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Self-dual polytopes and the chromatic number of distance graphs on the sphere

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Dedicated to Professor Béla Szőkefalvi-Nagy on his 70th birthday

0. Introduction

Let S^{n-1} denote the unit sphere in the n-dimensional euclidean space and let $0<\alpha<2$. Construct a graph $G(n, \alpha)$ on the points of S^{n-1} by connecting two of them iff their distance is exactly α. We shall study the chromatic number of the graph obtained this way and prove that this chromatic number is at least n. This answers a question of ERDŐS and GRAHAM [2], who conjectured that this chromatic number tends to infinity with n.

Let us modify the definition of the graph and construct another graph $B(n, \alpha)$ by connecting two points of S^{n-1} if and only if their distance is at least α. The graph $B(n, \alpha)$ obtained this way is often called Borsuk's graph because a classical theorem of BORSUK [1] implies (in fact, is equivalent to) the result that $B(n, \alpha)$ has chromatic number at least $n+1$. Since, however, $G(n, \alpha)$ is a proper subgraph of $B(n, \alpha)$, Borsuk's theorem has no immediate bearing on the chromatic number of $G(n, \alpha)$.

If α is larger than the side of a regular simplex inscribed in the unit ball, then it is easy to describe an $(n+1)$-coloration of $B(n, \alpha)$ (and so, a fortiori, of $G(n, \alpha)$). Let R be the regular simplex inscribed in S^{n-1} and use the facet of R intersected by the segment $0X$ as the "color" of $X \in S^{n-1}$. Hence if $\alpha > \sqrt{2(n+1)/n}$ then

$$\chi(G(n, \alpha)) \leqq \chi(B(n, \alpha)) = n+1.$$

It is easy to see that the colors of the vertices of R can be chosen different, and hence this is also true for $\alpha = \sqrt{2(n+1)/n}$.

In this paper we apply a lower bound on the chromatic number of a general graph, derived in [4], to an appropriate subgraph of $G(n, \alpha)$. It is interesting to

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remark that to prove this lower bound in [4], Borsuk's theorem was used. Thus in this sense we do establish a connection between the chromatic numbers of $B(n, \alpha)$ and $G(n, \alpha)$.

In section 1 we define, construct and study certain polyhedra called strongly self-dual. It seems that these polyhedra merit interest on their own right. In section 2 we state the general lower bound on the chromatic number mentioned above and apply it to prove our main result. In section 3 we discuss the question of sharpness of our results.

1. Strongly self-dual polytopes

Let P be a convex polytope in $\mathbf{R}^n$. We say that P is *strongly self-dual* if the following conditions hold.

(1) P is inscribed in the unit sphere S^{n-1} in $\mathbf{R}^n$ (so that all vertices of P lie on the sphere S^{n-1});

(2) P is circumscribed around the sphere S' with center 0 and radius r for some $0<r<1$ (so that S' touches every facet of P);

(3) There is a bijection σ between vertices and facets of P such that if v is any vertex then the facet $\sigma(v)$ is orthogonal to the vector v.

If $n=2$ then the strongly self-dual polytopes are precisely the odd regular polygons. If $n\geqq 3$ then there are strongly self-dual polytopes with a more complicated structure.

Let us start with proving some elementary properties of strongly self-dual polytopes.

Lemma 1. *If v_1, v_2 are vertices of a strongly self-dual polytope P and v_1 is a vertex of the facet $\sigma(v_2)$ then v_2 is a vertex of the facet $\sigma(v_1)$.*

Proof. Let v be any vertex of P. The inequality defining $\sigma(v)$ is $v\cdot x\geqq -r$. For $v=v_2$, the vector $x=v_1$ lies on the facet $\sigma(v_2)$, and so $v_2\cdot v_1=-r$. But by interchanging the role of v_1 and v_2, we obtain that v_2 lies on $\sigma(v_1)$.

Call a diagonal of a strongly self-dual polytope *principal* if it connects a vertex v to a vertex of the facet $\sigma(v)$. The proof of Lemma 1 implies:

Lemma 2. *Every principal diagonal of a strongly self-dual polytope is of the same length.*

This length α will be called the *parameter* of P. Clearly $\alpha=\sqrt{2+2r}$. As $r>0$, we have $\alpha>\sqrt{2}$. This trivial inequality can be improved. We show that the least possible value of the parameter of a strongly self-dual polytope in a given space is the side length of the regular simplex inscribed in the unit ball:

Lemma 3. *Let P be a strongly self-dual polytope in $\mathbf{R}^n$ with parameter α. Then* $\alpha \geqq \sqrt{2(n+1)/n}$.

Proof. We prove more generally that *if a polytope P is inscribed in S^{n-1} and contains the origin, then it has a pair of vertices at a distance at least $\sqrt{2(n+1)/n}$ apart.* Since the principal diagonals of a strongly self-dual polytope are obviously its longest diagonals, this will imply the Lemma.

Observe further that we may assume that P is a simplex, since if a polytope contains the origin then some of its vertices span a simplex which also contains it.

So let P be a simplex inscribed in S^{n-1} and containing the origin. Let P' be its facet nearest 0, and let z be the orthogonal projection of 0 on P'. It is easy to see that P' contains z. Let $t=|z|$. We claim that $t \leqq 1/n$. In fact, let $v_0, \ldots, v_n$ be the vertices of P. Then since 0 is in P, we can write

$$\sum_{i=0}^{n} \lambda_i v_i = 0 \quad \text{with} \quad \lambda_i \geqq 0, \quad \sum_{i=0}^{n} \lambda_i = 1.$$

We may assume without loss of generality that $\lambda_0 \leqq 1/(n+1)$. Consider the point

$$w_0 = \sum_{i=1}^{n} \frac{\lambda_i}{1-\lambda_0} v_i = \frac{-\lambda_0}{1-\lambda_0} v_0.$$

This point is on the boundary of P. Furthermore, $|w_0| = \lambda_0/(1-\lambda_0) \leqq 1/n$. Hence the facet of P nearest to the origin is at a distance at most $1/n$, which proves that $t \leqq 1/n$.

By induction on n, we may assume that the facet P' contains two vertices whose distance is at least

$$\sqrt{\frac{2n}{n-1}}\sqrt{1-t^2} \geqq \sqrt{\frac{2n}{n-1}}\sqrt{1-\frac{1}{n^2}} = \sqrt{\frac{2(n+1)}{n}}.$$

This proves the Lemma.

We do not know which values of α can be parameters of strongly self-dual polytopes, except in the trivial case $n=2$. But the following result will be sufficient for our purposes.

Theorem 1. *For each $n \geqq 2$ and $\alpha_1 < 2$ there exists a strongly self-dual polytope in $\mathbf{R}^n$ with parameter at least α_1.*

Proof. We give a construction by induction on n. For $n=2$ the assertion is obvious.

Let $n \geqq 3$ and let P_0 be a strongly self-dual polytope in dimension $n-1$ such that the parameter α_0 of P_0 satisfies $\alpha_0 > \alpha_1$. Thus the radius r_0 of the inscribed ball of P_0 satisfies $r_0 > r_1 = \alpha_1^2/2 - 1$.

We begin with an auxiliary construction in the plane. Let C be the unit circle in $\mathbf{R}^2$ and let E be an ellipse with axes 2 and $2r_0$, concentrical with C. Thus E touches C in two points x and y. Choose any t with $r_0 > t > r_1$ and let C_t denote the circle concentrical with C and with radius t. It is clear by a continuity argument that t can be chosen so that we can inscribe an odd polygon $Q = (x_0 = x, \ldots, x_{2k+1} = x)$ in E so that the sides of Q are tangent to C_t. Let α be an orthogonal affine transformation mapping E on C and let $y_0 = x_0, y_1, \ldots, y_{2k+1} = x_0$ be the images of $x_0, x_1, \ldots, x_{2k+1}$ under α.

Consider C as the "meridian" of S^{n-1} with x as the "north pole". Let S^{n-2} be the "equator" and suppose the P_0 is inscribed in the "equator". Let, for each vertex v of P_0; M_v be the "meridian" through v (so M_v is a one-dimensional semicircle). Let L_i denote the "parallel' through y_i $(i = 1, \ldots, k)$. We denote by $u(v, i)$ the intersection point of M_v and L_i. Further, let $u(v, 0) = x$ for all v. We define the polytope

$$P = \operatorname{conv}\{u(v, i)\colon v \in V(P_0);\ i = 0, \ldots, k\}.$$

(Here $V(P_0)$ denotes the set of vertices of P_0.) We prove that P is a strongly self-dual polytope with parameter $\sqrt{2+2r} > \alpha_1$.

Claim 1. *The facets of P are*

$$\operatorname{conv}\{u(v, k)\colon v \in V(P_0)\}$$

and

$$F^{(j)} = \operatorname{conv}\{u(v, i)\colon\ v \in V(F),\ \ i \in \{j, j+1\}\}$$

where F is a facet of P_0 and $0 \leqq j \leqq k-1$.

Proof. Consider the affine hull $A_F^{(j)}$ of the points $u(v, j)$ $(v \in V(F))$. Then $A_F^{(j)}$ and $A_F^{(j+1)}$ are parallel affine $(n-2)$-spaces $(1 \leqq j \leqq k-1)$ and so they span a unique hyperplane $B_F^{(j)}$. For $j = 0$, let $B_F^{(0)}$ denote the hyperplane through the affine $(n-2)$-space $A_F^{(1)}$ and x. We denote by $H_F^{(j)}$ the closed halfspace bordered by $B_F^{(j)}$ and containing the origin. Clearly $P \subset H_F^{(j)}$.

Let, further, B_0 be the affine hull of the points $u(v, k)$ $(v \in V(P_0))$ and let H_0 be the closed halfspace bordered by B_0 and containing the origin. Again, $P \subset H_0$. It is easy to see that

$$P = \bigcap_F \bigcap_{j=0}^{k-1} H_F^{(j)} \cap H_0.$$

This proves the Claim since each $B_F^{(j)}$ as well as B_0 are spanned by the vertices of P.

Claim 2. *The ball concentrical with S^{n-1} and with radius t touches every facet of P.*

Proof. This is clear for the facet B_0. Consider $B_F^{(j)}$. Let N be the 2-dimensional plane through 0 and x, and orthogonal to $B_F^{(j)}$; without loss of generality we may assume that N intersects S^{n-1} in the circle C featured in the auxiliary construction. Then since P_0 is a strongly self-dual polytope with inscribed ball radius r_0, it follows that N intersects $A_F^{(j)}$ and $A_F^{(j+1)}$ in the points x_j and x_{j+1}, respectively. Thus it intersects $B_F^{(j)}$ in the line through x_j and x_{j+1}. Since by construction, the circle C_t touches this line, it follows that the ball about 0 with radius t touches the hyperplane $B_F^{(j)}$.

Claim 3. *B_0 is orthogonal to the vector y_0. $B_{F_v}^{(k-j)}$ is orthogonal to the vector $u(v, j)$, where F_v is the facet of P_0 opposite to the vertex v.*

Proof. The first assertion is trivial. To prove the second, we use induction on j. Let w be any vertex of P_0. First we show that $u(w, k)$ is orthogonal to $B_{F_w}^{(0)}$. This follows easily on noticing that the plane D through x, 0 and $u(w, k)$ is orthogonal to $A_{F_w}^{(k)}$ by the hypothesis that P_0 is strongly self-dual, and since $A_{F_w}^{(k)} \| B_{F_w}^{(0)}$, it follows that D is also orthogonal to $B_{F_w}^{(0)}$. Since $|x-u(w, k)|=\alpha=$ $=\sqrt{2+2t}$, considering this plane D we see easily that $u(w, k)$ is orthogonal to $B_{F_w}^{(0)}$. Consequently, $u(w, k)$ is at a distance α from all vertices of the facet $B_{F_w}^{(0)}$.

We can repeat the same argument to show that $u(v, 1)$ is orthogonal to $B_{F_v}^{(k-1)}$, and then the same argument can be used to show that $u(v, k-1)$ is orthogonal to $B_{F_v}^{(1)}$, etc. This proves Claim 3 as well as Theorem 1.

2. The chromatic number of distance graphs

We now use the existence of strongly self-dual polytopes to derive lower bounds on the chromatic number of $G(n, \alpha)$, the graph obtained by connecting all pairs of points on the unit sphere S^{n-1} at distance α apart.

In [4] the following lower bound on the chromatic number of a graph was proved. Let G be a finite graph, and define its *neighborhood complex* $N(G)$ as the simplicial complex with vertex set $V(G)$, where a subset $A \subseteq V(G)$ forms a simplex if any only if the points of A have a neighbor in common.

Theorem A. *Let G be a graph and suppose that $N(G)$ is k-connected ($k \geqq 0$). Then $\chi(G) \geqq k+3$.*

The main result of this section is the following.

Theorem 2. *The graph formed by the principal diagonals of a strongly self-dual polytope in $\mathbf{R}^n$ has chromatic number $n+1$.*

One half of this Theorem follows immediately from Theorem A and the next Lemma.

Lemma 4. *Let P be a strongly self-dual polytope and let G_P be the graph formed by its vertices and principal diagonals. Then $N(G_P)$ is homotopy equivalent to the surface of P.*

Proof. Let $\overline{N(G_P)}$ denote the geometric realization of $N(G_P)$. Consider the natural bijection φ from the vertex set of $\overline{N(G_P)}$ onto the vertex set of P, and extend φ affinely over the simplices of $\overline{N(G_P)}$. This results in a continuous mapping $\bar{\varphi}\colon \overline{N(G_P)} \to \partial P$ since by the definition of the neighborhood complex and of G_P, each simplex of $\overline{N(G_P)}$ is mapped into a facet of P.

On the other hand, let $\psi = \varphi^{-1}$. Subdivide each facet of P into simplices without introducing new vertices, and let K denote the resulting simplicial complex. Then ∂P may be viewed as a geometric realization of K. Extend ψ affinely over the simplices in $\bar{K}$, to obtain a continuous mapping $\bar{\psi}\colon \partial P \to \overline{N(G_P)}$.

Now $\bar{\varphi} \circ \bar{\psi} = \mathrm{id}_{\partial P}$. Further, $\bar{\psi} \circ \bar{\varphi}$ is a simplicial map of $\overline{N(G_P)}$ into itself such that $(\bar{\psi} \circ \bar{\varphi})(S) \cup S$ is contained in a simplex of $\overline{N(G_P)}$, for every simplex S of $\overline{N(G_P)}$. Hence $\bar{\psi} \circ \bar{\varphi}$ is homotopic to $\mathrm{id}_{\overline{N(G_P)}}$, and the Lemma follows.

To complete the proof of Theorem 2, it suffices to remark that $G_P \subseteq G(n, \alpha) \subseteq B(n, \alpha)$, and even $B(n, \alpha)$ is $(n+1)$-colorable as $\alpha \geq \sqrt{2(n+1)/n}$ by Lemma 3.

Corollary 1. *If there exists a strongly self-dual polytope in $\mathbf{R}^n$ with parameter α, then $\chi(G(n, \alpha)) = n+1$.*

To treat the values α which are not parameters of strongly self-dual polytopes, we need a simple lemma.

Lemma 5. *Let $\alpha < \beta < 2$. Then $G(n-1, \beta)$ is isomorphic to a subgraph of $G(n, \alpha)$.*

Proof. Consider a hyperplane at distance $\sqrt{1-\alpha^2/\beta^2}$ from 0. This intersects the unit sphere in an $(n-2)$-sphere with radius α/β, and hence the restriction of $G(n, \alpha)$ to this hyperplane is isomorphic with $G(n-1, \beta)$.

By Theorem 1 and Lemma 5 we obtain the following.

Corollary 2. *For any $\alpha < 2$, $\chi(G(n, \alpha)) \geq n$.*

3. Concluding remarks

To determine the chromatic number of $G(n, \alpha)$ exactly appears to be a difficult question. For small values of α, $\chi(G(n, \alpha))$ grows probably exponentially fast with n; a similar result for euclidean spaces was proved by FRANKL and WILSON [3].

The situation is simpler when α is large; in this paper we have shown that for $\alpha > \sqrt{2(n+1)/n}$,

$$n \leqq \chi(G(n, \alpha)) \leqq n+1,$$

where the upper bound is attained by infinitely many values of α. If $n=2$, then the lower bound is attained for every α which is not the length of a diagonal of a regular odd polygon. We do not know if the lower bound is ever attained for $n \geqq 3$.

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