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Some weak-star ergodic theorems

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Dedicated to Professor Béla Szőkefalvi-Nagy on his 70th birthday

0. Introduction. Let M be a von Neumann algebra and let G be a group of $*$-automorphisms of M. It is proved in [3] that if the family of G-invariant normal states is faithful on M (i.e., M is G-finite), then for every $t \in M$, the w^*-closed convex hull of $\{gt : g \in G\}$ contains exactly one G-invariant element. In the present paper we prove the converse of this theorem in the case where M is σ-finite and G is abelian. We present our results in the more general setting of arbitrary Banach spaces.

1. Results. Throughout this paper B denotes a Banach space and B^* its dual space. We denote by $L_{w*}(B^*)$ the space of w^*-continuous linear operators in B^*, equipped with the topology of pointwise w^*-convergence. Every element g of $L_{w*}(B^*)$ is a bounded operator in B^* such that there exists a unique bounded linear operator g_* in B for which $(g_*)^* = g$. Throughout this paper G will denote a bounded commutative semigroup $G \subset L_{w*}(B^*)$. We shall study the implications of the following condition:

(U) For every $t \in B^*$, the w^*-closed convex hull of the orbit $\{gt : g \in G\}$ contains a unique G-invariant element, which will be denoted by t^G.

(The fact that this closed convex hull contains at least one G-invariant element follows from the Kakutani—Markov fixed point theorem (cf. [2], V. 10. 6), in view of the w^*-compactness of the unit ball of B^*.)

Theorem 1. *Suppose that condition* (U) *is satisfied and either B is a separable Banach space or G is a separable topological subspace of $L_{w*}(B^*)$. Then the mapping $t \to t^G$ $(t \in B^*)$ is a bounded linear projection P acting in B^*. We have $gP = Pg = P$ and P is the limit, in $L_{w*}(B^*)$, of a sequence of elements of the convex hull of G.*

Theorem 2. *Suppose that either B or G is separable. If condition* (U) *is satisfied and B is weakly complete, then the mapping $t \to t^G$ $(t \in B^*)$ is a w^*-continuous*

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linear projection P such that $gP=Pg=P$. The operator P belongs to the sequential closure, in $L_{w*}(B^*)$, of the convex hull co$G$ *of G. Moreover, for every $v_0 \in \mathrm{co}G$ and every w^*-neighborhood N of zero there exists $v_1 \in \mathrm{co}G$ such that $vv_1v_0t-t^G \in N$ for every $v \in \mathrm{co}G$ and $t \in B^*$ such that $\|t\| \leqq 1$.*

Proposition 1. *The hypotheses of Theorem 2 are satisfied if:*

(a) *$B=L^1(X, S, m)$, where (X, S, m) is a positive localizable measure space (then $B^*=L^\infty(X, S, m)$);*

(b) *G is a bounded commutative semigroup of w^*-continuous linear operators in $L^\infty(X, S, m)$, satisfying condition* (U);

(c) *Either $L^1(X, S, m)$ or G is separable.*

Proposition 2. *The assertions of Theorem 2 hold if:*

(a) *B^* is a W^*-algebra M;*

(b) *G is a bounded commutative semigroup of w^*-continuous linear mappings of M into itself, satisfying condition* (U).

(c) *Either M is σ-finite or G is separable.*

Corollary. *Let M be a von Neumann algebra and let G be a commutative group of $*$-automorphisms of M, satisfying condition* (U). *If M is σ-finite or G is separable, then M is G-finite* (for this notion, cf. [3]).

2. Proofs. For the proof of Theorem 1 we need the following two lemmas.

Lemma 1. *Let $G=\{g_1, g_2, \ldots\}$ be countable and let B be separable. Suppose that condition* (U) *is satisfied. Then for every $t \in B^*$, the sequence $\left\{\frac{1}{n^n} \sum_{i_1, \ldots, i_n=1}^{n} g_1^{i_1} \ldots g_n^{i_n} t\right\}_{n=1}^{\infty}$ w^*-converges to t^G.*

Lemma 2. *Let B_1 be a G_*-invariant closed subspace of B, i.e., let $g_*\varphi \in B_1$ for $g_* \in G_*$, $\varphi \in B_1$. Furthermore, let $B_1^\perp=\{t: (\varphi, t)=0$ for all $\varphi \in B_1\}$ and let the dual space B_1^* of B_1 be identified canonically with the quotient space $B^*/B_1^\perp$. If G acting on B^* satisfies condition* (U), *then G acting on B_1^* also satisfies condition* (U).

Proof of Lemma 1. Let $v_n=\frac{1}{n^n} \sum_{i_1, \ldots, i_n=1}^{n} g_1^{i_1} \ldots g_n^{i_n}$ and let $t \in B^*$. We have to prove that the sequence $\{v_n t\}$ w^*-converges to t^G. To prove this, we show that every subsequence $\{v_{n_k} t\}$ of $\{v_n t\}$ contains a subsequence $\{v_{n_{k_l}} t\}$ which w^*-converges to t^G. Since the sequence $\{v_n t\}$ is a bounded sequence in B^* and every closed ball in B^* is metrizable compact in the w^*-topology (cf. [2], V. 4.2, V. 5.1), this will imply that $v_n t \to t^G$ in the w^*-topology as $n \to \infty$. Let $\{v_{n_k} t\}$ be a subsequence of $\{v_n t\}$. Since $\{v_{n_k} t\}$ is a bounded sequence, it contains a w^*-convergent subsequence $\{v_{n_{k_l}} t\}$ (by the above remark). We have to prove that the limit of

$\{v_{n_{k_l}}t\}$ is t^G. Since the limit of $\{v_{n_{k_l}}t\}$ obviously belongs to the w^*-closed convex hull of $\{gt\colon g\in G\}$, we only have to prove that it is G-invariant. Pick a positive integer s. Let $n\geqq s$. Then g_s appears in v_n. By the commutativity of G we have:

$$\|g_s v_n t - v_n t\| =$$

$$= \left\| \frac{1}{n^n} \sum_{i_1, \dots, i_n = 1}^{n} g_g^{i_1} \dots g_{s-1}^{i_{s-1}} g_s^{i_s+1} g_{s+1}^{i_{s+1}} \dots g_n^{i_n} t - \frac{1}{n^n} \sum_{i_1, \dots, i_n = 1}^{n} g_1^{i_1} \dots g_n^{i_n} t \right\| =$$

$$= \left\| \frac{1}{n^n} \sum_{i_1, \dots, i_{s-1}, i_{s+1}, \dots, i_n = 1}^{n} (g_1^{i_1} \dots g_{s-1}^{i_{s-1}} g_s^{n+1} g_{s+1}^{i_{s+1}} \dots g_n^{i_n} t - g_1^{i_1} \dots g_{s-1}^{i_{s-1}} g_{s+1}^{i_{s+1}} \dots g_n^{i_n} t) \right\| \leqq$$

$$\leqq \frac{2n^{n-1}\|G\| \cdot \|t\|}{n^n} = \frac{2}{n} \|G\| \cdot \|t\|,$$

where $\|G\| = \sup \{\|g\|\colon g\in G\}$. If now $n = n_{k_l}$ and $l\to\infty$, then $n_{k_l}\to\infty$, and consequently, $\|g_s v_{n_{k_l}} t - v_{n_{k_l}} t\| \to 0$ by the above. Hence $g_s v_{n_{k_l}} t \to \lim_{l\to\infty} v_{n_{k_l}} t$. On the other hand, by the w^*-continuity of g_s we have $g_s v_{n_{k_l}} t \to g_s \lim_{l\to\infty} v_{n_{k_l}} t$. Consequently, $g_s \lim_{l\to\infty} v_{n_{k_l}} t = \lim_{l\to\infty} v_{n_{k_l}} t$. Since g_s was an arbitrary element of G, we have proved that $\lim_{l\to\infty} v_{n_{k_l}}$ is G-invariant, and consequently,

$$\lim_{l\to\infty} v_{n_{k_l}} t = t^G.$$

Proof of Lemma 2. Since $GB_1^\perp \subset B_1^\perp$, the semigroup G acts on $B_1^* = B^*/B_1^\perp$, and Lemma 2 makes sense. Let $f\in B_1^*$ and let f_0 be a G-invariant element of the w^*-closed convex hull of $\{gf\colon g\in G\}$. There exists a net v_n of elements of $\mathrm{co}G$ such that $v_n f\to f_0$ in the w^*-topology of B_1^*. The element $f\in B_1^*$ is canonically identified with a coset $t+B_1^\perp$ $(t\in B^*)$ and for every $g\in G$, the element gf is identified with $gt+B_1^\perp$. The convergence relation $vf\to f_0$ means that for every $\varphi\in B_1$, $(\varphi, v_n t)$ converges, the limit being (φ, f_0). For every $\varphi\in B_1$, $g\in G$ we have $(g_*\varphi, f_0) = (\varphi, g_*^* f_0) = (\varphi, gf_0) = (\varphi, f_0)$. Consequently, f_0 is a G_*-invariant bounded linear form on B_1.

Since closed balls are w^*-compact in B^*, there is a subnet v_1 of the net v_n for which $v_1 t$ converges in the w^*-topology of B^*. Let us denote the limit by t_0. The element $t_0\in B^*$ belongs to the w^*-closed convex hull of $\{gt\colon g\in G\}$ and

$$(*) \qquad (\varphi, t_0) = (\varphi, f_0) \quad \text{for} \quad \varphi\in B_1.$$

Since G acting on B^* satisfies condition (U); there is a net w_k in $\mathrm{co}G$ such that $w_k t_0 \to t^G$ in the w^*-topology of B^*. For $\varphi\in B_1$ we have: $(\varphi, t^G) = \lim_k (\varphi, w_k t_0) = \lim_k (w_{k*}\varphi, t_0) = \lim_k (w_{k*}\varphi, f_0) = \lim_k (\varphi, f_0) = (\varphi, f_0)$. (Here the next to the last equality holds because f_0 is G_*-invariant on B_1 and the equality before the next to the last equality holds because of ($*$) and the G_*-invariance of B_1.) Consequently, $(\varphi, t^G)=(\varphi, f_0)$ for $\varphi\in B_1$, i.e., f_0 is the restriction of t^G to B_1. Since f_0 was an arbitrary element in the w^*-closed convex hull of $\{gf: g\in G\}$, the lemma is proved.

Proof of Theorem 1. Throughout this proof we assume that condition (U) is satisfied for G acting on B^*.

(1) First we assume that B is separable. This implies the separability of G. Indeed, let $\{\varphi_n\}_{n=1}^{\infty}$ be a dense sequence in the unit ball of B. Let T be the set of B-valued sequences bounded by $\|G\|=\sup\{\|g\|: g\in G\}$. If $\alpha, \beta\in T$, we define $\varrho(\alpha, \beta)$ by the equality

$$\varrho(\alpha, \beta) = \sum_{n=1}^{\infty}\frac{1}{2^n}\frac{\|\alpha_n-\beta_n\|}{1+\|\alpha_n-\beta_n\|}.$$

Then ϱ is a metric on T. We have $\alpha^{(k)}\to\alpha$ in this metric if and only if $\alpha_n^{(k)}\to\alpha_n$ $(k\to\infty)$ for every $n=1, 2, \dots$. Since B is separable, so is T. Let $g\in G$ and let us define an element α^g of T by the equalities $\alpha_n^g=g_*\varphi_n$ $(n=1, 2, \dots)$. The mapping $g\to\alpha^g$ is a homeomorphism of G_* onto a subset of T if G_* is considered with the topology of pointwise strong convergence on B and T is considered with the topology induced by the metric ϱ. Since T has a countable dense subset, we may infer that so does G_* (because of the metrizability of T). Since taking adjoints of operators is a weak—weak* continuous operation, G contains a countable subset G_0 which is dense in G in the topology of $L_{w*}(B^*)$.

Now let G_0 be a countable dense subset of G in the topology of $L_{w*}(B^*)$. Then the G_0-invariant elements of B^* are the same as the G-invariant elements of B^* and for every $t\in B^*$, the w^*-closed convex hull of $\{gt: g\in G_0\}$ coincides with the w^*-closed convex hull of $\{gt: g\in G\}$. Consequently, if in addition, we choose G_0 to be a subsemigroup of G (for example, we replace G_0 by the subsemigroup generated by G_0), then G satisfies condition (U) if and only if G_0 does.

Now we can apply Lemma 1 to the separable Banach space B and countable semigroup G_0. We obtain that there exists a sequence $\{v_n\}_{n=1}^{\infty}$ in $\mathrm{co}\,G_0$ such that for every $t\in B^*$, $v_n t\to t^{G_0}=t^G$ in the w^*-topology as $n\to\infty$. Consequently, the mapping $t\to t^G$ is a bounded linear projection, to be denoted by P, acting in B^*. Since $(gt)^G=gt^G=t^G$ for $g\in G$, $t\in B^*$, we have: $gP=Pg=P$. This completes the proof of Theorem 1 in case B is separable.

(2) Suppose G is separable, i.e., there exists a countable subset G_0 of G which is dense in G in the topology of $L_{w*}(B^*)$. We may assume that G_0 is a subsemigroup of G. The first part of the proof shows that it is sufficient to prove the theorem for G_0. However, we cannot apply Lemma 1 because B may not be separable. Consequently, we also have to appeal to Lemma 2. Let $g_1, g_2, \dots$ be all different elements of G_0 and let $v_n=\frac{1}{n^n}\sum_{i_1,\dots,i_n=1}^{n} g_1^{i_1}\dots g_n^{i_n}$. We are going to prove that

for every $t \in B^*$, $v_n t \to t^G$ in the w^*-topology of B^* as $n \to \infty$. All the assertions of Theorem 1 will follow from this in the same way as in part (1) of this proof.

Let φ_0 be an arbitrary element of B. Let us denote by B_1 the Banach subspace spanned by the elements $g_{1*}\varphi_0, g_{2*}\varphi_0, \ldots$. The subspace B_1 is G_{0*}-invariant. We may apply Lemma 2 and obtain that G_0, acting on $B_1^* = B^*/B_1^\perp$, also satisfies condition (U). Since B_1 is separable and G_0 is countable, Lemma 1 may be applied. We obtain that for every $f \in B_1^*$, $v_n f \to f^G$ in the w^*-topology of B_1^* as $n \to \infty$. In view of the identification $B_1^* = B^*/B_1^\perp$, this implies that for every $t \in B^*$, the sequence $\{(\varphi_0, v_n t)\}_{n=1}^\infty$ is convergent. (It may be seen directly that it converges to (φ_0, t^{G_0}); however, we choose another way of proving this, which we think is easier to follow.) Since φ_0 was an arbitrary element of B and $\|v_n t\| \leqq \|G\| \cdot \|t\|$; we obtain that for every $t \in B^*$, the sequence $\{v_n t\}_{n=1}^\infty$ w^*-converges to an element Pt of B^*. It is easy to see that Pt is G_0-invariant. Therefore, $Pt = t^{G_0}$ $(= t^G)$.

Proof of Theorem 2. The hypotheses of Theorem 1 are satisfied. Consequently, there is a sequence $\{v_n\}_{n=1}^\infty$ in $\mathrm{co}G$ such that for every $t \in B^*$, $v_n t \to t^G$ in the w^*-topology of B^* as $n \to \infty$. Now let $\varphi \in B$ be given. For every $t \in B^*$, we have $(v_{n*}\varphi - v_{m*}\varphi, t) = (\varphi, (v_n - v_m)t) \to 0$ as $n, m \to \infty$. Consequently, the sequence $\{v_{n*}\varphi\}_{n=1}^\infty$ is a weak Cauchy sequence in B. Since B is assumed to be weakly complete, there exists an element of B, to be denoted by $P_*\varphi$, for which $(v_{n*}\varphi, t) \to$ $\to (P_*\varphi, t)$ $(n \to \infty)$ for every $t \in B^*$. It is easy to see that P_* is a bounded linear operator in B. As $n \to \infty$, we have: $(\varphi, v_n t) = (v_{n*}\varphi, t) \to (P_*\varphi, t) = (\varphi, P_*^* t)$ for $\varphi \in B$, $t \in B^*$. Consequently, $v_n \to P_*^*$ in $L_{w*}(B^*)$ as $n \to \infty$. Since P_*^* is obviously w^*-continuous, we obtain the assertions of Theorem 2 (except the last assertion) if we put $P = P_*^*$.

The last assertion of Theorem 2 may be proved as follows. First we prove that for every $\varphi \in B$, the closed convex hull of $\{g_*\varphi : g_* \in G_*\}$ contains exactly one G_*-invariant element (namely, $P_*\varphi$). Here we may take either weak or strong closure, because the strong closure of a convex subset of a Banach space coincides with its weak closure (cf. [2], V. 3.13). Let $\varphi \in B$ and let $\hat{\varphi}$ be a G_*-invariant element in the closure of $(\mathrm{co}G_*)\varphi$. Then there exist $w_n \in \mathrm{co}G_*$ such that $w_n\varphi \to \hat{\varphi}$ strongly as $n \to \infty$. We have $P_* w_n \varphi \to P_*\hat{\varphi}$. Here $P_* w_n \varphi = P_*\varphi$ (because $P_* g_* = P_*$ for $g \in G$) and $P_*\hat{\varphi} = \hat{\varphi}$ (because P_* is a weak limit of elements of $\mathrm{co}G_*$ and $\hat{\varphi}$ is G_*-invariant). Therefore, $\hat{\varphi} = P_*\varphi$. On the other hand, if $g \in G$, then $g_* P_* \varphi =$ $= P_*\varphi$, i.e., $P_*\varphi$ is G_*-invariant. Therefore, $P_*\varphi$ is the unique G_*-invariant element in the closure of $(\mathrm{co}G_*)\varphi$. Since this is true for every $\varphi \in B$, the following holds according to [1]: For every $\varphi \in B$, every $\varepsilon > 0$ and every $v_{0*} \in \mathrm{co}G_*$ there exists $v_{1*} \in \mathrm{co}G_*$ such that $\|v_* v_{1*} v_{0*}\varphi - P_*\varphi\| < \varepsilon$. This inequality is equivalent to the following: $|([v_* v_{1*} v_{0*} - P_*]\varphi, t)| < \varepsilon$ for all $t \in B^*$ such that $\|t\| \leqq 1$ or

$|(\varphi, [vv_1v_0-P]t)|<\varepsilon$ for all $t\in B^*$ such that $\|t\|\leqq 1$. The last assertion of Theorem 2 follows immediately from this.

Proofs of Propositions 1,2 and the corollary of Proposition 2. In Proposition 1, $L^1(X, S, m)$ is weakly complete (cf. [2], IV. 8.6); consequently, the hypotheses of Theorem 2 are satisfied. In Proposition 2, the predual of M is weakly complete (cf. [4], Proposition 1); consequently, the hypotheses of Theorem 2 are satisfied. The corollary to Proposition 2 is simply a special case of Proposition 2.

3. Remarks and problems.

Remark 1. It follows from the author's other results (to be published) that even if G is not commutative and G and B are not separable and condition (U) is satisfied, then the mapping $t\to t^G$ $(t\in B^*)$ is a bounded linear projection contained in the closure, in $L_{w*}(B^*)$, of the convex hull of G.

Remark 2. It follows from the author's other results (to be published) that even if B is not weakly complete and condition (U) is satisfied, then a weaker version of the last assertion of Theorem 2 holds.

Problem 1. Is Theorem 2 true without the hypothesis that B is weakly complete?

Problem 2. Is Theorem 2 true without the hypothesis of separability of B or G? (In this case we can only expect P to be in the closure of co$G$, instead of the sequential closure of co$G$.)

Problem 3. Are the results of this paper true without the hypothesis that G is commutative?

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