Acta Sci. Math., 45 (1983), 253—260

On arithmetic functions with regularity properties

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Dedicated to Professor Béla Szőkefalvi-Nagy on his 70th anniversary

1. We shall say that an additive function $f(n)$ is of finite support if $f(p^\alpha)=0$ whenever p is a large prime. Let

$$P(z)=\alpha_0+\alpha_1 z+\ldots+\alpha_k z^k, \quad \alpha_k=1, \quad \alpha_0\neq 0$$

be an arbitrary polynomial with complex coefficients. The operators E, Δ, I are defined by the following relations:

$$Ex_n=x_{n+1}, \quad \Delta x_n=x_{n+1}-x_n, \quad Ix_n=x_n.$$

We are interested in the following problem: What is the set of additive functions $f(n)$ satisfying the relation

$$P(E)f(n)\to 0 \quad (n\to\infty). \tag{1.1}$$

This question was raised in [1]. Recently we solved it for completely additive functions. Namely, from a famous result of E. Wirsing we deduced that if a completely additive function $f(n)$ satisfies the relation

$$\frac{P(E)f(n)}{\log n}\to 0, \tag{1.2}$$

then $f(n)$ is a constant multiple of $\log n$; $f(n)=c\log n$ satisfies (1.2) with $c\neq 0$ only if $P(1)=0$. In the same paper we proved that for a completely additive function $f(n)$,

$$\frac{1}{x}\sum_{n\leqq x}|P(E)f(n)|\to 0 \quad (x\to\infty) \tag{1.3}$$

implies that $f(n)=c\log n$. The method used there cannot be applied without change to additive functions. Now we shall show how we can modify the method so as to be suitable for additive functions.

Received January 27, 1982.

Theorem 1.1. *If* (1.3) *holds for a complex valued additive function* $f(n)$, *then* $f(n)=c\log n+f_1(n)$ *where* $f_1(n)$ *is an additive function of finite support satisfying the recursion*

$$P(E)f_1(n)=0 \quad (n=1,2,\ldots). \tag{1.4}$$

If $P(1)\neq 0$, *then* $c=0$.

Theorem 1.2. *If* $f(n)$ *is a complex valued additive function satisfying the linear recursion*

$$P(E)f(n)=0 \quad (n=1,2,\ldots), \tag{1.5}$$

then

1) $f(p^\alpha)=0$ *for every prime power* p^α *satisfying* $p>k+1$,

2) $f(p^{\gamma+1})=f(p^\gamma)$ *if* $p^{\gamma+1}-p^\gamma>k+1$,

3) $f(n)$ *is periodic with* B *where* $B=\prod_{p\leq k+1}p^{\gamma_p}$ *and* γ_p *is the smallest integer satisfying* $p^{\gamma_p+1}-p^{\gamma_p}>k+1$.

A modification of Theorem 1.2 was proved earlier by L. Lovász, A. Sárközy and M. Simonovits [2]. We shall deduce it immediately from Theorem 1.1.

Proof of Theorem 1.1. If the relation

$$\frac{1}{x}\sum_{n\leq a}|k(E)f(n)|\to 0 \quad (x\to\infty) \tag{1.6}$$

holds for a polynomial $k(z)$ then it holds for any other polynomial $K(z)$ that is a multiple of $k(z)$. Let $P(z)=\prod_{i=1}^{k}(z-\theta_i)$, and for a fixed integer $m>1$, let $Q_m(z^m)=\prod_{i=1}^{k}(z^m-\theta_i^m)$. Since $P(z)$ divides $Q_m(z^m)$, therefore

$$\frac{1}{x}\sum_{n\leq x}|Q_m(E^m)f(n)|\to 0,$$

and so

$$\frac{1}{x}\sum_{mn\leq x}|Q_m(E^m)f(nm)|\to 0. \tag{1.7}$$

Let $Q_m(z)=\beta_0+\beta_1 z+\ldots+\beta_k z^k$ $(\beta_k=1)$; $\Delta(m,n)=\sum_{j=0}^{k}\beta_j\{f(m(n+j))-f(n+j)\}$. Then

$$\Delta(m,n)=Q_m(E^m)f(nm)-Q_m(E)f(n). \tag{1.8}$$

Applying the operator $P(E)$ and taking into account that $P(E)|P(E)Q_m(E)$, we yet that

$$\frac{1}{x}\sum_{n\leq x}|P(E)\Delta(m,n)|\leq\frac{1}{x}\sum_{n\leq x}|P(E)Q_m(E^m)f(nm)|+\frac{1}{x}\sum_{n\leq x}|P(E)Q_m(E)f(n)|,$$

whence

$$\lim_{x\to\infty}\frac{1}{x}\sum_{n\le x}|P(E)\Delta(m,n)| = 0. \tag{1.9}$$

Let now $P>2k+1$ be a prime, and let n run over the set satisfying $P^\nu\|n$ with $\nu\geqq 1$ fixed. Then

$$\Delta(P,n) = \beta_0\{f(P^{\nu+1})-f(P^\nu)\}+(\beta_1+\dots+\beta_k)f(P) =$$
$$= \beta_0\{f(P^{\nu+1})-f(P^\nu)-f(P)\}+Q_m(1)f(P),$$
$$\Delta(P,n+h) = f(P)Q_m(1) \quad (0<h\leqq 2k).$$

Consequently

$$P(E)\Delta(m,n) = P(1)Q_m(1)f(P)+\alpha_0\beta_0\{f(P^{\nu+1})-f(P^\nu)-f(P)\}. \tag{1.10}$$

Observing that the set of n's has a positive density, we get that

$$P(1)Q_m(1)f(P)+\alpha_0\beta_0\{f(P^{\nu+1})-f(P^\nu)-f(P)\}. \tag{1.11}$$

Let now n run over the integers $\equiv 1 \pmod P$. Then we have $\Delta(P,n+h)=$ $=f(P)Q_m(1)$ $(0\leqq h\leqq 2k)$, and so $P(E)\Delta(m,n)=P(1)Q_m(1)f(P)$. Repeating the above argument we get $P(1)Q_m(1)f(P)=0$. Since $P(1)\neq 0$ implies that $Q_m(1)\neq 0$, we have $f(P)=0$ provided $P(1)\neq 0$. From (1.11) we get that

$$f(P^{\nu+1})-f(P^\nu)-f(P) = 0 \quad (\nu = 1, 2, \dots),$$

and hence $f(P^\nu)=\nu f(P)$ $(\nu\geqq 1)$.

Let P be an arbitrary prime, and let γ_0 be so large that $P^{\gamma_0}>2k+1$. Let $\varepsilon_1,\dots,\varepsilon_{2k}$ be fixed nonnegative integers such that $P^{\gamma_0}\|n$ and $P^{\varepsilon_i}\|n+i$ $(i=1,\dots,2k)$ hold for at least one n. Let A_γ denote the set of those n's for which $P^\gamma\|n$ and $P^{\varepsilon_i}\|n+i$ $(i=1,\dots,2k)$. The following assertion is obvious: A_γ is nonempty for $\gamma\geqq\gamma_0$ and it has a positive density.

Clearly $P(E)\Delta(n,P)$ is constant if n runs over the elements of A_γ; therefore it equals 0 on A_γ. Hence

$$P(E)\Delta(n_1,P)-P(E)\Delta(n_2,P) = 0$$

if $n_1\in A_{\gamma+1}$; $n_2\in A_\gamma$ $(\gamma\geqq\gamma_0)$. Consequently

$$\alpha_0\beta_0\{f(P^{\gamma+2})-f(P^{\gamma+1})\} = \alpha_0\beta_0\{f(P^{\gamma+1})-f(P^\gamma)\},$$

and from $\alpha_0\beta_0\neq 0$ we get that

$$\xi_{\gamma+1} = \xi_\gamma \quad (\gamma\geqq\gamma_0), \quad \xi_\gamma = f(P^{\gamma+1})-f(P^\gamma). \tag{1.12}$$

Now we write $f(n)$ as $f_1(n)+f_2(n)$ where $f_2(n)$ is a completely additive function defined as follows: $f_2(P^\alpha)=f(P^\alpha)$ if $P>2k+2$. Then $f_1(P^\alpha)=0$ if $P>2k+2$. For a smaller prime P we put $f_2(P)=\xi_{\gamma_0}$; which implies by (1.12) that $f_1(P^{j+1})=$ $=f_1(P^j)$ if $j\geqq\gamma_0$.

Now we have shown that $f_1(n)$ is a function of finite support; and it is periodic with a period B_1. Consequently

$$P(E)(E^{B_1}-I)f_1(n) = 0.$$

Taking into account the relation

$$(E^{B_1}-I)P(E)f(n) = P(E)(E^{B_1}-I)f_1(n)+(E^{B_1}-I)P(E)f_2(n) = (E^{B_1}-I)P(E)f_2(n),$$

we have

$$\frac{1}{x}\sum_{n\leqq x}|(E^{B_1}-I)P(E)f_2(n)| \to 0. \tag{1.13}$$

From the theorem cited above we get that $f_2(n)=c\log n$. Earlier we have proved that $f(P)=f_2(P)=0$ for every large P, provided $P(1)\neq 0$. This implies that $f_2(n)\equiv 0$; furthermore, from the periodicity of $f_1(n)$ and from (1.3) we get that $P(E)f_1(n)\equiv 0$ $(n=1, 2, \ldots)$.

Assume now that $P(1)=0$. Then $P(E)\,c\log n\to 0$, whence (1.3) yields that

$$\frac{1}{x}\sum_{n\leqq x}|P(E)f_1(n)| \to 0 \quad (x\to\infty).$$

Using the periodicity of $f_1(n)$ we get that $P(E)f_1(n)\equiv 0$ $(n=1, 2, \ldots)$. This finishes the proof of Theorem 1.1.

Proof of Theorem 1.2. Since (1.5) implies (1.3), we get that $f(n)=f_1(n)+ +c\log n$, $P(E)f_1(n)=0$; moreover, by (1.5), $P(E)c\log n=P(E)(f(n)-f_1(n))\equiv 0$, which is impossible for $c\neq 0$. Therefore we have that $f(n)=f_1(n)$ is of finite support. Then there exists a K such that $f(p^\alpha)=0$ for each prime $p>K$. For an integer n let $A_K(n)$ denote the product of all prime factors of n not greater than K. Let $\delta(n)$ be the exact exponent of p in $n: p^{\delta(n)}\|n$, and set $A'_K(n)=p^{-\delta(n)}A_K(n)$.

Let n_1 be chosen so that $\delta(n_1)=\gamma\geqq 0$, $n_1\equiv p^\gamma \pmod{p^{\gamma+1}}$, and let γ be so large that $p^{\gamma+1}-p^\gamma>k+1$. Then we can find an integer n_2 satisfying the following relations: $\delta(n_2)=\gamma+1$, $A'_K(n_1)=A'_K(n_2)$, $A_K(n_1+j)=A_K(n_2+j)$ $(j=1, \ldots, k)$. Taking into account the equality $f(n)=f(A_K(n))$, we get from (1.5) that

$$0 = P(E)f(n_2)-P(E)f(n_2) = \alpha_0\{f(P^{\gamma+1})-f(P^\gamma)\},$$

which by $\alpha_0\neq 0$ implies that $f(P^{\gamma+1})=f(P^\gamma)$.

Thus 1) and 2) are proved; 3) is an immediate consequence of them.

Remark. The assertion of Theorem 1.1 remains true if (1.3) is replaced by

$$\liminf\frac{1}{\log x}\sum_{n\leqq x}\frac{1}{n}|P(E)f(n)| = 0. \tag{1.3$'$}$$

2. Theorem 2.1. *Let f be a completely additive real valued function, and let P be a nonzero polynomial with rational coëfficients satisfying the relation*

$$A_P P(E) f(n) \equiv 0 \pmod 1 \tag{2.1}$$

with a suitable integer $A_P \neq 0$. Then there exists an integer B such that $f(n)=g(n)/B$, where $g(n)$ is an integer valued additive function.

First we prove the following

Lemma 2.1. *If $\Delta^k f(n) \equiv 0 \pmod 1$ $(n=1,2,\ldots)$ for a $k \geqq 1$, and $f(n)$ is completely additive, then $f(n) \equiv 0 \pmod 1$.*

Proof. Let us assume that $k=1$. Then summing the congruences $f(n+1)-f(n) \equiv 0 \pmod 1$ for $n=pu, \ldots, qu-1$, we have $f(q)-f(p) \equiv 0 \pmod 1$ for each pair p, q which by $q=np$ gives that $f(n) \equiv 0 \pmod 1$.

Now we use induction on k. Assume that our lemma is true for k, and consider the condition $\Delta^{k+1} f(n) \equiv 0 \pmod 1$. Starting from

$$\sum_{n=1}^{N-1} \Delta^{k+1} f(n) = \Delta^k f(N) - \Delta^k f(1) \equiv 0 \pmod 1,$$

we get

$$\Delta^k f(N) \equiv c \pmod 1, \quad c = \Delta^k f(1).$$

If Q is an arbitrary polynomial with integer coefficients, then

$$(E-I)^k Q(E) f(N) \equiv c Q(1) \pmod 1.$$

Let $Q(z)=Q_m(z)=\left(\frac{z^m-1}{z-1}\right)^k=(1+z+\ldots+z^{m-1})^k$. Then $(E-I)^k Q_m(E)=(E^m-I)^k$, consequently

$$(E^m-I)^k f(mN) \equiv c Q_m(1) \pmod 1;$$

furthermore;

$$(E^m-I)^k f(mN) \equiv (E-I)^k f(N) \equiv c \pmod 1,$$

whence $c(Q_m(1)-1) \equiv 0 \pmod 1$. Since $Q_m(1)=m^k$, we get $c(m^k-1) \equiv 0 \pmod 1$ $(m=2,3,\ldots)$. Therefore c is a rational number. Let $c=A/B$, where A, B are coprime integers. If $B \neq 1$, then by choosing $m=B$, we get $c(B^k-1) \equiv 0 \pmod 1$, $c \equiv 0 \pmod 1$, which is a contradiction. This completes the proof of the lemma.

Proof of Theorem 2.1. Let A be the set of all polynomials P with rational coefficients for which

$$A_P P(E) f(n) \equiv 0 \pmod 1$$

holds with a suitable integer A_P. Then A is an ideal in $R[x]$. Let $P(z)=\prod_{j=1}^{k}(z-\theta_j) \in A$. From the fundamental theorem of symmetric polynomials it follows that

$$k_m(z) = \prod_{j=1}^{k}{}' \frac{z^m-\theta_j^m}{z-\theta_j}$$

has rational coefficients; consequently

$$R_m(z^m) = \prod_{j=1}^{k} (z^m-\theta_j^m) \in A.$$

Furthermore, $R_m(E^m)f(nm)=R_m(1)f(m)+R_m(E)f(n)$. Let F be an integer such that $FR_m(E^m)f(n)\equiv 0 \pmod 1$. Then we have

$$FR_m(1)f(m)+FR_m(E)f(n) \equiv 0 \pmod 1.$$

If $R_m(1)=0$, then $R_m\in A$. If $R_m(1)\neq 0$, then applying the operator Δ we get that

$$FR_m(E)\Delta f(n) \equiv 0 \pmod 1,$$

whence $R_m(z)(z-1)\in A$.

Let P be the generator element of A, that is, a polynomial of minimum degree in A. Let $\deg P=k$. From (2.1) we get that A is not empty. If $k=0$, then our theorem is obviously true. For $k\geqq 1$ assume first that $P(1)=0$. Then

$$\delta(z) = (P(z), R_m(z))\in A, \tag{2.2}$$

implying $\deg \delta(z)=k$, i.e., $R_m(z)\equiv P(z)$,

$$\{\theta_1, \dots, \theta_k\} = \{\theta_1^m, \dots, \theta_k^m\} \quad (m = 2, 3, \dots), \tag{2.3}$$

whence it follows that $\theta_1=\dots=\theta_k=1$, $P(z)=(z-1)^k$. Assume now that $P(1)\neq 0$. Then

$$\delta(z) = (P(z), R_m(z)(z-1))\in A,$$

consequently $\deg\delta(z)=k$, and from $(z-1, P(z))=1$ we get that $P(z)=R_m(z)$ $(m=2, 3, \dots)$, which implies (2.3), and so $\theta_1=\dots=\theta_k=1$, which is impossible.

Thus we have proved the following assertion: If (2.1) holds with a suitable P then there exists an integer $F\neq 0$ and an integer $k>0$ such that

$$F\Delta^k f(n) \equiv 0 \pmod 1. \tag{2.4}$$

Using Lemma 2.1 with $Ff(n)$ instead of $f(n)$ we get that $Ff(n)$ is an integer for every n. This finishes the proof of the theorem.

3. Conjecture. Let $P(z)=1+\alpha_1 z+\dots+\alpha_k z^k$ $(k\geqq 1)$ be a polynomial with at least one irrational coefficient. If a completely additive function $f(n)$ satisfies the relation $P(E)f(n)\equiv 0 \pmod 1$ $(n=1, 2, \dots)$ then $f(n)$ is identically zero.

Theorem 3.1. *The conjecture is true for* $k=2$.

Proof. Let $\xi=f(2)$, $\eta=f(3)$. From $P(E)f(1)\equiv 0$ we get that

$$\alpha_1\xi \equiv -\alpha_2\eta \pmod 1, \tag{3.1}$$

and from $P(E)f(2)\equiv 0$ that

$$(2\alpha_2+1)\xi+\alpha_1\eta\equiv 0 \pmod 1. \tag{3.2}$$

Similarly, by considering $P(E)f(n)\equiv 0 \pmod 1$ for $n=7$ and $n=6$, and taking into account (3.1) we deduce:

$$f(7)\equiv -\alpha_1 f(8)-\alpha_2 f(9)\equiv -3\alpha_1\xi-2\alpha_2\eta \pmod 1,$$

$$f(7)\equiv \alpha_2\eta \pmod 1, \tag{3.3}$$

$$\alpha_1 f(7)\equiv -f(6)-\alpha_2 f(8) \pmod 1,$$

$$\alpha_1 f(7)\equiv -(1+3\alpha_2)\xi-\eta \pmod 1. \tag{3.4}$$

Similarly,

$$\alpha_1 f(5)\equiv -(2+\alpha_2)\xi-\alpha_2\eta \pmod 1 \quad (n=4). \tag{3.5}$$

Starting from $P(E)f(14)\equiv 0 \pmod 1$ we get

$$(\xi+4\alpha_2\xi)+f(7)+\alpha_1 f(3)+\alpha_1 f(5)\equiv 0 \pmod 1.$$

Substituting (3.3) and (3.5) into the left hand side, we get $(1+4\alpha_2)\xi+\alpha_2\eta+\alpha_1\eta-$ $-(2+\alpha_2)\xi-\alpha_2\eta\equiv 0 \pmod 1$, whence $-\xi+3\alpha_2\xi+\alpha_1\eta\equiv 0 \pmod 1$, and, by (3.2),

$$\alpha_2\xi\equiv 2\xi \pmod 1, \tag{3.6}$$

$$\alpha_1\eta\equiv -5\xi \pmod 1. \tag{3.7}$$

For $n=26$ and $n=13$ we have

$$f(2\cdot 13)+\alpha_1 f(3^3)+\alpha_2 f(2^2\cdot 7)\equiv 0 \pmod 1$$

$$f(13)+\alpha_1 f(2\cdot 7)+\alpha_2 f(3\cdot 5)\equiv 0 \pmod 1,$$

where by subtraction we get

$$\alpha_2 f(7)\equiv 3\xi-2\eta-2\alpha_1\xi \pmod 1. \tag{3.8}$$

Considering $n=5$ and taking into account (3.8) we get

$$f(5)\equiv \alpha_1\xi-\alpha_1\eta-3\xi+2\eta \pmod 1. \tag{3.9}$$

Putting now $n=25$ and $n=12$ we get that

$$f(5^2)+\alpha_1 f(2\cdot 13)+\alpha_2 f(3^3)\equiv 0 \pmod 1,$$

$$f(12)+\alpha_1 f(13)+\alpha_2 f(14)\equiv 0 \pmod 1.$$

Subtracting them and by using (3.8), (3.9) we deduce that

$$5\eta-3\xi+2\alpha_1\xi\equiv 0 \pmod 1. \tag{3.10}$$

From $n=3$ we get

$$\alpha_2 f(5)\equiv -\eta-2\alpha_1\xi \pmod 1. \tag{3.11}$$

Putting now $n=48$ we have

$$f(2^3\cdot 3)+\alpha_1 f(7^2)+\alpha_2 f(5^2\cdot 2)\equiv 0 \pmod 1,$$

and by (3.11) and (3.4) we get

(3.12) $$9\xi+3\eta+4\alpha_1\xi \equiv 0 \pmod 1.$$

Since $f(2^3)+\alpha_1 f(3^2)+\alpha_2 f(2)+\alpha_2 f(5)\equiv 0 \pmod 1$, we get that

$$\alpha_2 f(5) \equiv -5\xi-2\alpha_1 \pmod 1$$

(see (3.6), (3.7)) which implies by (3.11) that

(3.13) $$\eta+5\xi+2\alpha_1\xi \equiv 0 \pmod 1.$$

Now from (3.10), (3.12), and (3.13) we infer that

$$7\xi-7\eta \equiv 0 \pmod 1 \quad\text{and}\quad 4\eta-8\xi \equiv 0 \pmod 1,$$

which proves that ξ and η are rational numbers. Assume now that $\xi\neq 0$ and $\eta\neq 0$. Then (3.6) and (3.7) show that α_1 and α_2 are rational numbers, and the proof is finished. Let $\xi=0$ and $\eta\neq 0$. Then by (3.7) and (3.1) we get that α_1 and α_2 are rational numbers. In the case $\eta=0$, $\xi\neq 0$ we use (3.6) and (3.1) to derive the same result.

Finally, let us assume that $\xi=0$, $\eta=0$, and P is the smallest prime for which $f(P)\neq 0$. Since $P>3$, therefore $P+1$ is a composite number, $f(P+1)=0$, and so $\alpha_1 f(P+1)\equiv 0 \pmod 1$. Let us consider the relation

(3.14) $$f(P)+\alpha_1 f(P+1)+\alpha_2 f(P+2) \equiv 0 \pmod 1.$$

If $P+2$ is a composite number then $f(P+2)=0$, and so $f(P)\equiv 0 \pmod 1$. Using that $\alpha_1 f(P)\equiv 0 \pmod 1$, $\alpha_2 f(P)\equiv 0 \pmod 1$, and that $f(P)\neq 0$, we deduce that α_1 and α_2 are integers. Assume now that $P+2$ is a prime number. If $f(P+2)=0$ then we are done as before. Let $f(P+2)\neq 0$. Then

$$f(P+2)+\alpha_1 f(P+3)+\alpha_2 f(P+4) \equiv 0 \pmod 1,$$

and $P+3$, $P+4$ are composite numbers with prime factors smaller than P, whence it follows that $f(P+3)=f(P+4)=0$ and $f(P+2)\equiv 0 \pmod 1$. Since

$$f(P+1)+\alpha_1 f(P+2)+\alpha_2 f(P+3) \equiv 0 \pmod 1,$$

we have $\alpha_1 f(P+2)\equiv 0 \pmod 1$, and so α_1 is a rational number. (3.14) implies that α_2 is also rational. The proof is complete.

References

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