Acta Sci. Math., 45 (1983), 377—380

On the overconvergence of complex interpolating polynomials. II Domain of geometric convergence to zero

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To Professor B. Szőkefalvi-Nagy on his seventieth birthday

1. Introduction. We continue here with developments concerning extensions of Walsh's Theorem on the overconvergence of sequences of differences of interpolating polynomials. As the title suggests, we are interested in determining *precisely* those domains in the complex plane for which (cf. [1]) the sequence

$$\{p_{n-1}(z, Z, f) - Q_{n-1,l}(z, f)\}_{n=1}^{\infty} \tag{1.1}$$

converges geometrically to zero for all $f \in A_\varrho$, where A_ϱ is the set of functions analytic in the circle $|z| < \varrho$ and having singularity on $|z| = \varrho$ $(\varrho > 1)$. Here $p_{n-1}(z, Z, f)$ is the Lagrange interpolating polynomial of $f(z) = \sum_{k=0}^{\infty} a_k z^k$ of degree $\leq n-1$ based on the nodes determined by the n^{th} row of the infinite triangular matrix $Z = \{z_{k,n}\}_{k=1}^{n}{}_{n=1}^{\infty}$, and

$$Q_{n-1,l}(z, f) := \sum_{k=0}^{n-1} \left(\sum_{j=0}^{l-1} a_{k+jn} \right) z^k \quad (l = 1, 2, \ldots). \tag{1.2}$$

2. Constructions. As for Z, we now assume the stronger hypothesis (than that used in [1]) that there exists a real number ϱ' with $1 \leq \varrho' < \varrho$ for which

$$1 \leq |z_{k,n}| \leq \varrho' < \varrho \quad (k = 1, 2, \ldots, n;\ n = 1, 2, \ldots). \tag{2.1}$$

As in [1], we set

$$\omega_n(t, Z) := \prod_{k=1}^{n} (t - z_{k,n}) \quad (n = 1, 2, \ldots), \tag{2.2}$$

and

$$G_l(z, R) = G_l(z, R, Z) := \overline{\lim_{n \to \infty}} \left\{ \max_{|t|=R} \left| (1 - t^{-ln}) \frac{z^n - 1}{t^n - 1} - \frac{\omega_n(z, Z)}{\omega_n(t, Z)} \right| \right\}^{1/n}, \tag{2.3}$$

Received June 25, 1982.

for any $R>\varrho'$ and any complex number z. We also set

(2.4) $$\hat{G}_l(z,\varrho) := \inf_{\varrho'<R<\varrho} G_l(z,R).$$

With these definitions, we first establish

Proposition 1. *For any complex number $z\neq 1$ and any positive integer l, there holds*

(2.5) $$G_l(z,R) \leqq \frac{\max\{|z|;1\}}{R^{l+1}} \quad (z\neq 1,\ R>\varrho').$$

Proof. From the techniques of [1], we see, via the maximum principle, that

$$\max_{|t|=R}\left|(1-t^{-ln})\frac{z^n-1}{t^n-1}-\frac{\omega_n(z,Z)}{\omega_n(t,Z)}\right| =$$

$$= \max_{|t|=R}\left|\frac{(t^{(l-1)}+t^{(l-2)n}+\ldots+1)(z^n-1)\omega_n(t,Z)-t^{ln}\omega_n(z,Z)}{t^{ln}\omega_n(t,Z)}\right| =$$

$$= R^{-ln}\max_{|t|=R}\left|\frac{(t^{(l-1)n}+t^{(l-2)n}+\ldots+1)(z^n-1)\omega_n(t,Z)-t^{ln}\omega_n(z,Z)}{\prod_{k=1}^{n}\left(R-\frac{\bar{z}_{k,n}t}{R}\right)}\right| \geqq$$

$$\geqq R^{-ln}\frac{|(z^n-1)\omega_n(0,Z)|}{R^n} \geqq \frac{|z^n-1|}{R^{(l+1)n}},$$

as $|\omega_n(0,Z)|\geqq 1$ from (2.1) and (2.2). Thus from the definition of $G_l(z,R)$ in (2.3), (2.5) immediately follows. Q. E. D.

Now define

(2.6) $$\Delta_l(z)=\Delta_l(z,\varrho,Z) := \sup_{f\in A_\varrho}\overline{\lim_{n\to\infty}}\,|p_{n-1}(z,Z,f)-Q_{n-1,l}(z,f)|^{1/n}$$

for any complex number z. Then we have

Proposition 2. *For any z with $|z|>\varrho$,*

(2.7) $$\hat{G}_l(z,\varrho)\geqq \Delta_l(z)\geqq G_l(z,\varrho).$$

Proof. Let E denote the matrix of nodes of interpolation formed from the roots of unity. Then for any $f\in A_\varrho$ and $\varepsilon>0$, we have by [1, (1.9)]

$$|p_{n-1}(z,Z,f)-Q_{n-1,l}(z,f)| \leqq |p_{n-1}(z,Z,f)-p_{n-1}(z,E,f)|+|p_{n-1}(z,E,f)-$$

$$-Q_{n-1,l}(z,f)| \leqq \frac{1}{2\pi}\left|\int_\Gamma \frac{f(t)}{t-z}\left(\frac{\omega_n(z,Z)}{\omega_n(t,Z)}-\frac{z^n-1}{t^n-1}\right)dt\right|+\left(\frac{|z|}{\varrho^{l+1}}+\varepsilon\right)^n \leqq$$

$$\leqq \frac{M_fR}{|z|-R}\max_{|t|=R}\left|\frac{z^n-1}{t^n-1}-\frac{\omega_n(z,Z)}{\omega_n(t,Z)}\right|+\left(\frac{|z|}{\varrho^{l+1}}+\varepsilon\right)^n \leqq$$

$$\leqq \frac{M_fR}{|z|-R}\left\{(G_l(z,R)+\varepsilon)^n+R^{-ln}\frac{|z^n-1|}{R^n-1}\right\}+\left(\frac{|z|}{\varrho^{l+1}}+\varepsilon\right)^n \quad (\varrho'<R<\varrho<|z|),$$

where $\Gamma=\{t: |t|=R\}$, $M_f=\max_{z\in\Gamma}|f(z)|$, provided $n\geqq n_0=n_0(\varepsilon)$. Hence by Proposition 1,

$$\overline{\lim_{n\to\infty}}\,|p_{n-1}(z,Z,f)-Q_{n-1,l}(z,f)|^{1/n}\leqq\max\left\{G_l(z,R)+\varepsilon,\ \frac{|z|}{R^{l+1}},\ \frac{|z|}{\varrho^{l+1}}+\varepsilon\right\}\leqq$$

$$\leqq\max\left\{G_l(z,R),\ \frac{|z|}{\varrho^{l+1}}\right\}+\varepsilon.$$

But here $\varepsilon>0$ and R $(\varrho'<R<\varrho)$ were arbitrary. Thus again by Proposition 1

$$\overline{\lim_{n\to\infty}}\,|p_{n-1}(z,Z,f)-Q_{n-1,l}(z,f)|^{1/n}\leqq\inf_{\varrho'<R<\varrho}G_l(z,R)=:\hat{G}_l(z,\varrho).$$

As this inequality holds for all $f\in A_\varrho$, this gives from (2.6) that

$$\Delta_l(z)\leqq\hat{G}_l(z,\varrho),$$

the desired first inequality of (2.7).

Next, for any u with $|u|=\varrho$ and with $f_u(z):=(u-z)^{-1}\in A_\varrho$, a direct computation gives that

$$(2.8)\quad p_{n-1}(z,Z,f_u)-Q_{n-1,l}(z,f_u)=\frac{1}{u-z}\left\{(1-u^{-ln})\frac{z^n-1}{u^n-1}-\frac{\omega_n(z,Z)}{\omega_n(u,Z)}+u^{-ln}\right\}.$$

Now by Proposition 1, $G_l(z,\varrho)>\varrho^{-l}$ $(|z|>\varrho)$. Thus we may choose an $\varepsilon>0$ with

$$(2.9)\qquad \varrho^{-l}+\varepsilon<G_l(z,\varrho)\quad(|z|>\varrho).$$

Further let $\{n_j\}_{n=1}^{\infty}$ be an infinite sequence of positive integers with $n_1<n_1<\dots$ (dependent on z) such that

$$\max_{|t|=\varrho}\left|(1-t^{-ln_j})\frac{z^{n_j}-1}{t^{n_j}-1}-\frac{\omega_{n_j}(z,Z)}{\omega_{n_j}(t,Z)}\right|>(G_l(z,\varrho)-\varepsilon)^{n_j}\quad(j=1,2,\dots)$$

(cf. Definition (2.3)). Now, choose u_j with $|u_j|=\varrho$ (which is also dependent on z) so that

$$(2.10)\qquad \begin{aligned}&\left|(1-u_j^{-ln_j})\frac{z^{n_j}-1}{u_j^{n_j}-1}-\frac{\omega_{n_j}(z,Z)}{\omega_{n_j}(u_j,Z)}\right|=\\&=\max_{|t|=\varrho}\left|(1-t_j^{-ln_j})\frac{z^{n_j}-1}{t^{n_j}-1}-\frac{\omega_{n_j}(z,Z)}{\omega_{n_j}(t,Z)}\right|>(G_l(z,\varrho)-\varepsilon)^{n_j},\end{aligned}$$

for each $j=1,2,\dots$. With $n=n_j$ and $u=u_j$, it follows from (2.8) and (2.10) that

$$|p_{n_j-1}(z,Z,f_{u_j})-Q_{n_j-1,l}(z,f_{u_j})|>\frac{1}{|z|+\varrho}\{(G_l(z,\varrho)-\varepsilon)^{n_j}-\varrho^{-ln_j}\},$$

for all $j=1,2,\dots$. Now, following the construction of [1], there is an $\tilde{f}$ (dependent on z) in A_ϱ for which

$$(2.11)\quad |p_{n_j-1}(z,Z,\tilde{f})-Q_{n_j-1,l}(z,\tilde{f})|\geqq\frac{1}{3(|z|+\varrho)n_j}\{(G_l(z,\varrho)-\varepsilon)^{n_j}-\varrho^{-ln_j}\}$$

for all $j=1, 2, \ldots$. Thus, by (2.9)

$$\overline{\lim_{n\to\infty}} |p_{n-1}(z, Z, \tilde{f})-Q_{n-1,l}(z, \tilde{f})|^{1/n} \geqq G_l(z, \varrho)-\varepsilon,$$

and as $\tilde{f}$ is some element in A_ϱ, then from the definition in (2.6),

$$\Delta_l(z) \geqq G_l(z, \varrho)-\varepsilon.$$

But, as this holds for every $\varepsilon>0$ with $\varrho^{-l}+\varepsilon<G_l(z, \varrho)$, then

$$\Delta_l(z) \geqq G_l(z, \varrho),$$

the desired last inequality of (2.7). Q. E. D.

As an obvious consequence of (2.7) of Proposition 2, we have

Corollary 3. *Let z be any complex number with $|z|>\varrho$ for which $\hat{G}_l(z, \varrho)<1$. Then, the sequence* (1.1) *converges geometrically to zero for each $f\in A_\varrho$.*

As a consequence of the proof of Proposition 2, we further have

Corollary 4. *Let z be any complex number with $|z|>\varrho$ for which $G_l(z, \varrho)>1$. Then, there is a function $\tilde{f}$ (depending on z) in A_ϱ for which the sequence* (1.1) *(with f replaced by $\tilde{f}$) is unbounded.*

Proof. If $G_l(z, \varrho)=1+2\eta$ where $\eta>0$, choose $\varepsilon>0$ sufficiently small so that $G_l(z, \varrho)-\varepsilon>1+\eta>1$. Then, (2.11) directly shows that the sequence (1.1) (with f replaced by $\tilde{f}$) is unbounded. Q. E. D.

Obviously, Corollary 4 and Proposition 1 imply that the sequence (1.1) is necessarily unbounded for some $\tilde{f}$ in A_ϱ, whenever $|z|>\varrho^{l+1}$. The same conclusion was deduced in [1].

Open questions. 1. Is $\hat{G}_l(z, \varrho)=G_l(z, \varrho)$?

2. Assuming the answer is "yes" for the previous question, then $\mathfrak{G}:=$ $:=\{z: G_l(z, \varrho)=1\}$ divides the complex plane into sets where either one has geometric convergence to zero for all f in A_ϱ or unboundedness of the sequence (1.1) for some f in A_ϱ. What does $\mathfrak{G}$ look like?

3. In general, one would not suspect that $\mathfrak{G}$ is a circle, even though this is the case for all examples treated in the literature. Can one construct cases (i.e. matrices Z) where indeed $\mathfrak{G}$ is not a circle? This suggests considering $Z=\{z_{k,n}\}$ where $\{z_{k,n}\}_{k=1}^n$ are not uniformly distributed, as $n\to\infty$.

Reference

[1] J. Szabados and R. S. Varga, On the overconvergence of complex interpolating polynomials, *J. Approx. Theory*, **36** (1982), 346—363.

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