*Acta Sci. Math.*, 45 (1983), 213—218

# Von Neumann's coordinatization theorem [1]

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*In Honour of Béla Szőkefalvi-Nagy on his 70th birthday*

**1. Notation.** $L$ denotes a complemented, modular lattice with homogeneous basis $a_1, \ldots, a_N$, $N \geqq 4$ [2, Part II, Def. 3.1]; $A^j \equiv a_1 \vee \ldots \vee a_j$; $ab$ means $a \wedge b$; $a \dot{\vee} b$ means $a \vee b$ if $ab=0$; $L_{ji} \equiv (b \in L: b \dot{\vee} a_j = a_i \dot{\vee} a_j)$.

If $\mathscr{R}$ is a ring and $m \leqq N$, then $\mathscr{R}^N(m)$ denotes the *right* $\mathscr{R}$-module $((\alpha_1, \ldots, \alpha_N)$: all $\alpha_i \in \mathscr{R}$ and $\alpha_i = 0$ for $m < i \leqq N)$; $(\alpha_1, \ldots, \alpha_m)_N$ is an abbreviation for $(\alpha_1, \ldots, \alpha_m, 0, \ldots, 0) \in \mathscr{R}^N(m)$; $\mathscr{R}^N \equiv \mathscr{R}^N(N)$; $L(\mathscr{R}^N(m))$ denotes the set of finitely generated submodules of $\mathscr{R}^N(m)$, ordered by inclusion.

**2. Von Neumann's theorem.** *In each* $L_{ji}$ $(j \neq i)$, *addition and multiplication can be defined so that:*

(2.1) *The* $L_{ji}$ *become regular rings with unit, isomorphic to a common regular ring* $\mathscr{R}$ [2, Part II, Theorem 9.2].

(2.2) *For each* $j$ *the sublattice* $(b \in L: b \leqq a_j)$ *is isomorphic to* $L(\mathscr{R})$, *the lattice of principal right ideals of* $\mathscr{R}$ [2, Part II, Theorem 9.2].

(2.3) $L$ *is isomorphic to* $L(\mathscr{R}^N)$ [2, Part II, Theorem 14.1].

**3. Outline of von Neumann's proof.** (3.1) Choose $c_{1j} = c_{j1}$, $2 \leqq j \leqq N$, so that $c_{j1} \dot{\vee} a_j = c_{j1} \dot{\vee} a_1 = a_j \dot{\vee} a_1$; set $c_{ji} = (c_{j1} \vee c_{1i})(a_j \vee a_i)$ for $1, i, j$ all different.

(3.2) Call a family $\alpha \equiv (\alpha_{ji} \in L_{ji}: i \neq j)$ an $L$-number if $(\alpha_{ji} \vee c_{jk})(a_k \vee a_i) = \alpha_{ki}$ and $(\alpha_{ji} \vee c_{ik})(a_j \vee a_k) = \alpha_{jk}$. Note: For every $b \in L_{ji}$ there exists a unique $L$-number $\alpha$ with $\alpha_{ji} = b$ [2, Part II, Lemma 6.1].

(3.3) Let $\mathscr{R}$ denote the set of $L$-numbers with operations:

$$(\alpha + \beta)_{ji} = (\alpha_{jk} \vee (\beta_{ji} \vee a_k)(a_j \vee c_{ik}))(a_j \vee a_i),$$
$$(\alpha\beta)_{ji} = (\alpha_{jk} \vee \beta_{ki})(a_j \vee a_i).$$

(3.4) For each $\alpha \in \mathscr{R}$ and $1 \leqq j \leqq N$, define the *reach* of $\alpha$ into $a_j$ by $\alpha_j^{(r)} \equiv \equiv (\alpha_{ji} \vee a_i) a_j$ (does not depend on $i$, $i \neq j$).

Received August 20, 1981.

[1] A sketch of this paper appeared in *C. R. Math. Rep. Acad. Sci. Canada*, 3 (1981), 285—290.

(3.5) Prove: $\alpha\gamma=\beta$ has a solution $\gamma$ if and only if $\beta_j^{(r)}\leqq\alpha_j^{(r)}$ (holds for all $j$ if for some $j$) [2, Part II, Lemma 9.4] or [1, (3.2) with multiplication reversed].

(3.6) Prove: For each $b\leqq a_j$: $b=e_j^{(r)}$ for some idempotent $e\in\mathscr{R}$ [2, Part II, Theorem 9.3].

(3.7) Deduce: Parts (2.1), (2.2) of the theorem hold [2, Part II, Theorem 9.2].

$(3.8)_m$ Prove: For $1\leqq m\leqq N$ there exists an isomorphism

$$\varphi_m\colon (b\in L\colon\ b\leqq A^m)\to L(\mathscr{R}^N(m)) \quad\text{with}\quad \varphi_1\subset\varphi_2\subset\ldots\subset\varphi_N.$$

Note: $\varphi_N$ establishes Part (2.3) of the theorem. The outstanding difficulty in von Neumann's proof is to establish the $\varphi_m$.

**4. Von Neumann's strategy to prove $(3.8)_m$.** (4.1) Call $b$ an $m$-element if (i) $m=1$ and $b\leqq a_1$, or (ii) $2\leqq m\leqq N$ and $b\dot{\vee}A^{m-1}\leqq A^m$.

(4.2) For each $m$-element $b$ define $\varphi(b)$, a submodule of $L(\mathscr{R}^N(m))$, as follows:

(i) If $b\leqq a_1$ define $\varphi(b)\equiv(e,0,\ldots,0)\mathscr{R}$ with $e$ idempotent and $e_1^{(r)}=b$.

(ii) If $2\leqq m\leqq N$ define $\varphi(b)\equiv(-\alpha_1,\ldots,-\alpha_{m-1},1)_N e\mathscr{R}$ with $e$ idempotent and $e_m^{(r)}=(A^{m-1}\vee b)a_m$, with $b'\dot{\vee}e_m^{(r)}=a_m$ and $(\alpha_i)_{im}=(b\vee b'\vee A^{i-1}\vee a_{i+1}\vee\ldots\vee a_{m-1})(a_i\vee a_m)$.

Note: $\varphi(b)$ is determined uniquely by $b$ though $e, b'$, and the $\alpha_i$ may not be; also $(\alpha_i)_{im}(A^{m-1}\vee b)=(b\vee A^{i-1}\vee a_{i+1}\vee\ldots\vee a_{m-1})(a_i\vee a_m)$.

(4.3) For each $x\in L$ and decomposition $x=\bigvee_{i=1}^{N}x_i$ with $x_i$ an $i$-element, (such decompositions exist for all $x$), assign to $x$ the submodule $\varphi(x_1)+\ldots+\varphi(x_N)$.

$(4.4)_m$ Prove: the set $(\varphi(x_1)+\ldots+\varphi(x_m)\colon x\leqq A^m)=L(\mathscr{R}^N(m))$.

$(4.5)_m$ Prove: For decompositions $x=\bigvee_{i=1}^{m}x_i$, $y=\bigvee_{i=1}^{m}y_i$: $x\leqq y$ if and only if $\sum_{i=1}^{m}\varphi(x_i)\leqq\sum_{i=1}^{m}\varphi(y_i)$. Note: $(4.5)_m$ implies that $\varphi_m(x)\equiv\sum_{i=1}^{m}\varphi(x_i)$ has the same value for all decompositions of $x$; then $(4.4)_m$, $(4.5)_m$ establish $(3.8)_m$.

Von Neumann established $(4.4)_m$ without difficulty [2, Part II, Theorem 11.2]; $(4.5)_1$ follows immediately from (3.5), (3.6). But von Neumann's proof of $(4.5)_m$, $2\leqq m\leqq N$ [2, Part II, pages 168—208], is a virtuoso demonstration of mathematical technique.

**5. A new proof of $(4.5)_m$, $2\leqq m\leqq N$.** We use direct lattice calculations (for the case $m=2$, in particular) and reduce part of the case $m$ (to the case $m-1$) when $3\leqq m\leqq N$.

We require the following properties of $L$-numbers.

(5.1) $$(\alpha-\beta)_{jk}=(\alpha_{ji}\vee(a_k\vee\beta_{ji})(a_j\vee c_{ik}))(a_j\vee a_k)\quad [1,\ (2.3)],$$

hence

$$(\alpha-\beta)_j^{(r)} = (\alpha_{ji} \vee \beta_{ji}) a_j, \tag{5.2}$$

$$(\alpha+\beta\gamma)_{ji} = \big(\beta_{jk} \vee (\alpha_{ji} \vee a_k)(\gamma_{ki} \vee a_j)\big)(a_j \vee a_i) \tag{5.3}$$

[1, (5.2) with multiplication reversed].

**6. Proof of** $(4.5)_2$. We assume $x_1 \leqq a_1$, $y_1 \leqq a_1$, $x_2 \dot{\vee} a_1 \leqq a_2 \vee a_1$, $y_2 \dot{\vee} a_1 \leqq a_2 \vee a_1$ and we need to prove:

(6.1) $x_1 \vee x_2 \leqq y_1 \vee y_2$ if and only if $\varphi(x_1)+\varphi(x_2) \leqq \varphi(y_1)+\varphi(y_2)$. Because of modularity we need consider only the case $x_1=0$, $\varphi(x_1)=0$ (use $(4.5)_1$).

Now the inequality $\varphi(x_2) \leqq \varphi(y_1)+\varphi(y_2)$ is equivalent, in turn, to each of:

$$\big(-\alpha_1(x_2)e(x_2),\, e(x_2)\big)_N = \big(e(y_1),\, 0\big)_N \beta_1 + \big(-\alpha_1(y_2)e(y_2),\; e(y_2)\big)_N \beta_2 \tag{6.2}$$

for some $\beta_1, \beta_2 \in \mathscr{R}$;

$$e(y_2)e(x_2)=e(x_2) \quad \text{and} \quad \big(\alpha_1(y_2)e(x_2)-\alpha_1(x_2)e(x_2)\big)_1^{(r)} \leqq \big(e(y_1)\big)_1^{(r)}; \tag{6.3}$$

(6.4) $(a_1 \vee x_2)a_2 \leqq (a_1 \vee y_2)a_2$ and, (use (5.2)),

$$\big(\alpha_1(x_2)e(x_2)\big)_{13} \vee \big(\alpha_1(y_2)e(x_2)\big)_{13}\big) a_1 \leqq y_1;$$

(6.5) (i) $a_1 \vee x_2 \leqq a_1 \vee y_2$ and
(ii) $\big(\alpha_1(x_2)e(x_2)\big)_{13} \leqq y_1 \vee \big((\alpha_1(y_2)e(x_2)\big)_{13}$.

The inequality (6.5) (ii) is equivalent to each of:

$$\big((\alpha_1(x_2))_{12} \vee (e(x_2))_{23}\big)(a_1 \vee a_3) \leqq y_1 \vee \big((\alpha_1(y_2))_{12} \vee (e(x_2))_{23}\big), \tag{6.6}$$

$$\big((\alpha_1(x_2))_{12} \vee (e(x_2))_{23}\big)\big(a_1 \vee a_3 \vee (e(x_2))_{23}\big) \leqq y_1 \vee \big(\alpha_1(y_2)\big)_{12} \vee \big(e(x_2)\big)_{23}, \tag{6.7}$$

$$\big(\alpha_1(x_2)\big)_{12}\big(a_1 \vee (a_3 \vee (e(x_2))_{23}) a_2\big) \leqq y_1 \vee \big(\alpha_1(y_2)\big)_{12}, \tag{6.8}$$

$$\big(\alpha_1(x_2)\big)_{12}\big(a_1 \vee (a_1 \vee (e(x_2))_{21}) a_2\big) \leqq y_1 \vee \big(\alpha_1(y_2)\big)_{12}, \tag{6.9}$$

$$\big(\alpha_1(x_2)\big)_{12}(a_1 \vee x_2) \leqq y_1 \vee \big(\alpha_1(y_2)\big)_{12}. \tag{6.10}$$

Now (6.5) (i) and (6.10) together are equivalent to:

$$x_2 \leqq y_1 \vee y_2, \tag{6.11}$$

which establishes (6.1), i.e. $(4.5)_2$.

**7. Proof of** $(4.5)_m$ **assuming** $(4.5)_{m-1}$, $3 \leqq m \leqq N$. We assume $x_1 \leqq A^{m-1}$, $y_1 \leqq A^{m-1}$, $x_2 \dot{\vee} A^{m-1} \leqq A^m$, $y_2 \dot{\vee} A^{m-1} \leqq A^m$ and we must prove

(7.1) $x_1 \vee x_2 \leqq y_1 \vee y_2$ if and only if $\varphi_{m-1}(x_1)+\varphi(x_2) \leqq \varphi_{m-1}(y_1)+\varphi(y_2)$ where $\varphi_{m-1}$ is the isomorphism on $A^{m-1}$ determined by $\varphi$ (existing since $(4.5)_{m-1}$ is assumed to hold). We may assume that $x_1=y_1$ $(=z$, say).

We recall that [2, Part II, Lemma 13.2] states: *if* $a \leqq b$ *then every* $x$ *can be expressed as* $(x \vee a)(x \vee c)$ *for some* $c$ *with* $a \dot{\vee} c = b$. Repeated application of this lemma shows that our $z$ can be expressed as $z^{(1)} \wedge z^{(2)} \wedge \ldots \wedge z^{(m-1)}$ where, for each $j < (m-1)$: $z^{(j)} \vee a_j = A^{m-1}$, and $z^{(m-1)} \geqq A^{m-2}$.

It is clearly sufficient to establish (7.1) with $z$ replaced by $z^{(j)}, j = 1, \ldots, m-1$. Thus, in (7.1), we need consider only:

*case* (1): $z \geqq A^{m-2}$; and *case* (2): $z \vee a_j = A^{m-1}$ for some $j < (m-1)$.

The proof of (7.1) for case (1). We use lattice calculations as in the proof of $(4.5)_2$ in §6. With the present $z, x_2, y_2$,

$$\varphi_{m-1}(z) = u_1 \mathscr{R} + \ldots + u_{m-2} \mathscr{R} + u_{m-1} g \mathscr{R}$$

where $u_j$ is the vector in $\mathscr{R}^N$ with $j$-th component 1 and all other components 0, and $g$ is an idempotent with $(g)^{(r)}_{m-1} = z a_{m-1}$.

Let

$$\varphi(x_2) = (-\alpha_1, \ldots, -\alpha_{m-1}, 1)_N e \mathscr{R}, \quad \varphi(y_2) = (-\beta_1, \ldots, -\beta_{m-1}, 1)_N f \mathscr{R}.$$

Then the last inequality of (7.1) is equivalent to each of the following:

(7.2) (i) $ef = e$ and (ii) $(\beta_{m-1} - \alpha_{m-1}) e \in g \mathscr{R}$,

(7.3) (i) $(x_2 \vee A^{m-1}) a_m \leqq (y_2 \vee A^{m-1}) a_m$, i.e., $x_2 \vee A^{m-1} \leqq y_2 \vee A^{m-1}$, and (ii) $((\beta_{m-1} - \alpha_{m-1}) e)^{(r)}_{m-1} \leqq z a_{m-1}$.

Choose any $k \leqq N$ with $k$ different from $m-1$, $m$. Then (7.3) (ii) is equivalent to each of the following:

(7.4) $$\left((\beta_{m-1} e)_{m-1,k} \vee (\alpha_{m-1} e)_{m-1,k}\right) a_{m-1} \leqq z a_{m-1};$$

(7.5) $$(\alpha_{m-1} e)_{m-1,k} \leqq z a_{m-1} \vee (\beta_{m-1})_{m-1,m} \vee e_{mk};$$

(7.6) $$(\alpha_{m-1})_{m-1,m} (a_{m-1} \vee a_k \vee e_{mk}) \leqq z a_{m-1} \vee (\beta_{m-1})_{m-1,m} \vee e_{mk};$$

(7.7) $$(\alpha_{m-1})_{m-1,m} \left(a_{m-1} \vee a_m (a_k \vee e_{mk})\right) \leqq z a_{m-1} \vee (\beta_{m-1})_{m-1,m}.$$

The left hand side of (7.7) equals

$$(\alpha_{m-1})_{m-1,m} (a_{m-1} \vee e^{(r)}_m) = (\alpha_{m-1})_{m-1,m} (x_2 \vee A^{m-1}) = (x_2 \vee A^{m-2})(a_m \vee a_{m-1}).$$

In the presence of (7.3) (i), the right hand side of (7.7) may now be replaced by each of

$$(z a_{m-1} \vee (\beta_{m-1})_{m-1,m})(y_2 \vee A^{m-1}), \quad z a_{m-1} \vee (y_2 \vee A^{m-2})(a_m \vee a_{m-1}),$$

$$(y_2 \vee A^{m-2} \vee z a_{m-1})(a_m \vee a_{m-1}), \quad (y_2 \vee z)(a_m \vee a_{m-1}), \quad y_2 \vee z,$$

so (7.4) (ii) is equivalent to each of

$$(x_2\vee A^{m-2})(a_m\vee a_{m-1}\vee A^{m-2}) \leqq y_2\vee z, \quad x_2\vee A^{m-2}\leqq z\vee y_2.$$

Thus (7.4) is equivalent to: $x_2\leqq z\vee y_2$ and this establishes (7.1) for case (1).

The proof of (7.1) for case (2). Choose $z_{m-1}$ so that $z_{m-1}\dot{\vee} za_j = z(a_j\dot{\vee} a_{m-1})$. Then: $z_{m-1}\leqq z$; $z_{m-1}\dot{\vee} a_j=a_{m-1}\dot{\vee} a_j$; $z_{m-1}\dot{\vee} A^{m-2}=z\vee A^{m-2}=A^{m-1}$; and $\varphi_{m-1}(z_{m-1})=\varphi(z_{m-1})=(0, \ldots, -\beta, 0, \ldots, 1)_N\mathscr{R}$ with $-\beta$ in the $j$-th place and 1 in the $(m-1)$-th place.

Set $\bar{x}_2=(x_2\vee z_{m-1})(A^{m-2}\vee a_m)$, $\bar{y}_2=(y_2\vee z_{m-1})(A^{m-2}\vee a_m)$. Then $\bar{x}_2A^{m-1}=0=\bar{y}_2A^{m-1}$; $z\vee\bar{x}_2=z\vee x_2$; $z\vee\bar{y}_2=z\vee y_2$ and so the inequality $z\vee x_2\leqq z\vee y_2$ can be expressed as: $z\vee\bar{x}_2\leqq z\vee\bar{y}_2$.

If $\varphi(x_2)=(-\alpha_1, \ldots, -\alpha_{m-2}, -\alpha_{m-1}, 1)_N e\mathscr{R}$ and $\varphi(y_2)=(-\beta_1, \ldots, -\beta_{m-2}, -\beta_{m-1}, 1)_N f\mathscr{R}$ then (use (5.3)):

$$\varphi(\bar{x}_2) = (-\alpha_1, \ldots, -\alpha_{j-1}, -\alpha_j-\beta\alpha_{m-1}, -\alpha_{j+1}, \ldots, -\alpha_{m-2}, 0, 1)_N e\mathscr{R},$$

$$\varphi(\bar{y}_2) = (-\beta_1, \ldots, -\beta_{j-1}, -\beta_j-\beta\beta_{m-1}, -\beta_{j+1}, \ldots, -\beta_{m-2}, 0, 1)_N f\mathscr{R}$$

so the inequality

$$\varphi_{m-1}(z)+\varphi(x_2) \leqq \varphi_{m-1}(z)+\varphi(y_2)$$

can be expressed as:

$$\varphi_{m-1}(z)+\varphi(\bar{x}_2) \leqq \varphi_{m-1}(z)+\varphi(\bar{y}_2)$$

(use: $(0, \ldots, 0, -\beta, 0, \ldots, 1)_N(\beta_{m-1}-\alpha_{m-1})e$, with $-\beta$ in the $j$-th place and 1 in the $(m-1)$-th place, is in $\varphi_{m-1}(z)$).

Thus we need only prove (7.1) in case (2) with $z, x_2, y_2$ replaced by $z$, $\bar{x}_2$, $\bar{y}_2$ respectively. We may now also replace $z$ by $\bar{z}=zA^{m-2}$. Then we observe that all of $\bar{z}, \bar{x}_2, \bar{y}_2, \leqq A^{m-2}\dot{\vee} a_m$. Hence we can apply $(4.5)_{m-1}$ with $a_1, \ldots, a_{m-2}, a_{m-1}$ replaced by $a_1, \ldots, a_{m-2}, a_m$ (replacing $(\alpha_1, \ldots, \alpha_{m-1})_N$ in $\mathscr{R}^N(m-1)$ by $(\alpha_1, \ldots, \alpha_{m-2}, 0, \alpha_{m-1})_N$ in $\mathscr{R}^N(m)$); this replacement is permitted because it preserves the order of the $a_j$, and the functions $\varphi$, $\varphi_{m-2}$. This establishes (7.1) for the case (2) and completes the proof of $(4.5)_m$. This completes the proof of von Neumann's theorem.

**8. Supplementary remark.** Call $a_1, \ldots, a_N$, $N\geqq 3$ a *Desarguesian basis* for a complemented modular lattice $L$ if for some $c_{1j}, j>1$:

(i) (Bjarni Jónsson) $a_i$ is perspective to some $b_i\leqq a_1$ for $i\geqq 2$ with $b_2=b_3=a_1$;

(ii) $a_2a_1=a_3(a_2\dot{\vee} a_1)=0$ and $a_1\vee\ldots\vee a_N=1$, and

(iii) the formulae (3.3) make $\mathscr{R}$ a regular ring if, in the definition of $L$-number, $i, j$ are restricted to $\{1, 2, 3\}$.

If such a Desarguesian basis for $L$ exists, then the $a_i$, $i>3$ can be altered so that $\{a_1, \ldots, a_N\}$ becomes an independent basis for $L$ and, with some changes, the above proof of von Neumann's theorem holds; the condition (iii) can be replaced by certain Desarguesian-type lattice conditions (K. D. FRYER and I. HALPERIN, *Acta Sci. Math.*, **17** (1956), 203—249; B. JÓNSSON, *Trans. Amer. Math. Soc.*, **97** (1960), 64—94).

The proof is simplified when, in the definition of $L$-number, the $i, j$ are further restricted to $j<i$; but then the use of $e_N^{(r)}$ in (4.2) above and $(e(x))_{21}$ in (6.9) above, and the use of $k$ $(<m-1)$ in (7.5) above (when $m=N$) must be (and can be) adjusted.

## References

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