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Moment problem for dilatable semigroups of operators

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Dedicated to Professor Béla Szőkefalvi-Nagy on the occasion of his 70th birthday

0. Introduction

The Main Theorem in the dilation theory of operators on Hilbert space due to Sz.-Nagy appeared in the Appendix to the third edition of [1]. The applications presented also in [1] show its central role in operator theory. At the same time Stinespring [11] described the so-called completely positive (linear) maps between C^*-algebras as (in a general sense) dilatable operator valued (linear) functions. It is also a generalization of Neumark's theorem (see [1]) on the dilatability of positive operator measures, a source of dilation theory.

On the other hand, Sz.-Nagy proved (see [1]) a moment theorem for self-adjoint operators generalizing a result of R. V. Kadison concerning a Schwarz-inequality for operator valued functions. Although it is also a consequence of the Main Theorem we think it has a more general character. Namely, given a $*$-semigroup in a C^*-algebra and an operator valued function on this $*$-semigroup, a moment theorem for the existence of a (completely positive) linear operator-function on the whole C^*-algebra can be formulated. This generalizes also Stinespring's theorem. Moreover we treat the moment problem for operators in the general case, when we assume only that the restrictions of the operators in question to some given subset (not assumed to be a subspace) of the Hilbert space are given. It is a new aspect for the existence of a single positive (hence for a self-adjoint) operator on Hilbert space and a self-adjoint semibounded operator also. The familiar Krein and Friedrichs extension is thus generalized and joined to moment and dilation problems.

The scalar valued case gives also a new insight into the classical Hausdorff moment problem, giving a solution analogous to that of the trigonometric moment

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problem by Riesz and Fejér. In any case, our solution differs from those of Hausdorff and Riesz—Fejér.

We give a new characterization of subnormal operators, too, along the lines of our argument.

For other applications, e.g., factorization questions for operators, moment problems for contraction and subnormal operators and their generalizations, see [4, 5, 2, 3, 10].

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1. Main problems and results

Given a *-semigroup G, a subset X of a Hilbert space H, and a function $f: G\times X\to H$, it is natural to seek an operator valued function F on G assuming its values in $B(H)$, the C^*-algebra of all bounded linear operators on H, such that

$$F(g)x = f(g, x) \quad (g\in G, x\in X) \tag{1}$$

holds. In this case F is called an operator representation of f.

We shall treat only the case when F in (1) is *dilatable* (in a general sense), i.e., when there is a Hilbert space K with a continuous linear operator $V: K\to H$ and a *-representation S of G on K such that

$$F(g) = VS(g)V^* \quad (g\in G). \tag{2}$$

Here F is *strongly dilatable* if V satisfies $VV^*=I_H$. A C^*-seminorm p on a *-semigroup G is a submultiplicative function $p: G\to \mathbf{R}^+$ with $p(g^*g)=p(g)^2$ implying $p(g^*)=p(g)$ $(g\in G)$.

Theorem 1.1. *A given H-valued function f on $G\times X$ has a dilatable operator representation F if and only if there exist $M\geqq 0$ and a C^*-seminorm p on G such that*

$$\Big\|\sum_{h,x} c_{h,x} f(h, x)\Big\|^2 \leqq M \sum_{h,x}\sum_{k,y} c_{h,x}\bar{c}_{k,y}(f(k^*h, x), y) \tag{3}$$

holds for each finite sequence $\{c_{h,x}\}$ of complex numbers indexed by elements of $G\times X$, and

$$(f(g^*g, x), x) \leqq Mp(g)^2\|x\|^2 \quad (g\in G;\ x\in X). \tag{4}$$

Theorem 1.2. *Assume that G has an identity e and that the function f on $G\times X$ satisfies*

$$f(e, x) = x \quad (x\in X), \tag{5}$$

$$\vee\{f(g, x):\ g\in G,\ x\in X\} = H. \tag{6}$$

This function f is of the form (1) with strongly dilatable F if and only if there is an H-valued function φ on $G\times f(G\times X)$ such that

$$\varphi(g, f(e, x)) = \varphi(e, f(g, x)) = f(g, x) \quad (g\in G;\ x\in X), \tag{7}$$

$$\|\varphi(g, f(g', x))\| \leqq p(g)\|f(g', x)\| \quad (g, g'\in G; x\in X) \tag{8}$$

for some C^*-seminorm p on G, and

$$\Big\|\sum_{h,\xi} c_{h,\xi}\varphi(h, \xi)\Big\|^2 \leqq \sum_{h,\xi}\sum_{k,\eta} c_{h,\xi}\bar{c}_{k,\eta}(\varphi(k^*h, \xi), \eta) \tag{9}$$

for each sequence $\{c_{h,\xi}\}$ of complex numbers indexed by elements of $G\times f(G\times X)$.

Let now G be a (multiplicative) *-semigroup in a given C^*-algebra A. In this case a $B(H)$-valued operator function F on G is *A-dilatable* if there is a *-representation S of the C^*-algebra A on some Hilbert space K with a continuous linear operator $V: K\to H$ such that (2) holds. In other words F has a completely positive (linear) extension to the whole A (hence a "moment" F is given for this completely positive map). We shall treat a more general setting by restricting the data to a subset of the Hilbert space.

Theorem 1.3. *Let G be a (multiplicative) *-semigroup in the C^*-algebra A whose linear span is norm dense in A. An H-valued function f on $G\times X$ is of the form* (1) *with an A-dilatable operator function F if and only if there is a constant $M\geqq 0$ such that*

$$\begin{aligned}&\Big\|\sum_{h,x} c_{h,x}f(h, x)\Big\|^2 \leqq M\sum_{h,x}\sum_{k,y} c_{h,x}\bar{c}_{k,y}(f(k^*h, x), y),\\ &\sum_{h}\sum_{k} c_h\bar{c}_k(f(k^*h, x), x) \leqq M\|x\|^2\Big\|\sum_{h} c_h h\Big\|^2 \quad (x\in X)\end{aligned} \tag{10}$$

hold for each finite sequence $\{c_{h,x}\}$ or $\{c_h\}$ of complex numbers indexed by elements of $G\times X$ and G, respectively.

Theorem 1.4. *Assume that A has an identity e such that the *-subsemigroup G of A which spans A, contains e too. An H-valued function f on $G\times X$ with* (5)—(6) *is of the form* (1) *with a strongly A-dilatable operator function F if and only if there is an H-valued function φ on $G\times f(G\times X)$ with* (7) *and such that*

$$\begin{aligned}&\Big\|\sum_{h,\xi} c_{h,\xi}\varphi(h, \xi)\Big\|^2 \leqq \sum_{h,\xi}\sum_{k,\eta} c_{k,\xi}\bar{c}_{k,\eta}(\varphi(k^*h, \xi), \eta),\\ &\sum_{h}\sum_{k} c_h\bar{c}_k(\varphi(k^*h, \xi), \xi) \leqq \|\xi\|^2\Big\|\sum_{h} c_h h\Big\|^2 \quad (\xi\in f(G\times X))\end{aligned} \tag{11}$$

hold for any finite sequence $\{c_{h,\xi}\}$ or $\{c_h\}$ of complex numbers indexed by elements of $G\times f(G\times X)$ and G, respectively.

Proof of the necessity. (1.1) Assuming that the H-valued function f on $G\times X$ has form (1) with dilatable operator function F on G, we have, for any finite sequence $\{c_{h,x}\}$ $(h\in G, x\in X)$ of complex numbers,

$$\Big\|\sum_{h,x} c_{h,x}f(h, c)\Big\|^2 = \Big\|\sum_{h,x} c_{h,x}F(h)x\Big\|^2 = \Big\|V\sum_{k,x} c_{h,x}S(h)V^*x\Big\|^2 \leqq$$

$$\leqq \|V\|^2 \sum_{h,x}\sum_{k,y} c_{h,x}\bar{c}_{k,y}(VS(k^*h)V^*x, y) = \|V\|^2 \sum_{h,x}\sum_{k,y} c_{h,x}\bar{c}_{k,y}\big(f(k^*h, x), y\big).$$

Moreover,

$$\big(f(g^*g, x), x\big) = \big(F(g^*g)x, x\big) = \big(VS(g^*g)V^*x, x\big) =$$

$$= \|S(g)V^*x\|^2 \leqq \|S(g)\|^2\|V^*\|^2\|x\|^2 = \|V\|^2\|S(g)\|^2\|x\|^2$$

holds for each $g\in G$, $x\in X$. This yields (4) with the C^*-seminorm p on G defined (by the *-representation S of G on the Hilbert space K) as $p(g):=\|S(g)\|$ $(g\in G)$.

(1.2) Defining φ on $G\times f(G\times X)$ by

$$\varphi\big(g, f(g', x)\big) := F(g)f(g', x) \quad (g, g'\in G;\ x\in X) \tag{12}$$

we deduce (7) and (8) from (5) and (1) as follows:

$$\varphi\big(g, f(e, x)\big) = F(g)f(e, x) = F(g)x = f(g, x),$$

$$\varphi\big(e, f(g, x)\big) = F(e)f(g, x) = VS(e)V^*f(g, x) = VV^*f(g, x) = f(g, x),$$

$$\big\|\varphi\big(g, f(g', x)\big)\big\| \leqq \|F(g)\|\|f(g', x)\| \leqq \|V\|\|S(g)\|\|V^*\|\|f(g', x)\| \leqq \|S(g)\|\|f(g', x)\|.$$

To prove (9), let $\{c_{h,\xi}\}$ be any finite sequence of complex numbers indexed by elements of $G\times f(G\times X)$. We have then

$$\Big\|\sum_{h,\xi} c_{h,\xi}\varphi(h, \xi)\Big\|^2 = \Big\|\sum_{h,\xi} c_{h,\xi}F(h)\xi\Big\|^2 = \Big\|V\sum_{k,\xi} c_{h,\xi}S(h)V^*\xi\Big\|^2 \leqq$$

$$\leqq \|V\|^2 \sum_{h,\xi}\sum_{k,\eta} c_{h,\xi}\bar{c}_{k,\eta}\big(VS(k^*h)V^*\xi, \eta\big) = \sum_{h,\xi}\sum_{k,\eta} c_{h,\xi}\bar{c}_{k,\eta}\big(\varphi(k^*\eta), \xi), \eta\big).$$

(1.3) Assuming that F is A-dilatable, we know that S is a *-representation of the C^*-algebra A. Hence $\|S\|\leqq 1$. From (1.1) we see furthermore

$$\frac{1}{\|V\|^2}\Big\|\sum_{h,x} c_{h,x}f(h, x)\Big\|^2 \leqq \sum_{h,x}\sum_{k,y} c_{h,x}\bar{c}_{k,y}\big(f(k^*h, x), y\big),$$

$$\sum_{h}\sum_{k} c_h\bar{c}_k\big(f(k^*h, x), x\big) \leqq \Big\|\sum_{h} c_hS(h)V^*x\Big\| = \Big\|S\big(\sum_{h} c_hh\big)V^*x\Big\|^2 \leqq$$

$$\leqq \|S\|^2\Big\|\sum_{h} c_hh\Big\|^2\|V^*\|^2\|x\|^2 \leqq \|x\|^2\Big\|\sum_{h} c_hh\Big\|^2$$

for any $x\in H$, whence (10) follows.

(1.4) Applying φ defined above in (12) we have by (12) and by the relations $\|V\|=1, \|S\|\leqq 1$ that

$$\Big\|\sum_{h,\xi} c_{h,\xi}\varphi(h,\xi)\Big\|^2 \leqq \sum_{h,\xi}\sum_{k,\eta} c_{h,\xi}\bar{c}_{k,\eta}(\varphi(k^*h,\xi),\eta),$$

$$\sum_h\sum_k c_h\bar{c}_k(\varphi(k^*h,\xi),\xi) = \Big\|\sum_h c_h S(h)V^*\xi\Big\|^2 = \Big\|S\big(\sum_h c_h h\big)V^*\xi\Big\|^2 \leqq$$

$$\leqq \|S\|^2\Big\|\sum_h c_h h\Big\|^2\|V^*\|^2\|\xi\|^2 \leqq \Big\|\sum_h c_h h\Big\|^2\|\xi\|^2 \quad (\xi\in f(G\times X)).$$

Proof of the sufficiency. (1.1) Let f be an H-valued function on $G\times X$ satisfying (3) and (4) with a C^*-seminorm p on G. Further let K_0 be the linear space of all finitely supported complex valued functions on $G\times X$. Each element of K_0 has the form $\sum_{h,x} c_{h,x}\delta(h,x)$, where $\delta(h,x)$ denotes the function assuming the value 1 in $(h,x)\in G\times X$ and 0 otherwise, and $\{c_{h,x}\}$ $(h\in G, x\in X)$ is a finite sequence of complex numbers. With this notation we can define two operations on K_0, the first of which is the linear map V into H given by

(13) $$V\Big(\sum_{h,x} c_{h,x}\delta(h,x)\Big) := \sum_{h,x} c_{h,x}f(h,x),$$

and the second one the translation operation on K_0 by elements of G, defined for any g in G by

(14) $$S(g)\Big(\sum_{h,x} c_{h,x}\delta(h,x) := \sum_{h,x} c_{h,x}\delta(gh,x).$$

Remark also that the map $g\mapsto S(g)$ constitutes an endomorphism of G. Lastly, we define a semi-inner product $\langle\cdot,\cdot\rangle$ on K_0 by

(15) $$\Big\langle\sum_{h,x} c_{h,x}\delta(h,x), \sum_{k,y} d_{k,y}\delta(k,y)\Big\rangle := \sum_{h,x}\sum_{k,y} c_{h,x}\bar{d}_{k,y}(f(k^*h,x),y).$$

Observe the nonnegativity of the right hand side in (5) in view of (3). Now we are in a position to construct a Hilbert space K with a continuous linear map $V: K\to H$ and a *-representation S of G on K such that (1) holds with F satisfying (2), too. We obtain a pre-Hilbert space by factorizing K_0 with respect to the null space $N:=\{\xi\in K_0: \langle\xi,\xi\rangle=0\}$ and by taking the induced inner product on K_0/N. The completion of K_0/N is a Hilbert space, say K. For simplicity, denote also by $\delta(h,x)$ the image of $\delta(h,x)\in K_0$ in K under the factorization. Thus K_0 is viewed as a norm dense subset of K and V is a densely defined bounded (cf. (3)) linear operator from K into H. Thus V has a unique continuous extension to K which is denoted naturally also by V. Lastly we have to show that $S(g)$ $(g\in G)$ induces a bounded linear operator also denoted by $S(g)$ on K such that $S_g^*= =S(g^*)$ for any g in G. To this end let $\xi=\sum_{h,x} c_{h,x}\delta(h,x)$ be taken from a dense subset of K and we show

(16) $$\|S(g)\xi\| \leqq p(g)\|\xi\| \quad (g\in G)$$

which is enough for our purpose. Indeed, we have

$$\|S(g)\xi\|^2 = \Big\|\sum_{h,x} c_{h,x}\delta(gh, x)\Big\|^2 = \sum_{h,x}\sum_{k,y} c_{h,x}\bar{c}_{k,y}\big(f(k^*g^*gh, x), y\big) =$$
$$= \langle S(g^*g)\xi, \xi\rangle \leqq \|S(g^*g)\xi\|\|\xi\|,$$

and hence, by induction on n,

$$\|S(g)\xi\|^{2^n} \leqq \|S((g^*g)^{2^{n-1}})\xi\|^2\|\xi\|^{2^n-2} = \|\xi\|^{2^n-2}\Big\|\sum_{h,x} c_{h,x}\delta\big((g^*g)^{2^n-2}h, x\big)\Big\|^2 \leqq$$
$$\leqq \|\xi\|^{2^n-2}\big(\sum_{h,x} |c_{h,x}|\|\delta((g^*g)^{2^n-2}h, x)\|\big)^2 = \|\xi\|^{2^n-2}\big(\sum_{h,x} |c_{h,x}|(f(h^*(g^*g)^{2^n-1}h, x), x^{1/2}\big)^2 \leqq$$
$$\leqq \|\xi\|^{2^n-2}\big(\sum_{h,x} |c_{h,x}|M^{1/2}p((g^*g)^{2^{n-2}}h)\|x\|\big)^2 \leqq \qquad \text{(by (4))}$$
$$\leqq \|\xi\|^{2^n-2}Mp(g)^{2^n}p(h)^2\big(\sum_{h,x} |c_{h,x}|\|x\|\big)^2 \quad (n = 1, 2, \ldots).$$

This implies (16) for $n\to\infty$.

Now $S(g)^*=S(g^*)$ follows for any g in G by observing

(17) $$\langle S(g^*)\xi, \eta\rangle = \langle \xi, S(g)\eta\rangle \quad \text{for each} \quad \xi, \eta\in K_0.$$

Indeed, if $\xi=\sum_{h,x} c_{h,x}\delta(h, x)$, $\eta=\sum_{k,y} d_{k,y}\delta(k, y)$, then both sides of (17) are equal to

$$\sum_{h,x}\sum_{k,y} c_{h,x}\bar{d}_{k,y}\big(f(k^*g^*h, x), y\big).$$

To complete the proof of (1.1) we show

(18) $$VS(g)V^*x = f(g, x) \quad (g\in G;\ x\in X).$$

By (13), it suffices to see that $S(g)V^*x=\delta(g, x)$. But

$$\langle S(g)V^*x, \sum_{k,y} d_{k,y}\delta(k, y)\rangle = \big(x, VS(g^*)\big(\sum_{k,y} d_{k,y}\delta(ky)\big)\big) =$$
$$= \big(x, V\big(\sum_{k,y} d_{k,y}\delta(g^*k, y)\big)\big) = \big(x, \sum_{k,y} d_{k,y}f(g^*k, y)\big) = \big\langle\delta(g, x), \sum_{k,y} d_{k,y}\delta(k, y)\big\rangle$$

holds for any $\sum_{k,y} d_{k,y}\delta(k, y)\in K_0$, verifying our assertion.

(1.2) We shall adopt the argument used in the proof of (1.1) by replacing X by $f(G\times X)$ and f by φ. (9) is a translation of (3) into the new situation and (4) implies (8) with $M\equiv 1$ since

$$\big\|\varphi(g, f(g', x))\big\|^2 \leqq \big(\varphi(g^*g, f(g', x)), f(g', x)\big) \leqq$$
$$\leqq \big\|\varphi(g^*g, f(g', x))\big\|\|f(g', x)\| \leqq p(g)^2\|f(g', x)\|^2$$

for any $g, g'\in G$, $x\in X$. Now we define $V, S, \langle\ ,\ \rangle$ by

(13') $$V\big(\sum_{h,\xi} c_{h,\xi}\delta(h, \xi)\big) := \sum_{h,\xi} c_{h,\xi}\varphi(h, \xi),$$

(14') $$S(g)\big(\sum_{h,\xi} c_{h,\xi}\delta(h, \xi)\big) := \sum_{h,\xi} c_{h,\xi}\delta(gh, \xi),$$

(15') $$\big\langle\sum_{h,\xi} c_{h,\xi}\delta(h, \xi), \sum_{k,\eta} d_{k,\eta}\delta(k, \eta)\big\rangle := \sum_{h,\xi}\sum_{k,\eta} c_{h,\xi}\bar{d}_{k,\eta}\big(\varphi(k^*h, \xi), \eta\big).$$

In consequence, we have a *-representation S of G on some suitable Hilbert space K with a continuous linear operator $V\colon K\to H$ such that

(19) $$VS(g)V^*f(g',x)=\varphi(g,f(g',x))\quad (g,g'\in G;\ x\in X).$$

By (3) and (7) this implies (18) since

$$VS(g)V^*x=VS(g)V^*f(e,x)=\varphi(g,f(e,x))=f(g,x)\quad (g\in G;\ x\in X).$$

Finally, since $S(e)=I_K$, by (7) we have

$$VV^*f(g,x)=VS(e)V^*f(g,x)=\varphi(e,f(g,x))=f(g,x)\quad (g\in G;\ x\in X)$$

proving $VV^*=I_K$ (cf. (16)).

(1.3) First of all (10) implies (4) if we take the C^*-(semi)norm p on G defined in terms of the norm $\|\cdot\|$ of A as $p(g)=\|g\|$ $(g\in G)$ since

$$(f(g^*g,x),x)\leqq M\|x\|^2\|g\|^2\quad (g\in G,x\in X).$$

As a consequence of the proof of (1.1), we have a *-representation of G on a suitable Hilbert space K such that (18) holds true. This proves (1) for a dilatable operator function F satisfying (2). But we need the A-dilatability of F. The key step is at hand: we shall prove the extendibility of S from G to A. To this end we have only to show

(20) $$\Big\|\sum_g \lambda_g S(g)\Big\|\leqq\Big\|\sum_g \lambda_g g\Big\|$$

for each finite sequence $\{\lambda_g\}$ of complex numbers indexed by elements of G (because G spans A). Putting $a=\sum_g \lambda_g g\in A$, $\xi=\sum_{h,x} c_{h,x}\delta(h,x)\in K$ we have for $S(a)=\sum_g \lambda_g S(g)$

$$\|S(a)\xi\|^2=\langle S(a^*a)\xi,\xi\rangle\leqq\|S(a^*a)\xi\|\,\|\xi\|,$$

and thus by induction, for any $n=0,1,2,\ldots,$

$$\|S(a)\xi\|^{2^n}\leqq\|S((a^*a)^{2^{n-2}})\xi\|^2\|\xi\|^{2^n-2}=\|\xi\|^{2^n-2}\Big\|\sum_{h,x} c_{h.x}S((a^*a)^{2^{n-2}})\delta(h,x)\Big\|^2=$$

$$=\|\xi\|^{2^n-2}\Big\|\sum_{h,x}\sum_s c_{h,x}\lambda_s\delta(g_sh,x)\Big\|^2\leqq\|\xi\|^{2^n-2}\Big(\sum_x\Big\|\sum_{h,s} c_{h,x}\lambda_s\delta(g_sh,x)\Big\|\Big)^2=$$

$$=\|\xi\|^{2^n-2}\Big\{\sum_x\Big(\sum_{h,s}\sum_{k,t} c_{h,x}\bar c_{k,x}\lambda_s\bar\lambda_t(f(k^*g_t^*g_sh,x),x)\Big)^{1/2}\Big\}^2\leqq$$

$$\leqq\|\xi\|^{2^n-2}\Big(\sum_x M^{1/2}\|x\|\Big\|\sum_{h,s} c_{h,x}\lambda_s g_sh\Big\|\Big)^2\leqq\|\xi\|^{2^n-2}M\Big(\sum_x\|x\|\Big\|\sum_h c_{h,x}h\Big\|\Big)^2\Big\|\sum_s\lambda_sg_s\Big\|^2=$$

$$=\|(a^*a)^{2^{n-2}}\|^2\|\xi\|^{2^n-2}M\Big(\sum_x\|x\|\Big\|\sum_h c_{h,x}h\Big\|\Big)^2=\|a\|^{2^n}\|\xi\|^{2^n-2}M\Big(\sum_x\|x\|\Big\|\sum_h c_{h,x}h\Big\|\Big)^2.$$

By passing to $n\to\infty$ we see that $\|S_a\xi\|\leqq\|a\|\,\|\xi\|$, which proves (20). Here the notation $(a^*a)^{2^n}=\sum_s\lambda_sg_s$ was used for $a=\sum_g\lambda_gg$.

(1.4) Similarly as before we adopt the preceding argument for our purpose such that X is replaced by $f(G\times X)$ and f by φ. (11) is then a simple translation of (10) into the new setting. The definitions (13′), (14′) and (15′) yield a *-representation of A on a suitable Hilbert space K with a continuous linear operator $V: K\to H$ such that (19) holds also true. The proof of (18) and $VV^*=I_H$ is the same as in (1.2).

2. Applications

(i) Let G be the trivial semigroup $G=\{e\}$. A familiar identification $G\times X\approx X$ implies the following results.

Theorem 2. 1. *Let f be an H-valued function given on a subset X of the Hilbert space H. There exists a self-adjoint operator F on H with $mI_H\leqq F\leqq MI_H$ and extending f if and only if*

$$\Big\|\sum_x c_x(f(x)-mx)\Big\|^2\leqq (M-m)\Big(\sum_x c_x(f(x)-mx),\ \sum_x c_x x\Big) \tag{21}$$

holds for any finite sequence $\{c_x\}$ $(x\in X)$ of complex numbers.

Proof. Since a self-adjoint operator F is bounded by m and M from below and above, respectively, if and only if $0\leqq F-mI_H$ and $\|F-mI_H\|\leqq M-m$, we have the assertion by Theorem 1.1. Indeed, (3) is the same as (21) ((4) is immediately satisfied with $p(e)=1$ if $M-m$ is replaced by M) for $f-mI_x$.

Corollary 2.1 (Krein (see [1])). *Let f be a symmetric and bounded linear operator from a linear subspace X of the Hilbert space H. Then there exists a self-adjoint operator F on H extending f and with the same norm.*

Theorem 2.1.1. *Let b be an H-valued function given on a subset Y of the Hilbert space H with norm dense linear hull in H. There exists a semi-bounded self-adjoint operator B with bound 1 from below and extending b if and only if*

$$\Big\|\sum_x c_x x\Big\|^2\leqq \Big(\sum_x c_x x,\ \sum_x c_x b(x)\Big) \tag{22}$$

holds for each finite sequence $\{c_x\}$ $(x\in Y)$ of complex numbers.

Proof. The necessity of (22) is evident so we omit the proof. To prove the sufficiency of (22) let f be the inverse map of b (the existence of which is an easy consequence of (22)). Since (22) is the same as (21) with $m=0$, $M=1$ we have by Theorem 2.1 a positive operator F with norm $\leqq 1$ which extends f. Hence F has an inverse $B=F^{-1}$ too. B is the desired operator. The proof is complete.

Corollary 2.1.1 (FRIEDRICHS (see [1])). *Let b be a symmetric operator bounded from below by* 1 *defined on a dense linear subspace Y of the Hilbert space H. Then there exists a self-adjoint operator B bounded from below by* 1, *and extending b.*

(ii) If H is a one dimensional Hilbert space, Theorem 1.3 (with the usual identifications $B(H)\cong \mathbf{C}$, $X=\{1\}$) gives us a new solution to the classical Hausdorff moment theorem differing also from Riesz' solution.

Theorem 2.2. *Let G be a (multiplicative) *-semigroup of a C^*-algebra A, spanning a norm dense *-subalgebra in A. A complex valued function f given on G has a (necessarily unique) positive linear extension to A if and only if there is a constant $M>0$ for which*

$$(1/M)\Big|\sum_g c_g f(g)\Big|^2 \leqq \sum_g \sum_h c_g \bar{c}_h f(h^* g) \leqq M \Big\|\sum_g c_g g\Big\|^2 \tag{23}$$

holds for each finite sequence $\{c_g\}$ $(g\in G)$ of complex numbers.

Corollary 2.2.1. *Let f be a complex valued function on a C^*-algebra A. f is a positive linear functional on A if and only if* (23) *holds with $G=A$.*

Corollary 2.2.2. *Let Ω be a compact subset of the real line and let $\{\mu_n\}_{n=0}^{\infty}$ be a given sequence of complex numbers. There is a positive (bounded) measure μ on Ω such that*

$$\int_\Omega t^n \, d\mu = \mu_n \quad \text{for} \quad n = 0, 1, 2, \ldots$$

if and only if

$$0 \leqq \sum_m \sum_n c_m \bar{c}_n \mu_{m+n} \leqq \mu_0 \max_{t\in\Omega} \Big|\sum_n c_n t^n\Big|^2 \tag{24}$$

holds for each finite sequence $\{c_n\}_{n\geqq 0}$ of complex numbers.

Corollary 2.2.3. *Let Ω be a compact subset of the complex plane and let $\{\mu_{m,n}\}_{m,n=0}^{\infty}$ be a given double sequence of complex numbers. There is a (necessarily unique) positive (bounded) measure μ on Ω such that*

$$\int_\Omega (\bar{\lambda})^m \lambda^n \, d\mu(\lambda) = \mu_{m,n} \quad \text{for} \quad m, n = 0, 1, 2, \ldots$$

if and only if

$$0 \leqq \sum_j \sum_k c_j \bar{c}_k \mu_{m_j+n_k, m_k+n_j} \leqq \mu_{0,0} \max_{\lambda\in\Omega} \Big|\sum_j c_j (\bar{\lambda})^{m_j} \lambda^{n_j}\Big|^2 \tag{25}$$

holds for each finite sequence $\{c_j\}_{j\geqq 0}$ of complex numbers.

3. Operator problems

Theorem 3.1. *Let G be a (multiplicative) *-semigroup in a C^*-algebra A generating a norm dense *-subalgebra of A. An operator valued function $f: G\to B(H)$ is A-dilatable if and only if there is a constant $M\geqq 0$ such that*

$$\Big\|\sum_g f(g)x_g\Big\|^2 \leqq M\sum_g\sum_h (f(h^*g)x_g, x_h) \tag{26}$$

and

$$\sum_g\sum_h c_g\bar{c}_h(f(h^*g)x, x) \leqq M\|x\|^2\Big\|\sum_g c_g g\Big\|^2 \quad (x\in H) \tag{27}$$

hold for every finite sequence $\{x_g\}_{g\in G}$ and $\{c_g\}_{g\in G}$ in H and $\mathbf{C}$, respectively.

Sketch of the proof (for details see [9]). The following function f on $G\times X$ given by

$$f(g, x) = f(g)x \quad (g\in G;\ x\in X),$$

where $X=H$, produces (26) and (27) along (10) with $x_g=\sum_x c_{g,x}x$. The necessity of (26) and (27) thus follows. For the proof of the sufficiency we have to change the argument used in the proof of Theorem 1.3 replacing K_0 by the linear space of H-valued functions on G with finite support, that is, δ_g is replaced by $\delta_g x_g$ with $x_g\in H$ for $g\in G$. An easy analysis of the proof of the sufficiency part of Theorem 1.3 shows our statement.

Corollary 3.1.1. *Let G be a (multiplicative) *-semigroup in a commutative C^*-algebra A generating a norm dense *-subalgebra in A. An operator valued function $f: G\to B(H)$ is A-dilatable if and only if there is a constant $M>0$ such that*

$$\Big|\sum_g c_g(f(g)x, x)\Big|^2 \leqq \|x\|^2\sum_g\sum_h c_g\bar{c}_h(f(h^*g)x, x) \leqq M\|x\|^4\Big\|\sum_g c_g g\Big\|^2 \quad (x\in H) \tag{28}$$

holds for each finite sequence $\{c_g\}_{g\in G}$ of complex numbers.

Proof. Since (26) and (27) imply (28) (by setting $x_g=c_g x$ for $g\in G, x\in X$), the necessity of (28) is obvious. For the sufficiency, the function $g\mapsto(f(g)x, x)$ on G has a (unique) positive linear extension by Theorem 2.2 for any fixed x in H. The norm of this extension is $\leqq M^{1/2}\|x\|^2$. Hence we obtain a positive linear extension F of f to the whole of A. But a result of Stinespring [11, Theorem 4] ensures that F is automatically A-dilatable.

The next result solves the operator moment problem of Sz.-Nagy in a new way.

Corollary 3.1.2. *Let Ω be a compact subset of the real line and let $\{A_n\}_{n=0}^\infty$ be a sequence of operators on a Hilbert space H. There is a positive (bounded)* *operator measure* F *on* Ω *such that*

$$\int_{\Omega} t^n (F(dt)x, x) = (A_n x, x) \quad \textit{for} \quad n = 0, 1, 2, \ldots; \quad x \in H$$

if and only if

$$0 \leqq \sum_m \sum_n c_m \bar{c}_n (A_{m+n} x, x) \leqq \|A_0\| \|x\|^2 \max_{t \in \Omega} \left| \sum_n c_n t^n \right|^2 \quad (x \in H) \tag{29}$$

holds for any finite sequence $\{c_n\}_{n \geqq 0}$ *of complex numbers.*

Proof. (29) is a version of (28) for $G=\{t^n\}_{n=0}^{\infty}$ $(t \in \Omega)$ and $A=C(\Omega)$ with $f(t^n)=A_n$ $(n=0, 1, 2, \ldots)$. Thus Corollary 3.1.1 implies the statement.

Corollary 3.1.3. *Let* Ω *be a compact subset of the complex plane and let* $\{A_{m,n}\}_{m,n=0}^{\infty}$ *be a double sequence of operators on a Hilbert space* H. *There is a positive (bounded) operator measure* F *on* Ω *such that*

$$\int_{\Omega} (\bar{\lambda})^m \lambda^n (F(dt)x, x) = (A_{m,n} x, x) \quad (m, n = 0, 1, 2, \ldots; \ x \in H)$$

if and only if

$$0 \leqq \sum_j \sum_k c_j \bar{c}_k (A_{m_j+n_k, m_k+n_j} x, x) \leqq \|A_{0,0}\| \|x\|^2 \max_{\lambda \in \Omega} \left| \sum_j c_j (\bar{\lambda})^{m_j} \lambda^{n_j} \right|^2 \quad (x \in H) \tag{30}$$

holds for any finite sequence $\{c_j\}_{j \geqq 0}$ *of complex numbers.*

Proof. (30) is a version of (28) for $G=\{(\bar{\lambda})^m \lambda^n\}_{m,n=0}^{\infty}$ $(\lambda \in \Omega)$, $A=C(\Omega)$ with $f((\bar{\lambda})^m \lambda^n)=A_{m,n}$ $(m, n=0, 1, 2, \ldots)$. Thus Corollary 3.1.1 implies Corollary 3.1.3.

It follows a new characterization of subnormal operators (for the definition see [1]).

Corollary 3.1.4. *Let* B *be an operator in* $B(H)$ *for a Hilbert space* H. B *is subnormal if and only if*

$$0 \leqq \sum_{k,l} \sum_{m,n} c_{k,l} \bar{c}_{m,n} (B^{*(k+n)} B^{l+m} x, x) \leqq \|x\|^2 \max_{\lambda \in \Omega} \left| \sum_{m,n} c_{m,n} (\bar{\lambda})^m \lambda^n \right|^2 \quad (x \in H) \tag{31}$$

holds for any finite double sequence $\{c_{m,n}\}_{m,n \geqq 0}$ *of complex numbers, where* Ω *denotes the spectrum of* B.

Proof. B is subnormal if and only if the function

$$(\bar{\lambda})^m \lambda^n \mapsto B^{*m} B^n \quad (m, n = 0, 1, 2, \ldots; \ \lambda \in \Omega)$$

is $C(\Omega)$-dilatable (for details see [9]). But this is equivalent to (31).

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