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On the homotopy type of some spaces occurring in the calculus of variations

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Dedicated to Professor B. Sz.-Nagy on the occasion of his 70th birthday

1. Let $n\in\mathbf{N}$ and let $D\subset\mathbf{R}\times\mathbf{R}^n$ be an open region. Suppose $\xi_0, \xi_1\in\mathbf{R}^n$ are given such that $(0, \xi_0)$, $(1, \xi_1)\in D$. Denote by $M(D)$ the class of continuous functions $x: [0, 1]\to\mathbf{R}^n$ such that

$$(1)\qquad x(0) = \xi_0,\ x(1) = \xi_1, \quad \text{and} \quad \Gamma(x) := \{(t, x(t))\,|\,t\in[0, 1]\} \subset D.$$

The space of $\mathbf{R}^n$-valued continuous functions over $[0, 1]$ will be denoted by $C_n[0, 1]$. Thus $M(D)$ is a subspace of $C_n[0, 1]$. Endow $M(D)$ with the relative topology of $C_n[0, 1]$.

The global methods of the calculus of variations (see [1], [3], [5] and [6]) lead us to the following problem: how can the homotopy type of $M(D)$ be described from that of D? In this paper we establish a connection between the homotopy types of the spaces D and $M(D)$ for a rather wide class of regions D. We shall define a class of admissible regions and for this class we shall prove the following theorem.

Theorem. *Suppose $D\subset\mathbf{R}\times\mathbf{R}^n$ is an admissible region and its homotopy type is the one point union $S^{r_1}\vee S^{r_2}\vee\ldots\vee S^{r_k}$ of the spheres S^{r_i} of dimension $r_i\geqq 1$ $(i=1, 2, \ldots, k)$. Then the homotopy type of $M(D)$ is the one point union $S^{r_1-1}\vee S^{r_2-1}\vee\ldots\vee S^{r_k-1}$ of the spheres S^{r_i-1} $(i=1, 2, \ldots, k)$.*

2. In this section the necessary definitions and constructions will be given.

Definition 1. The regions $D_1, D_2\subset\mathbf{R}^{n+1}$ satisfying (1) will be called t-invariantly homeomorphic, if there exists a uniformly continuous homeomorphism $\varphi: D_1\to D_2$ such that

$$\text{a)}\qquad \varphi(0, \xi_0) = (0, \xi_0),\ \varphi(1, \xi_1) = (1, \xi_1),$$

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b) the diagram

$$\begin{array}{ccc} \varphi\colon D_1 & \to & D_2 \\ & {\scriptstyle \mathrm{pr}_1}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_1} & \\ & \mathbf{R} & \end{array}$$

is commutative where $\mathrm{pr}_1\colon \mathbf{R}^1\times\mathbf{R}^n\to\mathbf{R}^1$ is the projection of the space $\mathbf{R}^1\times\mathbf{R}^n$ onto the first factor.

Denote by $I_n\subset\mathbf{R}^n$ the n-dimensional open unit interval $\bigtimes_{i=1}^{n}]0,1[$. Let k, i ($i\leqq k$) and r ($r\leqq n$) be positive integers and $\delta\in]0,1/2[$ a real number. For the ordered quadruple (k,i,r,δ) define the set $Q(k,i,r,\delta)$ as the product

$$\Big(\bigtimes_{j=1}^{n-r}]0,1[\Big)\times\Big(\bigtimes_{j=n-r+1}^{n-1}\Big[\frac{1}{2}-\delta,\frac{1}{2}+\delta\Big]\Big)\times\Big[\frac{2i-1-\delta}{2k},\frac{2i-1+\delta}{2k}\Big].$$

Now, suppose that the positive integers n, k are given. Let $r=(r_1, r_2, \ldots, r_k)\in\mathbf{N}^k$ ($r_i\leqq n$ for $i=1,2,\ldots,k$), $\alpha,\beta,\delta\in I_k$. Suppose that $\alpha_i<\beta_i$ for all $i=1,2,\ldots,k$ and $2\delta\in I_k$. The set $D(k,r,\alpha,\beta,\delta)\subset\mathbf{R}\times\mathbf{R}^n$ will be given in the following manner:

$$D(k,r,\alpha,\beta,\delta):=\{(t,x)\in\mathbf{R}\times\mathbf{R}^n \mid t\in[0,1],\ x\in I_n,\ \text{and if } t\in[\alpha_i,\beta_i] \text{ then}$$
$$x\notin Q(k,i,r_i,\delta_i)\}.$$

Definition 2. A region $D\subset\mathbf{R}^{n+1}$ is said to be admissible if there exist $k\in\mathbf{N}$, $r\in\mathbf{N}^k$ ($r_i\leqq n$, $i=1,2,\ldots,k$), $\alpha,\beta,\delta\in I_k$ ($\alpha_i<\beta_i$, $i=1,2,\ldots,k$, $2\delta\in I_n$), such that the intersection $\bigcap_{i=1}^{k}]\alpha_i,\beta_i[$ is nonempty, and D and $D(k,r,\alpha,\beta,\delta)$ are t-invariantly homeomorphic regions.

Remark. It can be easily seen that the homotopy type of the regions $D(k,r,\alpha,\beta,\delta)$ (and thus that of D) is the one point union $S^{r_1}\vee S^{r_2}\vee\ldots\vee S^{r_k}$.

Now, choose real numbers $\alpha_0, \beta_0, \alpha', \beta', t_0$ such that $0<\alpha'<\alpha_0<t_0<\beta_0<\beta'<1$. Define the function $f\colon[0,1]\to[0,1]$ in the following way:

$$f(t):=\begin{cases} t, & t\in[0,\alpha']\cup[\beta',1],\\ \alpha'-\dfrac{t-\alpha'}{\alpha_0-\alpha'}(t_0-\alpha'), & t\in[\alpha',\alpha_0],\\ t_0, & t\in[\alpha_0,\beta_0],\\ t_0+\dfrac{t-\beta_0}{\beta'-\beta_0}(\beta'-t_0), & t\in[\beta_0,\beta']. \end{cases}$$

The restriction of f to the set $[0, \alpha_0]\cup]\beta_0, 1[$ is invertible and the inverse also can be easily given:

$$(f|_{[0,\alpha_0]\cup]\beta_0,1[})^{-1}(t) = \begin{cases} t, & t\in[0, \alpha']\cup[\beta', 1], \\ \alpha'+\dfrac{t-\alpha'}{t_0-\alpha'}(\alpha_0-\alpha'), & t\in[\alpha', t_0], \\ \beta_0+\dfrac{t-t_0}{\beta'-t_0}(\beta'-\beta_0), & t\in]t_0, \beta']. \end{cases}$$

Let $k\in\mathbf{N}$, $r\in\mathbf{N}^k$ ($r_i\leqq n$, $i=1,2,\ldots,k$), $\alpha', \beta'\in]0,1[$ ($\alpha'<\beta'$), $\delta\in\mathbf{R}^k$ ($2\delta\in I_k$) be given. Define the subspace $M(k, r, \alpha', \beta', \delta)\subset C_n[0, 1]$ as follows:

$$M(k, r, \alpha', \beta', \delta) := \{x\in C_n[0, 1] \mid x|_{[\alpha',\beta']} \text{ const.}, \ \xi(x) =: x(\alpha'),$$

$$\xi(x)\in I_n\setminus\bigcap_{i=1}^{k} Q(k, i, r, \delta), \quad x(t) = \xi_0+\frac{t}{\alpha'}(\xi(x)-\xi_0) \quad (t\in[0, \alpha']),$$

$$x(t) = \xi(x)+\frac{t-\beta'}{1-\beta'}(\xi_1-\xi(x)) \quad (t\in[\beta', 1])\}.$$

Finally, denote by j the identity map of $[0, 1]$.

3. We start with a simple observation.

Lemma 1. *If the regions $D_1, D_2\subset\mathbf{R}^{n+1}$ satisfying the condition* (1) *are t-invariantly homeomorphic, then $M(D_1)$ and $M(D_2)$ are homeomorphic.*

Proof. Let $\varphi: D_1\to D_2$ be a t-invariant homeomorphism (in this case, obviously, the inverse $\varphi^{-1}: D_2\to D_1$ is also a t-invariant homeomorphism). Define the desired homeomorphism $\Phi: M(D_1)\to M(D_2)$ as follows:

$$(\Phi(x))(t) := \mathrm{pr}_2\varphi(t, x(t)) \quad (t\in[0, 1]),$$

where $\mathrm{pr}_2: \mathbf{R}^1\times\mathbf{R}^n\to\mathbf{R}^n$ is the projection of the product space $\mathbf{R}^1\times\mathbf{R}^n$ onto the second factor. From the t-invariance of the homeomorphism φ it follows immediately, that Φ is a homeomorphism. It is also clear that Φ^{-1} has a form similar to that of Φ:

$$(\Phi^{-1}(x))(t) = \mathrm{pr}_2\,\varphi^{-1}(t, x(t)) \quad (t\in[0, 1]).$$

From Lemma 1 it follows that it is sufficient to determine the homotopy type of the spaces $M(D(k, r, \alpha, \beta, \delta))$. We now turn to the calculation of the homotopy type of the space $M(k, r, \alpha', \beta', \delta)$. For this purpose we shall prove the following

Lemma 2. *The homotopy type of the space $M(k, r, \alpha', \beta', \delta)$ is the one point union $S^{r_1-1}\vee S^{r_2-1}\vee\ldots\vee S^{r_k-1}$ of the spheres S^{r_i-1} $(i=1, 2, \ldots, k)$.*

Proof. It is obvious that the space $M(k, r, \alpha', \beta', \delta)$ is homeomorphic to the n-dimensional region $I_n\setminus\bigcup_{i=1}^{n} Q(k, i, r, \delta)$. The desired homeomorphism Ψ can be given by $(\xi|_{M(k,r,\alpha',\beta',\delta)})^{-1}$, where ξ is the function from the end of the 2nd point. Now, by the definition of the sets $Q(k,i,r,\delta)$ the region $I_n \setminus \bigcup_{i=1}^{k} Q(k,i,r,\delta)$ is homotopically equivalent to the one point union $S^{r_1-1} \vee S^{r_2-1} \vee \ldots \vee S^{r_k-1}$ of the spheres S^{r_i-1} $(i=1,2,\ldots,k)$.

Choose numbers $t_0 \in \bigcap_{i=1}^{k}]\alpha_i, \beta_i[$ and $\alpha_0, \beta_0; \alpha', \beta' \in]0,1[$ such that the inequalities

$$0 < \alpha' < \alpha_0 < \min_{1 \le i \le k} \{\alpha_i\} < \max_{1 \le i \le k} \{\beta_i\} < \beta_0 < \beta' < 1$$

are satisfied.

Lemma 3. *The space* $M(k,r,\alpha',\beta',\delta)$ *is a deformation retract of the space* $M(D(k,r,\alpha,\beta,\delta))$.

Proof. A homotopy

$$F\colon [0,1] \times M(D(k,r,\alpha,\beta,\delta)) \to M(D(k,r,\alpha,\beta,\delta))$$

is defined by

$$F(\tau,x) := \begin{cases} x \circ (2\tau f + (1-2\tau) j), & \tau \in [0, 1/2], \\ (2-2\tau) x \circ f + (2\tau-1) \Psi(x(t_0)), & \tau \in [1/2, 1], \end{cases}$$

where ψ is the function from the preceding proof.

The restriction of $x \mapsto F(\tau,x)$ to $M(k,r,\alpha',\beta',\delta)$ is the identity, because the elements of $M(k,r,\alpha',\beta',\delta)$ are constant over $[\alpha',\beta']$ and linear over the rest of $[0,1]$, consequently

$$x \circ (2\tau j + (1-2\tau) f)|_{[\alpha',\beta']} = \xi(x) = x|_{[\alpha',\beta']},$$

$$x \circ (2\tau j + (1-2\tau) f)|_{[0,\alpha'] \cup [\beta',1]} = x \circ (2\tau j + (1-2\tau) j)|_{[0,\alpha'] \cup [\beta',1]} = x|_{[0,\alpha'] \cup [\beta',1]},$$

for $\tau \in [0, 1/2]$, and

$$[(2-2\tau) x \circ f + (2\tau-1) \Psi(x(t_0))]_{[\alpha',\beta']} = (2-2\tau) x(t_0) + (2\tau-1) x(t_0) = x|_{[\alpha',\beta']},$$

$$[(2-2\tau) x \circ f + (2\tau-1) \Psi(x(t_0))]_{[0,\alpha'] \cup [\beta',1]} = [(2-2\tau) x + (2\tau-1) x]|_{[0,\alpha'] \cup [\beta',1]} =$$

$$= x|_{[0,\alpha'] \cup [\beta',1]},$$

for $\tau \in [1/2, 1]$.

The function $x \mapsto F(0,x)$ is the identity over $M(D(k,r,\alpha,\beta,\delta))$. The function $x \mapsto F(1,x)$ is a retract of $M(D(k,r,\alpha,\beta,\delta))$ onto $M(k,r,\alpha',\beta',\delta)$. The proof of Theorem follows immediately from Lemmas 1—3.

If $n=2$ and the region D is $I_3 \setminus \{(t, 1/2, 1/2) \mid t \in [0,1]\}$, furthermore $\xi_0 = \xi_1 = (1/2, 1/3)$, then the homotopy type of $M(D)$ is the one point union $\bigvee_{i=-\infty}^{\infty} S^0$ of infinitely many 0-dimensional spheres. There are as many spheres as there are different ways to wind the graphs of the functions around the omitted segment.

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