_{n=1}^\infty a_n c^{-n}\sum_{k=0}^{n-1} c^k \leqq \frac{c+1}{c-1}\sum_{n=0}^\infty a_n.$$

Moreover,

$$b_{k+1}=\sum_{n=0}^{\infty}a_n c^{-|k+1-n|}=\sum_{n=0}^{k}a_n c^{-(k+1-n)}+\sum_{n=k+1}^{\infty}a_n c^{k+1-n}=$$

$$=c^{-1}\sum_{n=0}^{k}a_n c^{-(k-n)}+c\sum_{n=k+1}^{\infty}a_n c^{(k-n)}.$$

Therefore, we find that $b_{k+1}\leqq cb_k$ and $b_{k+1}\geqq c^{-1}b_k$. Trivially $a_k\leqq b_k$. The proof is complete.

Proof of Lemma 3.3. Let $g(x)x^b$ be an increasing function of x. Then, for $2^k\leqq x\leqq 2^{k+1}$, $k=0,1,2,\ldots,$

$$2^{-b}g(2^k)\leqq g(x)\leqq 2^b g(2^{k+1}). \tag{3.3}$$

Therefore

$$\sum_0^{\infty}g(2^k)2^k\leqq 2^b\sum_0^{\infty}\int_{2^k}^{2^{k+1}}g(x)\,dx\leqq 2^b\int_1^{\infty}g(x)\,dx<\infty. \tag{3.4}$$

Now we can use Lemma 3.2 with $c=2^\delta$ to obtain real numbers d_k, $k=0,1,2,\ldots,$ such that $d_k\geqq g(2^k)$, $\sum_0^{\infty}d_k2^k<\infty$,

$$2^{-(1+\delta)}\leqq d_{k+1}/d_k\leqq 2^{-1+\delta}, \tag{3.5}$$

and

$$\sum_0^{\infty}d_k2^k\leqq\frac{2^\delta+1}{2^\delta-1}\sum_0^{\infty}g(2^k)2^k. \tag{3.6}$$

We define the function g_1 in the following way:

$$g_1(x)=g_1(2^{k+u})=2^b(d_k)^{1-u}(d_{k+1})^u,\quad k=0,1,2,\ldots,\ 0\leqq u\leqq 1.$$

Observe that, for $0\leqq u_1\leqq u_2\leqq 1$,

$$2^{-(\delta+1)(u_2-u_1)}\leqq\frac{g_1(2^{k+u_2})}{g_1(2^{k+u_1})}=\left(\frac{d_{k+1}}{d_k}\right)^{u_2-u_1}\leqq 2^{(\delta-1)(u_2-u_1)} \tag{3.7}$$

and, for $k_2>k_1$,

$$2^{-(\delta+1)(k_2-k_1)}\leqq\frac{g_1(2^{k_2})}{g_1(2^{k_1})}=\frac{d_{k_2}}{d_{k_1}}\leqq 2^{(\delta-1)(k_2-k_1)}. \tag{3.8}$$

According to the estimates (3.7)—(3.8) we find that our function g_1 satisfies the growth conditions (3.1) and (3.2).

We may, without loss of generality, assume that $\delta<1$. Then, by (3.3), (3.5), and the fact that $d_{k+1}\geqq g(2^{k+1})$, we get

$$g_1(x)=g_1(2^{k+u})=2^b(d_k)^{1-u}(d_{k+1})^u\geqq 2^b2^{(1-\delta)(1-u)}d_{k+1}\geqq 2^bd_{k+1}\geqq 2^bg(2^{k+1})\geqq g(x).$$

Finally, by (3.4), (3.6), and (3.7), we have

$$\int_1^\infty g_1(x)\,dx = \sum_0^\infty \int_{2^k}^{2^{k+1}} g_1(x)\,dx \leqq \sum_0^\infty g_1(2^k)2^k =$$

$$= 2^b \sum_0^\infty d_k 2^k \leqq 2^b \frac{2^\delta+1}{2^\delta-1} \sum_0^\infty g(2^k)2^k \leqq 2^{2b} \frac{2^\delta+1}{2^\delta-1} \int_1^\infty g(x)\,dx.$$

The case when $g(x)x^b$ is a decreasing function of x can be carried out analogously. The proof is complete.

4. Proof of Theorem 2.1; the case $p>q$

In this case part a) of Theorem 2.1 is identical with Theorem A so it is sufficient to prove part b) of the theorem.

First we assume that

$$\int_1^\infty (f^*)^q t^{q/p-1}\,dt < \infty,$$

and choose ε satisfying $0<\varepsilon<q/p$. We can now use Lemma 3.3 to find a function $g(t)$, such that $g(t)\geqq f^*(t)$,

$$(g(t))^q t^{q/p+\varepsilon} \text{ is increasing,} \tag{4.1}$$

$$(g(t))^q t^{q/p-\varepsilon} \text{ is decreasing,} \tag{4.2}$$

and

$$\int_1^\infty (g(t))^q t^{q/p-1}\,dt < \infty. \tag{4.3}$$

For $k=0, 1, 2, \ldots$ we set $b_k=(g(2^k)2^{k/p})^q$ and observe that, by (4.1)—(4.3), the series $\sum_0^\infty b_k$ converges. We also note that we may, without loss of generality, assume that $g(t)\leqq 1$.

We define the function h at the points $x_k=\log\left(1/g(2^k)\right)$ by $h(x_k)=b_k^{(q-p)/pq}$, $k=0, 1, 2, \ldots$. According to (4.1)—(4.2) we find, for $0\leqq u\leqq 1$ and $k=0, 1, 2, \ldots$,

$$g^q(2^k)2^{-u(q/p+\varepsilon)} \leqq g^q(2^{k+u}) \leqq g^q(2^k)2^{u(\varepsilon-q/p)}. \tag{4.4}$$

We can now use (4.4) and make some elementary calculations to obtain the following useful estimates:

$$2^{-\varepsilon} \leqq \frac{b_{k+1}}{b_k} = \left(\frac{g(2^{k+1})2^{(k+1)/p}}{g(2^k)2^{k/p}}\right)^q \leqq 2^{\varepsilon}, \tag{4.5}$$

$$2^{-\varepsilon(p-q)/pq} \leqq \frac{h(x_{k+1})}{h(x_k)} \leqq 2^{\varepsilon(p-q)/pq}, \tag{4.6}$$

and

$$0 < \frac{1}{q}\left(\frac{q}{p}-\varepsilon\right) \leqq x_{k+1}-x_k \leqq \frac{1}{q}\left(\frac{q}{p}+\varepsilon\right). \tag{4.7}$$

We extend the definition of the function h by setting

$$h(x) = \left((h(x_k))^{x-x_k}(h(x_{k+1}))^{x_{k+1}-x}\right)^{1/(x_{k+1}-x_k)}$$

for $x_k \leqq x \leqq x_{k+1}$, $k=0, 1, 2, \dots$. We can make some elementary (but rather laborious) calculations and find, for some $\delta > 0$, that

$$h(x)2^{\delta x} \text{ is increasing} \tag{4.8}$$

and

$$h(x)2^{-\delta x} \text{ is decreasing.} \tag{4.9}$$

(We can for example choose $\delta = \varepsilon(p-q)/(q-p\varepsilon)$.)

According to (4.5)—(4.9) we obtain, for $x_k \leqq x \leqq x_{k+1}$, $k=0, 1, 2, \dots$, and for some $\delta_0 > 0$,

$$2^{-\delta_0} h(x_k) \leqq h(x) \leqq 2^{\delta_0} h(x_k).$$

(If we choose $\delta = \varepsilon(p-q)/(q-p\varepsilon)$, then we can have $\delta_0 = \varepsilon(p-q)/pq$.) Therefore, by (4.7), we have

$$\int_{x_0}^{\infty} (h(x))^{pq/(q-p)}\,dx \leqq \sum_0^{\infty} \int_{x_k}^{x_{k+1}} (h(x))^{pq/(q-p)}\,dx \leqq$$

$$\leqq 2^{\delta_0 pq/(q-p)} \sum_0^{\infty} (h(x_k))^{pq/(q-p)}(x_{k+1}-x_k) \leqq 2^{\delta_0 pq/(q-p)} \sum_0^{\infty} b_k(x_{k+1}-x_k) \leqq \tag{4.10}$$

$$\leqq 2^{\delta_0 pq/(q-p)} \frac{1}{q}\left(\frac{q}{p}+\varepsilon\right) \sum_0^{\infty} b_k < \infty.$$

We use (4.4) once more and obtain, for $2^k \leqq t \leqq 2^{k+1}$, $k=0, 1, 2, \dots,$

$$g(2^k)2^{-(q+p\varepsilon)/pq} \leqq g(t) \leqq g(2^k). \tag{4.11}$$

Hence we can use (4.8)—(4.9) to obtain that, for $2^k \leqq t \leqq 2^{k+1}$,

$$\begin{aligned} h\left(\log\frac{1}{g(t)}\right) &\leqq h\left(\log\left(\frac{1}{g(2^k)}+\frac{q+p\varepsilon}{pq}\right)\right) 2^{\delta(\log(g(t)/g(2^k))+(q+p\varepsilon)/pq)} \leqq \\ &\leqq h\left(\log\frac{1}{g(2^k)}\right) 2^{2\delta(q+p\varepsilon)/pq}. \end{aligned} \tag{4.12}$$

Furthermore, according to (4.11)—(4.12),

$$\int_1^\infty \left(g(t)\,h\left(\log\frac{1}{g(t)}\right)\right)^p dt = \sum_0^\infty \int_{2^k}^{2^{k+1}} \left(g(t)\,h\left(\log\frac{1}{g(t)}\right)\right)^p dt \leqq$$

(4.13)

$$\leqq K_0 \sum_0^\infty (g(2^k)h(x_k))^p 2^k = K_0 \sum_0^\infty b_k^{p/q} 2^{-k} b_k^{1-p/q} 2^k = K_0 \sum_0^\infty b_k < \infty.$$

(We can for example choose $K_0 = 2^{2\delta(q+p\varepsilon)/q}$.)

By choosing ε small enough and using the growth condition (4.8) we see that $yh(\log(1/y))$ is an increasing function of y, $0<y<1$. Therefore, by (4.13) and the fact that $f^*(t) \leqq g(t)$, we have

$$\int_1^\infty \left(f^* h\left(\log^+\frac{1}{f^*}\right)\right)^p dt < \infty.$$

Since the function h satisfies (4.8)—(4.10) we conclude that $f \in E_\infty(p, q)$.

In order to prove the converse implication we assume that $f \in E_\infty(p, q)$. Let $(\alpha_k)_0^\infty$ be the nondecreasing sequence of the least real numbers α_k such that $2^{-k-1} \leqq f^*(t) \leqq 2^{-k}$, when $\alpha_{k-1} \leqq t < \alpha_k$, $k=0, 1, 2, \dots$. Let $h(x)$ be the function associated with the definition of $E_\infty(p, q)$. We assume that $h(x)2^{\delta x}$, for some $\delta>0$, is an increasing function of x. Therefore, if $\alpha_{k-1} \leqq t \leqq \alpha_k$, then

$$h(k)2^{-\delta} \leqq h\left(\log\frac{1}{f^*(t)}\right) \leqq h(k+1)\,2^\delta.$$

Thus the assumption

$$\int_1^\infty \left(f^* h\left(\log^+\frac{1}{f^*}\right)\right)^p dt < \infty$$

implies that

$$\sum_{k=0}^\infty 2^{-pk}(h(k))^p(\alpha_k - \alpha_{k-1}) < \infty. \tag{4.14}$$

Moreover,

$$\int_{\alpha_0}^\infty (f^*)^q t^{q/p-1}\,dt = \sum_1^\infty \int_{\alpha_{k-1}}^{\alpha_k} (f^*)^q t^{q/p-1}\,dt \leqq \frac{p}{q} \sum_1^\infty 2^{-qk}(\alpha_k^{q/p} - \alpha_{k-1}^{q/p}). \tag{4.15}$$

We use Hölder's inequality and an elementary estimate and obtain

$$\sum_1^\infty 2^{-qk}(\alpha_k^{q/p} - \alpha_{k-1}^{q/p}) \leqq \sum_1^\infty 2^{-qk}(\alpha_k - \alpha_{k-1})^{q/p} \leqq$$

(4.16)

$$\leqq \left(\sum_1^\infty 2^{-pk}(h(k))^p(\alpha_k - \alpha_{k-1})\right)^{q/p} \left(\sum_1^\infty (h(k))^{pq/(q-p)}\right)^{1-q/p}.$$

From the growth and integrability properties of h we deduce that the series $\sum_1^\infty (h(k))^{pq/(q-p)}$ converges. Hence, by (4.14)—(4.16), we obtain

$$\int_1^\infty (f^*)^q t^{q/p-1}\,dt < \infty.$$

The case when $h(x)2^{-\delta x}$ is a decreasing function of x can be handled analogously. The proof is complete.

5. Proof of Theorem 2.1; the case $p<q$

We assume

$$\int_0^1 (f^*)^q t^{q/p-1}\,dt < \infty.$$

Let h be any function on $[0, \infty[$ such that for some $\delta, 0<\delta<p$,

$$h(x)2^{\delta x} \text{ is increasing,} \tag{5.1}$$

$$h(x)2^{-\delta x} \text{ is decreasing} \tag{5.2}$$

and

$$\int_1^\infty (h(x))^{pq/(q-p)}\,dx < \infty. \tag{5.3}$$

Let $(\beta_k)_0^\infty$ be the nonincreasing sequence of the least real numbers β_k, such that $2^{k-1} \leqq f^*(t) \leqq 2^k$, when $\beta_k \leqq t < \beta_{k-1}$, $k=0, 1, 2, \ldots$. Then

$$\int_0^{\beta_0} (f^*)^q t^{q/p-1}\,dt = \sum_1^\infty \int_{\beta_k}^{\beta_{k-1}} (f^*)^q t^{q/p-1}\,dt \geqq \frac{p}{q} 2^{-q} \sum_1^\infty 2^{qk}(\beta_{k-1}^{q/p} - \beta_k^{q/p}). \tag{5.4}$$

Moreover, by (5.1),

$$\begin{aligned}\int_0^{\beta_0} (f^* h(\log^+ f^*))^p\,dt &= \sum_1^\infty \int_{\beta_k}^{\beta_{k-1}} (f^* h(\log^+ f^*))^p\,dt \leqq \\ &\leqq 2^{\delta p} \sum_1^\infty 2^{pk}(h(k))^p(\beta_{k-1} - \beta_k).\end{aligned} \tag{5.5}$$

We use Hölder's inequality and find

$$\sum_1^\infty 2^{pk}(h(k))^p(\beta_{k-1} - \beta_k) \leqq \left(\sum_1^\infty 2^{qk}(\beta_{k-1} - \beta_k)^{q/p}\right)^{p/q} \left(\sum_1^\infty (h(k))^{pq/(q-p)}\right)^{1-p/q}. \tag{5.6}$$

Since $(\beta_{k-1}-\beta_k)^{q/p} \leqq \beta_{k-1}^{q/p}-\beta_k^{q/p}$ we can use (5.4) and the integrability assumption on f^* to obtain

$$\sum_1^\infty 2^{qk}(\beta_{k-1}-\beta_k)^{q/p} < \infty. \tag{5.7}$$

The conditions (5.1)—(5.3) imply that the series $\sum_1^\infty (h(k))^{pq/(q-p)}$ converges. Therefore, according to (5.6)—(5.7), $\sum_1^\infty 2^{pk}(h(k))^p(\beta_{k-1}-\beta_k) < \infty$. In view of (5.5) we conclude that

$$\int_0^1 (f^* h(\log^+ f^*))^p \, dt < \infty$$

for every function h satisfying (5.1)—(5.3).

Finally we suppose that the conditions (5.1) and (5.2) on the function h, are replaced by the general condition that, for some $a>0, h(x)a^x$ is increasing or decreasing. Then we can use Lemma 3.3 to obtain a function $h_1 \geqq h$ satisfying (5.1)—(5.3). We have just proved that

$$\int_0^1 (f^* h_1(\log^+ f^*))^p \, dt < \infty$$

and, thus, since $h_1 \geqq h$,

$$\int_0^1 (f^* h(\log^+ f^*))^p \, dt < \infty$$

so that $f \in E_0(p, q)$.

In order to prove the converse implication we assume that $f \in E_0(p, q)$. Let h be an arbitrary function satisfying (5.1)—(5.3). Then

$$\begin{aligned} \int_0^{\beta_0} (f^* h(\log^+ f^*))^p \, dt &= \sum_1^\infty \int_{\beta_k}^{\beta_{k-1}} (f^* h(\log^+ f^*))^p \, dt \geqq \\ &\geqq 2^{-p(1+\delta)} \sum_1^\infty 2^{pk}(h(k))^p (\beta_{k-1}-\beta_k). \end{aligned} \tag{5.8}$$

Hence, by assumption and (5.8), the series $\sum_1^\infty 2^{pk}(h(k))^p(\beta_{k-1}-\beta_k)$ converges. We make an Abelian transformation on this series and find

$$\sum_1^\infty 2^{pk}(h(k))^p \beta_k < \infty. \tag{5.9}$$

Since

$$\int_0^{\beta_0} (f^*)^q t^{q/p-1} \, dt = \sum_1^\infty \int_{\beta_k}^{\beta_{k-1}} (f^*)^q t^{q/p-1} \, dt \leqq \frac{p}{q} \sum_1^\infty 2^{qk} \beta_{k-1}^{q/p},$$

it is sufficient if we can prove that $\sum_1^\infty 2^{qk}\beta_k^{q/p}<\infty$. We assume the contrary, viz. $\sum_1^\infty 2^{qk}\beta_k^{q/p}=\infty$. For $k=1, 2, 3, \dots$ we set $c_k=2^{qk}\beta_k^{q/p}$ and $d_k=2^{k(q-p)}\beta_k^{q/p-1}$. By assumption the series $\sum_1^\infty c_k$ diverges so we can use Lemma 3.1 and obtain

$$\sum_1^\infty 2^{pk}\beta_k\frac{d_k}{S_k}=\sum_1^\infty\frac{c_k}{S_k}=\infty \tag{5.10}$$

and, for $a=p/(q-p)$,

$$\sum_1^\infty\left(\frac{d_k}{S_k}\right)^{q/(q-p)}=\sum_1^\infty\frac{c_k}{S_k^{1+a}}<\infty.$$

We choose δ, $0<\delta<p$, and set $a_k=d_k/S_k$. We apply Lemma 3.2 to the series $\sum_1^\infty a_k^{q/(q-p)}$ to obtain a sequence $(b_k)_1^\infty$ such that $b_k\geqq a_k$,

$$(b_k 2^{\delta pk})_1^\infty \text{ is an increasing sequence,} \tag{5.11}$$

$$(b_k 2^{-\delta pk})_1^\infty \text{ is a decreasing sequence,} \tag{5.12}$$

$$\sum_1^\infty b_k^{q/(q-p)}<\infty, \tag{5.13}$$

and, by (5.10),

$$\sum_1^\infty 2^{pk}\beta_k b_k=\infty. \tag{5.14}$$

For $k=1, 2, 3, \dots$ and $0\leqq u\leqq 1$ we define $h(x)=h(k+u)=(b_k^{1-u}b_{k+1}^u)^{1/p}$. Then, by (5.11)—(5.14), we can see that there exists a function h satisfying (5.1)—(5.3) but

$$\sum_1^\infty 2^{pk}\beta_k(h(k))^p=\infty.$$

This fact contradicts the condition (5.9). We conclude that our assumption is false so that

$$\int_0^1 (f^*)^q t^{q/p-1}\,dt<\infty.$$

The proof of part a) of the theorem is complete.

In order to prove part b) we study the nondecreasing sequence $(\alpha_k)_0^\infty$ of the least real numbers α_k such that $2^{-k-1}\leqq f^*(t)\leqq 2^{-k}$, when $\alpha_{k-1}\leqq t<\alpha_k$, $k=0, 1, 2, \dots$. The proof of part b) can now be carried out by arguing exactly as in the proof of part a). Therefore we leave out the details.

6. A description of the spaces $L^{p,q}(\log L)^\alpha$

Theorem 2.1 can be generalized in the following way.

Theorem 6.1. *Let* $0<p<\infty$, $0<q<\infty$ *and* $-\infty<\alpha<\infty$.

a) *Let* $p>q$. *Then*

$$\int_0^1 (f^* t^{1/p}(|\log t|+1)^\alpha)^q \, dt/t < \infty \tag{6.1}$$

if and only if

$$\int_0^1 (f^*(\log^+ f^*+1)^\alpha h(\log^+ f^*))^p \, dt < \infty \tag{6.2}$$

for some function h, *such that, for some real number* a,

(6.3) $h(x)a^x$ *is a decreasing or an increasing function of* x

and

$$\int_1^\infty (h(x))^{pq/(q-p)} \, dx < \infty. \tag{6.4}$$

b) *Let* $p<q$. *Then* (6.1) *holds if and only if* (6.2) *holds for every function* h *satisfying* (6.3) *and* (6.4).

c) *Let* $p>q$. *Then*

$$\int_1^\infty (f^* t^{1/p}(|\log t|+1)^\alpha)^q \, dt/t < \infty \tag{6.5}$$

if and only if

$$\int_1^\infty \left[f^*\left(\log^+\frac{1}{f^*}+1\right)^\alpha h\left(\log^+\frac{1}{f^*}\right)\right]^p dt < \infty \tag{6.6}$$

for some function h *satisfying* (6.3) *and* (6.4).

d) *Let* $p<q$. *Then* (6.5) *holds if and only if* (6.6) *holds for every function* h *satisfying* (6.3) *and* (6.4).

The proof of Theorem 6.1 can be carried out in a similar way as the proof of Theorem 2.1 so we omit the details. Moreover, we can use Theorem 6.1 and argue in a similar way as before to obtain the following exact characterization of the Lorentz—Zygmund spaces.

Theorem 6.2. *Let* $0<p<\infty$, $0<q<\infty$ *and* $-\infty<\alpha<\infty$.

a) *Let* $p>q$. *Then* $f\in L^{p,q}(\log L)^\alpha$ *if and only if* $f\in L^p h(\log L)$ *for some function* h *satisfying* (6.3) *and*

$$\int_1^\infty (h(x)x^{-\alpha})^{pq/(q-p)} \, dx < \infty. \tag{6.7}$$

b) *Let* $p<q$. *Then* $f\in L^{p,q}(\log L)^\alpha$ *if and only if* $f\in L^p h(\log L)$ *for every function* h *satisfying* (6.3) *and* (6.7).

The following recent embedding result by BENNETT and RUDNICK [2, p. 31] can be deduced from Theorem 6.2.

Corollary 6.3. *Let* $0<p<\infty$, $0<q<\infty$, $0<q_1<\infty$, $-\infty<\alpha<\infty$ *and* $-\infty<\alpha_1<\infty$. *Then*

$$L^{p,q}(\log L)^\alpha \subseteqq L^{p,q_1}(\log L)^{\alpha_1} \tag{6.8}$$

whenever either

$$q > q_1 \quad \text{and} \quad \alpha+1/q > \alpha_1+1/q_1 \tag{6.9}$$

or

$$q \leqq q_1 \quad \text{and} \quad \alpha \geqq \alpha_1. \tag{6.10}$$

Remark. It is easy to find elementary examples showing that the inclusion (6.8) does not hold in general if we permit some α satisfying $\alpha\leqq\alpha_1+1/q_1-1/q$ when $q>q_1$ or some α satisfying $\alpha<\alpha_1$ when $q\leqq q_1$ (see [2, p. 33]).

In our introduction we have noted that $L(p,q)\subset L^p$ when $p>q$ and $L(p,q)\supset L^p$ when $p<q$. Therefore, by applying Corollary 6.3 with $q=p$, $\alpha=0$ and $q_1=p$, $\alpha_1=0$ and by using the inclusions (2.7) and (2.8), we obtain the following chains of inclusions: If $0<q<p<\infty$, then, for every $\varepsilon>0$,

$$L^p(\log L)^{1/q-1/p+\varepsilon} \subset L(p,q) \subset L^p \subset L^{p,q}(\log L)^{1/p-1/q-\varepsilon}$$

and if $0<p<q<\infty$, then, for every $\varepsilon>0$,

$$L^{p,q}(\log L)^{1/p-1/q+\varepsilon} \subset L^p \subset L(p,q) \subset L^p(\log L)^{1/q-1/p-\varepsilon}.$$

All inclusions are the sharpest possible in the sense that we can nowhere permit that $\varepsilon=0$.

Proof of the corollary. We assume that $f\in L^{p,q}(\log L)^\alpha$ and $q>q_1$. First we consider the case $p>q$. Then, by Theorem 6.2 a), $f\in L^p h(\log L)$ for some function h satisfying (6.3) and

$$\int_1^\infty (h(x)x^{-\alpha})^{pq/(q-p)}\,dx = \int_1^\infty \left(\frac{x^\alpha}{h(x)}\right)^{pq/(p-q)} dx < \infty. \tag{6.11}$$

We put $a=q(p-q_1)/q_1(p-q)$ and use Hölder's inequality to obtain

$$\int_1^\infty \left(\frac{x^{\alpha_1}}{h(x)}\right)^{pq_1/(p-q_1)} dx \leqq \left(\int_1^\infty \left(\frac{x^\alpha}{h(x)}\right)^{pq/(p-q)} dx\right)^{1/a} \left(\int_1^\infty x^{(\alpha_1-\alpha)qq_1/(q-q_1)}\,dx\right)^{1-1/a}.$$

The assumption $\alpha+1/q>\alpha_1+1/q_1$ implies that $(\alpha_1-\alpha)qq_1/(q-q_1)<-1$. Therefore, according to (6.11),

$$\int_1^\infty \left(h(x)x^{-\alpha_1}\right)^{pq_1/(q_1-p)}\,dx = \int_1^\infty \left(\frac{x^{\alpha_1}}{h(x)}\right)^{pq_1/(p-q_1)} dx < \infty. \tag{6.12}$$

We have just proved that $f\in L^p h(\log L)$ for some function h satisfying (6.3) and (6.12). Thus, by Theorem 6.2 a), $f\in L^{p,q_1}(\log L)^{\alpha_1}$.

For the case $p<q_1$ we assume that h is an arbitrary function satisfying (6.3) and (6.12). We put $a=q_1(p-q)/q(p-q_1)$ and use Hölder's inequality and the assumption that $(\alpha_1-\alpha)qq_1/(q-q_1)<-1$ to see that h also satisfies the condition (6.11). Therefore, according to Theorem 6.2 b), $f\in L^p h(\log L)$. By using Theorem 6.2 b) once more we conclude that $f\in L^{p,q_1}(\log L)^{\alpha_1}$.

For the case $p=q$ our assumption means that $f\in L^p h(\log L)$ for $h(x)=x^\alpha$. We note that the function h satisfies (6.3) and (6.12). We use Theorem 6.2 a) and conclude that $f\in L^{p,q_1}(\log L)^{\alpha_1}$.

When $p=q_1$ we can use Theorem 6.2 b) to see that $f\in L^p h(\log L)$ for every function h satisfying (6.3) and (6.11). We note that the function $h(x)=x^{\alpha_1}$ satisfies these conditions. Thus, $f\in L^p(\log L)^{\alpha_1}$ which in this case is equivalent to that $f\in L^{p,q_1}(\log L)^{\alpha_1}$.

Finally we suppose that $q_1<p<q$. Then we can use Theorem 6.2 b) to see that $f\in L^p h(\log L)$ for every function h satisfying the conditions (6.3) and (6.11). In particular, the assumption $(\alpha_1-\alpha)qq_1/(q-q_1)<-1$ implies that the function

$$h(x) = x^{((\alpha_1-\alpha)qq_1/p+(\alpha q-\alpha_1 q_1))/(q-q_1)}$$

satisfies these conditions. But this function $h(x)$ satisfies also the condition (6.12) so we can use Theorem 6.2 a) to conclude that $f\in L^{p,q_1}(\log L)^{\alpha_1}$. Thus the proof of the case $q_1<q$ is complete.

If $q_1\geqq q$ we may, without loss of generality, assume that $\alpha_1=\alpha$. The proof of this case is analogous and even simpler so we leave out the details.

7. Some concluding remarks

Professor Jaak Peetre has made me aware of the fact that our description of the $L(p,q)$-spaces is similar to the definition of the spaces $B^p_{\theta,q}(\omega)$, defined by Peetre [10] and Gilbert [3, pp. 242—243] in the following way: Let ω be a nonnegative weight function, $0<\theta<1$, $1\leqq p<\infty$, $1\leqq q\leqq\infty$ and $\gamma=1/p-1/q$. Let Φ_G be the set of nonnegative functions φ on $[0,\infty[$, such that

$$\|\varphi\|_{L^*} = \int_0^\infty \varphi(t)\frac{dt}{t} = 1, \tag{7.1}$$

and

(7.2) $t^{\theta}\varphi^{\gamma}(t)$ is nondecreasing.

Then

$$B^p_{\theta,q}(\omega) = \begin{cases} \bigcup_{\varphi\in\Phi_G} \{L^p_\sigma | \sigma = \omega^\theta \varphi^\gamma(\omega)\}, & \text{when } \gamma \leqq 0, \\ \bigcap_{\varphi\in\Phi_G} \{L^p_\sigma | \sigma = \omega^\theta \varphi^\gamma(\omega)\}, & \text{when } \gamma \geqq 0. \end{cases}$$

In particular, when the underlying measure space is (R^n, dx) we obtain the usual Beurling—Herz spaces

$${}^pL^q = \begin{cases} B^p_{-\gamma,q}(|x|^n), & \text{when } q < p, \\ B^p_{\gamma,q}\left(\frac{1}{|x|^n}\right), & \text{when } q > p. \end{cases}$$

The Beurling spaces A^p and B^p are the special cases ${}^pL^1$ and ${}^pL^\infty$, respectively (see [3, p. 247] and [5, pp. 298—300]).

We can use our Theorem 2.2 and make some elementary calculations to see that the $L(p, q)$-spaces can be characterized in similar terms. More exactly, we can in fact *define* the $L(p, q)$-spaces in the following way: Let $0<p<\infty$, $0<q<\infty$ and $\gamma=1/p-1/q$. Let Φ_P be the set of nonnegative functions φ on $[0, \infty[$, satisfying (7.1) and, for some real number a,

(7.2)′ $t^a\varphi(t)$ is nondecreasing (or nonincreasing).

Then

$$L(p, q) = \begin{cases} \bigcup_{\varphi\in\Phi_P} \{L^p(\varphi(L))^\gamma\}, & \text{when } \gamma \leqq 0, \\ \bigcap_{\varphi\in\Phi_P} \{L^p(\varphi(L))^\gamma\}, & \text{when } \gamma \geqq 0. \end{cases}$$

It is also interesting to compare how the spaces $L(p, q)$ (or, equivalently, $E(p, q)$) and $B^p_{\theta,q}(\omega)$ (and, thus, the Beurling—Herz spaces ${}^pL^q$) occur as intermediate spaces in analogous situations in the theory of interpolation. For example we have

$$(L^{p_0}, L^{p_1})_{\theta,q;K} = L(p, q) \; (=E(p, q))$$

when $1/p=(1-\theta)/p_0+\theta/p_1$ (see e.g. [13, p. 134]) and

$$(L^p, L^p_\omega)_{\theta,q;K} = B^p_{\theta,q}(\omega)$$

(see [3, p. 243] and [10, pp. 64—66]).

Lorentz has in [7] defined that a function f belongs to the space $\Lambda(\varphi, q)$ if

$$\int_0^\infty (f^*)^q \varphi \, dt < \infty.$$

Here φ is a nonnegative and integrable function on $[0, \infty[$. Lorentz has also given an exact characterization of the spaces $\Lambda(\varphi, 1)$ which are also Orlicz spaces (see [8, pp. 130—132]. Roughly speaking, the result of Lorentz shows that this can happen if and only if we impose integrability conditions on φ such that the space $\Lambda(\varphi, 1)$ is fairly close to L^1.

In this context we also note that it is feasible to generalize Theorem 6.1 for example by replacing the factor $(\log —)^a$ in the conditions (6.1)—(6.2) and (6.5)—(6.6) by any "logarithmic varying" function φ. (We say that a function φ is logarithmic varying if there exist x_0 and a such that, for $x \geqq x_0$, $\varphi(x)(\log x)^a$ is a decreasing or an increasing function of x.) We can still use essentially the same techniques as in the proofs in Sections 4 and 5.

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