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On a representation of deterministic frontier-to-root tree transformations

FERENC GÉCSEG

To Professor B. Sz.-Nagy on his 70th birthday

In [8] M. STEINBY introduced the concept of the product of tree automata (the product of universal algebras, if we disregard the initial vectors and final states), and gave an algorithm to decide for every finite system of algebras whether or not it is isomorphically complete with respect to the product. So far, no similar result has been proved for homomorphic completeness. Moreover, by the knowledge of the author, there are no investigations concerning a system K of algebras which is complete for a system L of tree transformations in the following sense: every transformation from L can be induced by a tree transducer built (in an obvious way) on a product of algebras from K.

In this paper we introduce special types of products which are the tree automata theoretic generalizations of α_i-products of finite automata introduced in [3]. Moreover, we shall study a weaker form of the last-mentioned completeness (to be called m-completeness) with respect to the product and the α_i-products for the class of all deterministic tree transformations.

1. Notions and notations

By an *operator domain* we mean a set Σ together with a mapping $r\colon \Sigma \to N_0$ which assigns to every $\sigma \in \Sigma$ an *arity*, or *rank* $r(\sigma)$, where N_0 is the set of all nonnegative integers. For any $m \geqq 0$, $\Sigma_m = \{\sigma \in \Sigma \mid r(\sigma) = m\}$ is the set of the *m-ary operators* (or *operational symbols*). If Σ is finite then it is called a *ranked alphabet*. In the sequel we shall generally omit r in the definition of an operator domain Σ. Moreover, we shall suppose that if an operator belongs to more than one operator domain then it has the same rank in all of them.

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A finite subset $R \subseteq N_0$ is a *rank type*. It is said that the *rank type* of a ranked alphabet Σ is R if $r(\Sigma)=R$; that is R consists of all $m \in N_0$ for which $\Sigma_m \neq \emptyset$.

The set of *Σ-trees over* Z (or *Σ-polynomial symbols* with variables from Z) will be denoted by $F_\Sigma(Z)$. Moreover, for every $m \geqq 0$, $F_\Sigma^m(Z)$ is the set consisting of all trees $p \in F_\Sigma(Z)$ with $h(p) \leqq m$, where $h(p)$ is the *height* of p.

In the sequel we shall use the terms "node of a tree" and "subtree at a given node of a tree" in an informal and obvious way. Moreover, relabeling of nodes of a tree will mean that every label of a tree which is an operator is replaced by an arbitrary operator of the same rank.

The symbol X will stand for the countable set $\{x_1, x_2, \ldots\}$ of variables, and for every $n \geqq 0, X_n = \{x_1, \ldots, x_n\}$.

Let R be a rank type. Take an operator domain Σ of rank type R and a tree $p \in F_\Sigma(X_n)$ for some $n \geqq 0$. Consider another operator domain Ω of rank type R (not necessarily different from Σ) and a tree $q \in F_\Omega(X_n)$. We say that q is *similar* to p if the following conditions are satisfied:

(i) there exist relabelings of the nodes of p and q such that the resulting trees coincide,

(ii) if at two nodes d_1 and d_2 of p the subtrees coincide then q also has the same subtree at d_1 and d_2.

The class of all trees similar to p will be denoted by $[p]$.

Take a class S of trees. We say that S is a *shape* of rank type R if there exist a ranked alphabet Σ of rank type R, a non-negative integer $n \geqq 0$ and a tree $p \in F_\Sigma(X_n)$ such that $S=[p]$. The *height* $h(S)$ of S is $h(p)$. A shape S is *trivial* if $S=\{x_i\}$ for some $x_i \in X$. Otherwise S is called *nontrivial*. If we want to emphasize that all the frontier variables occurring in trees from S belong to X_n then we write $S(n)$ for S.

Let Σ be an operator domain. A Σ-algebra $\mathscr{A}$ is a pair consisting of a nonempty set A and a mapping that assigns to every operator $\sigma \in \Sigma$ an m-ary operation $\sigma^{\mathscr{A}}: A^m \to A$, where m is the arity of σ. The operation $\sigma^{\mathscr{A}}$ is called the *realization* of σ in $\mathscr{A}$. The mapping $\sigma \to \sigma^{\mathscr{A}}$ will not be mentioned explicitly, but we write $\mathscr{A}=(A, \Sigma)$. The Σ-algebra $\mathscr{A}$ is *finite* if A is finite and Σ is a ranked alphabet. Moreover, $\mathscr{A}$ has *rank type* R if Σ is of rank type R. Finally, if p is a Σ-tree then the realization of p in $\mathscr{A}$ will be denoted by $p^{\mathscr{A}}$. If there is no danger of confusion then we omit $\mathscr{A}$ in $\sigma^{\mathscr{A}}$ and $p^{\mathscr{A}}$.

A *frontier-to-root ΣX_n-recognizer* or an *$F\Sigma X_n$-recognizer,* for short, is a system $\mathbf{A}=(\mathscr{A}, \mathbf{a}, X_n, A')$ where

(1) $\mathscr{A}=(A, \Sigma)$ is a finite Σ-algebra,

(2) $\mathbf{a}=(a^{(1)}, \ldots, a^{(n)}) \in A^n$ is the *initial vector*,

(3) $A' \subseteq A$ is the set of final states.

If Σ and X_n are not specified then we speak about an *F-recognizer*. Moreover, let us note that in [7] we use a mapping $\alpha\colon X_n \to A$ instead of an initial vector.

Next we recall the concept of a tree transducer. To this we need one more set of variables $Y=\{y_1, y_2, \ldots\}$, and let $Y_n=\{y_1, \ldots, y_n\}$ for every $n \geqq 0$. Moreover, $\Xi=\{\xi_1, \xi_2, \ldots\}$ is the set of auxiliary variables, and $\Xi_n=\{\xi_1, \ldots, \xi_n\}$ for arbitrary $n \geqq 0$.

A *frontier-to-root tree transducer (F-transducer)* is a system $\mathfrak{A}=(\Sigma, X_n, A, \Omega, Y_m, P, A')$, where

(1) Σ and Ω are ranked alphabets,

(2) X_n and Y_m are the *frontier alphabets*,

(3) A is a ranked alphabet consisting of unary operators, the *state set* of $\mathfrak{A}$. (It is assumed that A is disjoint with all other sets in the definition of $\mathfrak{A}$, except A'.)

(4) $A' \subseteq A$ is the set of *final states*,

(5) P is a finite set of *productions* of the following two types:

(i) $x \to aq$ $(x \in X_n,\ a \in A,\ q \in F_\Omega(Y_m))$,

(ii) $\sigma(a_1, \ldots, a_l) \to aq(\xi_1, \ldots, \xi_l)$ $(\sigma \in \Sigma_l,\ l \geqq 0,\ a_1, \ldots, a_l,\ a \in A,\ q(\xi_1, \ldots, \xi_l) \in F_\Omega(Y_m \cup \Xi_l))$.

The *transformation* induced by $\mathfrak{A}$ will be denoted by $\tau_{\mathfrak{A}}$. Moreover, deterministic totally defined F-transducers will be called DTF-*transducers,* too. One can easily show, that for every deterministic F-transducer $\mathfrak{A}$ there is a DTF-transducer $\mathfrak{B}$ with $\tau_{\mathfrak{A}}=\tau_{\mathfrak{B}}$. Accordingly, in this paper we deal transformations induced DTF-transducers.

To a DTF-transducer $\mathfrak{A}=(\Sigma, X_n, A, \Omega, Y_m, P, A')$ we can correspond an $F\Sigma X_n$-recognizer $\mathbf{A}=(\mathscr{A}, \mathbf{a}, X_n, A')$ with $\mathscr{A}=(A, \Sigma)$ and $\mathbf{a}=(a^{(1)}, \ldots, a^{(n)})$, where

(1) $a^{(i)}=a$ if $x_i \to aq \in P$ for some q $(i=1, \ldots, n)$, and

(2) for arbitrary $l \geqq 0$, $\sigma \in \Sigma_l$ and $a_1, \ldots, a_l \in A$, $\sigma^{\mathscr{A}}(a_1, \ldots, a_l)=a$ if $\sigma(a_1, \ldots, a_l) \to aq \in P$, for some q.

This uniquely determined recognizer will be denoted by $\operatorname{rec}(\mathfrak{A})$.

Now take an $F\Sigma X_n$-recognizer $\mathbf{A}=(\mathscr{A}, \mathbf{a}, X_n, A')$ with $\mathscr{A}=(A, \Sigma)$ and $\mathbf{a}=(a^{(1)}, \ldots, a^{(n)})$. Define an F-transducer $\mathfrak{A}=(\Sigma, X_n, A, \Omega, Y_m, P, A')$ by

$$P = \{x_i \to a^{(i)}q^{(i)} \mid q^{(i)} \in F_\Omega(Y_m),\ i = 1, \ldots, n\} \cup$$
$$\cup \{\sigma(a_1, \ldots, a_l) \to \sigma^{\mathscr{A}}(a_1, \ldots, a_l)\, q^{(\sigma, a_1, \ldots, a_l)} \mid \sigma \in \Sigma_l,$$
$$l \geqq 0,\ a_1, \ldots, a_l \in A,\ q^{(\sigma, a_1, \ldots, a_l)} \in F_\Omega(Y_m \cup \Xi_l)\},$$

where the ranked alphabet Ω, the integer m and the trees in the right sides of the productions in P are fixed arbitrarily. Obviously, $\mathfrak{A}$ is a DTF-transducer. Denote by $\operatorname{tr}(\mathbf{A})$ the class of all DTF-transducers obtained in the above way. It is easy to see that for arbitrary DTF-transducer $\mathfrak{A}$ the inclusion $\mathfrak{A} \in \operatorname{tr}(\operatorname{rec}(\mathfrak{A}))$ holds. Therefore, we have

Statement 1. *For every DTF-transducer* $\mathfrak{A}$ *there exists an F-recognizer* $\mathbf{A}$ *such that* $\mathfrak{A}\in\operatorname{tr}(\mathbf{A})$.

Before recalling the definition of products of algebras, we note that in the sequel if $\mathbf{a}$ is an n-dimensional vector then $\operatorname{pr}_i(\mathbf{a})$ $(1\leqq i\leqq n)$ will denote its i^{th} component. Moreover, we suppose that every finite index set $I=\{i_1, \dots, i_k\}$ is given together with a (fixed) ordering of its elements. Furthermore, for arbitrary system $\{a_{i_j}|i_j\in I\}$, $(a_{i_j}|i_j\in I)$ is the vector $(a_{i_1}, a_{i_2}, \dots, a_{i_k})$ if $i_1<i_2<\dots<i_k$ is the ordering of I.

From now on we shall deal with a fixed rank type R. To exclude trivial cases, it will be assumed that for an $m>0$, $m\in R$.

Let $\Sigma, \Sigma^1, \dots, \Sigma^k$ be ranked alphabets of rank type R, and consider the Σ^i-algebras $\mathscr{A}_i=(A_i, \Sigma^i)$ $(i=1, \dots, k)$. Furthermore, let

$$\psi = \{\psi_m\colon (A_1\times\dots\times A_k)^m\times\Sigma_m \to \Sigma^1_m\times\dots\times\Sigma^k_m|m\in R\}$$

be a family of mappings. Then by the *product* of $\mathscr{A}_1, \dots, \mathscr{A}_k$ with respect to ψ we mean the Σ-algebra

$$\psi(\mathscr{A}_1, \dots, \mathscr{A}_k, \Sigma) = \mathscr{A} = (A, \Sigma)$$

with $A=A_1\times\dots\times A_k$ and for arbitrary $m\in R$, $\sigma\in\Sigma_m$ and $\mathbf{a}_1, \dots, \mathbf{a}_m\in A$,

$$\sigma^{\mathscr{A}}(\mathbf{a}_1, \dots, \mathbf{a}_m) = \big(\sigma_1^{\mathscr{A}_1}(\operatorname{pr}_1(\mathbf{a}_1), \dots, \operatorname{pr}_1(\mathbf{a}_m)), \dots, \sigma_k^{\mathscr{A}_k}(\operatorname{pr}_k(\mathbf{a}_1), \dots, \operatorname{pr}_k(\mathbf{a}_m))\big),$$

where $(\sigma_1, \dots, \sigma_k)=\psi_m(\mathbf{a}_1, \dots, \mathbf{a}_m, \sigma)$.

(Sometimes we shall consider ψ_m to be an $(mk+1)$-ary function in an obvious sense.)

Consider the above product $\psi(\mathscr{A}_1, \dots, \mathscr{A}_k, \Sigma)=\mathscr{A}$, and define the mappings $\psi^i\colon A^n\times F_\Sigma(X_n)\to F_{\Sigma^i}(X_n)$ $(i=1, \dots, k;\ n\geqq 0)$ in the following way: for arbitrary $\mathbf{a}=(\mathbf{a}_1, \dots, \mathbf{a}_n)\in A^n$ and $p\in F_\Sigma(X_n)$

(1) if $p=x_j$ $(1\leqq j\leqq n)$ then $\psi^i(\mathbf{a}, p)=x_j$,

(2) if $p=\sigma(p_1, \dots, p_m)$ $(\sigma\in\Sigma_m)$ then $\psi^i(\mathbf{a}, p)=\sigma^i(\psi^i(\mathbf{a}, p_1), \dots, \psi^i(\mathbf{a}, p_m))$, where $(\sigma_1, \dots, \sigma_k)=\psi_m(p_1^{\mathscr{A}}(\mathbf{a}), \dots, p_m^{\mathscr{A}}(\mathbf{a}), \sigma)$.

One can easily see that the equation

$$p^{\mathscr{A}}(\mathbf{a}) = \big(\psi^1(\mathbf{a}, p)^{\mathscr{A}_1}(\operatorname{pr}_1(\mathbf{a}_1), \dots, \operatorname{pr}_1(\mathbf{a}_n)), \dots, \psi^k(\mathbf{a}, p)^{\mathscr{A}_k}(\operatorname{pr}_k(\mathbf{a}_1), \dots, \operatorname{pr}_k(\mathbf{a}_n))\big)$$

holds. Moreover, for arbitrary $i(1\leqq i\leqq k)$, $\mathbf{a}\in A^n$ and $p\in F_\Sigma(X_n)$, $\psi^i(\mathbf{a}, p)\in[p]$.

We now define special types of products. First of all let us write ψ_m in the form $\psi_m=(\psi_m^{(1)}, \dots, \psi_m^{(k)})$, where for arbitrary $\mathbf{a}_1, \dots, \mathbf{a}_m\in A$ and $\sigma\in\Sigma_m$,

$$\psi_m(\mathbf{a}_1, \dots, \mathbf{a}_m, \sigma) = \big(\psi_m^{(1)}(\mathbf{a}_1, \dots, \mathbf{a}_m, \sigma), \dots, \psi_m^{(k)}(\mathbf{a}_1, \dots, \mathbf{a}_m, \sigma)\big).$$

We say that $\mathscr{A}$ is an α_i-product $(i=0, 1, \dots)$ if for any j $(1\leqq j\leqq k)$ and $m\in R$, $\psi_m^{(j)}$ is independent of its u^{th} components if $(v-1)k+j+i\leqq u\leqq vk$ $(v=1, \dots, m)$. (Here

$\psi_m^{(j)}$ is considered an $(mk+1)$-ary function.) In the case of an α_i-product in $\psi_m^{(j)}$ we shall indicate only those variables on which $\psi_m^{(j)}$ may depend. For instance, we write $\psi_m^{(1)}(\sigma)$ for $\psi_m^{(1)}(\mathbf{a}_1, \ldots, \mathbf{a}_m, \sigma)$ if $i=0$.

By the above definition, $\mathcal{A}$ is an α_i-product if for arbitrary j $(1 \leqq j \leqq k)$ and $m \in R$, $\psi_m^{(j)}$ is independent of component algebras $\mathcal{A}_u$ with $i+j \leqq u \leqq k$. If $i=0$ then we speak about a *loop-free product,* too. Moreover, if for every $m \in R$, ψ_m may depend on its last variable only then $\mathcal{A}$ is a *quasi-direct product.* If in addition, $\mathcal{A}_1 = \ldots = \mathcal{A}_k = \mathcal{B}$ then we speak about a *quasi-direct power* of $\mathcal{B}$.

One can see easily that the formation of the product, α_0-product and quasi-direct product is associative. (This is not true for the α_i-product with $i>0$.)

Let $\mathfrak{A} = (\Sigma, X_u, A, \Omega, Y_v, P, A')$ and $\mathfrak{B} = (\Sigma, X_u, B, \Omega, Y_v, P', B')$ be two DTF-transducers and $m \geqq 0$ an integer. We write $\tau_{\mathfrak{A}} \overset{m}{=} \tau_{\mathfrak{B}}$ if $\tau_{\mathfrak{A}}(p) = \tau_{\mathfrak{B}}(p)$ for every $p \in F_\Sigma^m(X_u)$.

Take a class K of algebras of rank type R. We say that K is *metrically complete* (*m-complete,* for short) with respect to the product (α_i-product) if for arbitrary DTF-transducer $\mathfrak{A} = (\Sigma, X_u, A, \Omega, Y_v, P, A')$ and integer $m \geqq 0$ there exist a product (α_i-product) $\mathcal{B} = (B, \Sigma)$ of algebras from K, a vector $\mathbf{b} \in B^u$ and a subset $B' \subseteq B$ such that $\tau_{\mathfrak{A}} \overset{m}{=} \tau_{\mathfrak{B}}$ for some $\mathfrak{B} \in \mathrm{tr}(\mathbf{B})$, where $\mathbf{B} = (B, \mathbf{b}, X_u, B')$. (The name metrical completeness comes from the fact that such systems are the tree automata theoretic generalizations of metrically complete systems of finite automata introduced in [1].)

Let $\mathcal{A} = (A, \Sigma)$ be an algebra, $n \geqq 0$ an integer and $\mathbf{a} \in A^n$ a vector. For arbitrary $m \geqq 0$, set $A_{\mathbf{a}}^{(m)} = \{p^{\mathcal{A}}(\mathbf{a}) \mid p \in F_\Sigma^m(X_n)\}$. The system $(\mathcal{A}, \mathbf{a})$ is called *m-free* if $|A_{\mathbf{a}}^{(m)}| = |F_\Sigma^m(X_n)|$, i.e., $p \neq q$ implies $p(\mathbf{a}) \neq q(\mathbf{a})$ whenever $p, q \in F_\Sigma^m(X_n)$.

Now let $\mathcal{A} = (A, \Sigma)$, $\mathcal{B} = (B, \Sigma)$ be algebras, $n, m \geqq 0$ integers and $\mathbf{a} \in A^n$, $\mathbf{b} \in B^n$ vectors. We say that $(\mathcal{B}, \mathbf{b})$ is an *m-homomorphic image* of $(\mathcal{A}, \mathbf{a})$ if there is a mapping φ of $A_{\mathbf{a}}^{(m)}$ onto $B_{\mathbf{b}}^{(m)}$ such that

(1) $\varphi(\mathrm{pr}_i(\mathbf{a})) = \mathrm{pr}_i(\mathbf{b})$ for all $i = 1, \ldots, n$,

(2) $\varphi(\sigma^{\mathcal{A}}(a_1, \ldots, a_l)) = \sigma^{\mathcal{B}}(\varphi(a_1), \ldots, \varphi(a_l))$ for arbitrary $l \in R$, $\sigma \in \Sigma_l$ and $a_1, \ldots, a_l \in A_{\mathbf{a}}^{(m-1)}$.

If in addition φ is one-to-one then we speak about an *m-isomorphic image,* or we say that $(\mathcal{A}, \mathbf{a})$ and $(\mathcal{B}, \mathbf{b})$ are *m-isomorphic,* in notation $(\mathcal{A}, \mathbf{a}) \overset{m}{\cong} (\mathcal{B}, \mathbf{b})$.

We obviously have the following statements.

Statement 2. *Let $\mathcal{A} = (A, \Sigma)$ and $\mathcal{B} = (B, \Sigma)$ be algebras. Take two integers $m, n \geqq 0$ and two vectors $\mathbf{a} \in A^n, \mathbf{b} \in B^n$. If $(\mathcal{A}, \mathbf{a})$ is m-free then*

(i) *$(\mathcal{B}, \mathbf{b})$ is an m-homomorphic image of $(\mathcal{A}, \mathbf{a})$, and*

(ii) *for arbitrary $\mathbf{B} = (\mathcal{B}, \mathbf{b}, X_n, B')$ and $\mathfrak{B} \in \mathrm{tr}(\mathbf{B})$ there exist $\mathbf{A} = (\mathcal{A}, \mathbf{a}, X_n, A')$ and $\mathfrak{A} \in \mathrm{tr}(\mathbf{A})$ such that $\tau_{\mathfrak{A}} \overset{m}{=} \tau_{\mathfrak{B}}$.*

Statement 3. *Let* $\mathcal{A}=(A, \Sigma)$ *and* $\mathcal{B}=(B, \Sigma)$ *be algebras. Take two integers* $m, n \geqq 0$ *and two vectors* $\mathbf{a} \in A^n, \mathbf{b} \in B^n$. *If* $(\mathcal{A}, \mathbf{a})$ *and* $(\mathcal{B}, \mathbf{b})$ *are m-free then they are m-isomorphic. Conversely, if* $(\mathcal{A}, \mathbf{a})$ *is m-free and m-isomorphic to* $(\mathcal{B}, \mathbf{b})$ *then* $(\mathcal{B}, \mathbf{b})$ *is also m-free.*

Let $(\mathcal{A}, \mathbf{a})$ $(\mathcal{A}=(A, \Sigma), \mathbf{a} \in A^n)$ be a system, $\mathcal{B}=(B, \Sigma)$ an algebra and $m \geqq 0$ integer. We say that $(\mathcal{A}, \mathbf{a})$ can be represented *m-isomorphically* by $\mathcal{B}$ if there exists a $\mathbf{b} \in B^n$ such that $(\mathcal{A}, \mathbf{a}) \overset{m}{\cong} (\mathcal{B}, \mathbf{b})$.

Finally, we say that the α_i-product and the α_j-product $(i, j \geqq 0)$ are *metrically equivalent* (*m-equivalent*) if a system of algebras is m-complete with respect to the α_i-product if and only if it is m-complete with respect to the α_j-product. The m-equivalence between an α_i-product and the product is defined similarly. (Let us note that in [4] the term "metrical equivalence" is used in a stronger sense.)

For notions not defined here we refer the reader to [5] and [6] or [7].

2. Metrically complete systems of algebras

In this section we shall give necessary and sufficient conditions for a system of algebras to be m-complete with respect to the α_i-products $(i=0, 1, \ldots)$ and the product. It will turn out that all the α_i-products are m-equivalent to each other and they are m-equivalent to the product.

First we prove

Theorem 1. *A system* K *of algebras of rank type* R *is m-complete with respect to the* α_i*-product (product) if and only if for arbitrary* $m, n \geqq 0$ *and ranked alphabet* Σ *of rank type* R *every m-free system* $(\mathcal{A}, \mathbf{a})$ *with* $\mathcal{A}=(A, \Sigma)$ *and* $\mathbf{a} \in A^n$ *can be represented m-isomorphically by an* α_i*-product (product) of algebras from* K.

Proof. The sufficiency is obvious by Statements 1 and 2.

To prove the necessity take an m-free system $(\mathcal{A}, \mathbf{a})$ with $\mathcal{A}=(A, \Sigma)$ and $\mathbf{a}=(a^{(1)}, \ldots, a^{(n)})$, where $\mathcal{A}$ is of rank type R. Moreover, let Ω be a ranked alphabet such that for every $l \in R$, $|\Omega_l| \geqq |F_\Sigma^{m+1}(X_n)|$. Consider the DTF-transducer $\mathfrak{A}= =(\Sigma, X_n, A, \Omega, X_n, P, A)$, where P consists of the productions

$$x_i \to a^{(i)} x_i \quad (i = 1, \ldots, n), \tag{1}$$

$$\sigma(a_1, \ldots, a_l) \to \sigma^{\mathcal{A}}(a_1, \ldots, a_l)\omega(\xi_1, \ldots, \xi_l) \quad (\sigma \in \Sigma_l, \omega \in \Omega_l, a_1, \ldots, a_l \in A,\ l \geqq 0) \tag{2}$$

such that $n+|\{\omega | \sigma(a_1, \ldots, a_l) \to \sigma^{\mathcal{A}}(a_1, \ldots, a_l)\omega(\xi_1, \ldots, \xi_l) \in P, a_1, \ldots, a_l \in \{p^{\mathcal{A}}(\mathbf{a}) | p \in \in F_\Sigma^m(X_n)\}\}|=|F_\Sigma^{m+1}(X_n)|$. (Since $(\mathcal{A}, \mathbf{a})$ is m-free, by our assumptions about the cardinality of Ω, P can be chosen thus.)

Now let $\mathscr{B}=(B, \Sigma)$ be an α_i-product (product) of algebras from K, $\mathbf{b}=(b^{(1)}, \ldots, b^{(n)})\in B^n$ a vector and suppose that for some $\mathbf{B}=(\mathscr{B}, \mathbf{b}, X_n, B')$ and DTF-transducer $\mathfrak{B}=(\Sigma, X_n, B, \Omega, X_n, B')\in \operatorname{tr}(\mathbf{B})$ the relation $\tau_{\mathfrak{B}} \overset{m+1}{=} \tau_{\mathfrak{A}}$ holds. We shall show that $(\mathscr{A}, \mathbf{a}) \overset{m}{\cong} (\mathscr{B}, \mathbf{b})$. To this, by Statement 3, it is enough to prove that $(\mathscr{B}, \mathbf{b})$ is m-free.

Suppose that for two trees $p_1, p_2 \in F_\Sigma^m(X_n)$ we have $p_1 \neq p_2$ and $p_1^{\mathscr{B}}(\mathbf{b}) = p_2^{\mathscr{B}}(\mathbf{b})$. For an $l\in R$ with $l>0$ take a $\sigma\in\Sigma_l$ and arbitrary $r_2, \ldots, r_l \in F_\Sigma^m(X_n)$. Set $t_1=\sigma(p_1, r_2, \ldots, r_l)$ and $t_2=\sigma(p_2, r_2, \ldots, r_l)$. Then the trees q_1 and q_2 obtained by $t_1 \Rightarrow_{\mathfrak{B}}^* t_1^{\mathscr{B}}(\mathbf{b})q_1$ and $t_2 \Rightarrow_{\mathfrak{B}}^* t_2^{\mathscr{B}}(\mathbf{b})q_2$ have the same label at their roots. Moreover, by $\tau_{\mathfrak{A}} \overset{m+1}{=} \tau_{\mathfrak{B}}$, the derivations $t_1 \Rightarrow_{\mathfrak{A}}^* t_1^{\mathscr{A}}(\mathbf{a})q_1$ and $t_2 \Rightarrow_{\mathfrak{A}}^* t_2^{\mathscr{A}}(\mathbf{a})q_2$ hold, too. Thus, by the choice of P, q_1 and q_2 should have distinct labels at their roots, which is a contradiction. This ends the proof of Theorem 1.

Next we give necessary conditions for a system of algebras to be m-complete with respect to the product.

Theorem 2. *Let K be a system of algebras of rank type R which is m-complete with respect to the product. Then for arbitrary integers $m, n\geqq 0$ and nontrivial shape $S(n)$ with rank type R and height less than or equal to m, there is an algebra $\mathscr{A}=(A, \Sigma)\in K$, a vector $\mathbf{a}\in A^n$, a tree $\sigma(p_1, \ldots, p_l)\in S\cap F_\Sigma(X_n)$ $(\sigma\in\Sigma_l)$ and an operator $\sigma'\in\Sigma_l$ such that $\sigma(p_1(\mathbf{a}), \ldots, p_l(\mathbf{a})) \neq \sigma'(p_1(\mathbf{a}), \ldots, p_l(\mathbf{a}))$.*

Proof. Assume that there exist integers $m, n\geqq 0$ and a nontrivial $S(n)$ with $h(S(n))=k$ $(0\leqq k\leqq m)$ such that for arbitrary $\mathscr{A}=(A, \Sigma)\in K$, $\mathbf{a}\in A^n$ and $\sigma(p_1, \ldots, p_l)$, $\sigma'(p_1, \ldots, p_l)\in S(n)\cap F_\Sigma(X_n)$ $(\sigma, \sigma'\in\Sigma_l)$ the equation $\sigma(p_1(\mathbf{a}), \ldots, p_l(\mathbf{a})) = \sigma'(p_1(\mathbf{a}), \ldots, p_l(\mathbf{a}))$ holds. Consider a k-free system $(\mathscr{B}=(B, \Omega), \mathbf{b})$, where the ranked alphabet Ω has rank type R, $|\Omega_l|\geqq 2$ and $\mathbf{b}=(b^{(1)}, \ldots, b^{(n)})$. We show that $(\mathscr{B}, \mathbf{b})$ cannot be represented k-isomorphically by any product of algebras from K. Indeed, let

$$\mathscr{C} = (C, \Omega) = \psi(\mathscr{A}_1, \ldots, \mathscr{A}_r, \Omega) \quad (\mathscr{A}_i\in K, \quad i = 1, \ldots, r)$$

be an arbitrary product and $\mathbf{c}=(\mathbf{c}_1, \ldots, \mathbf{c}_n)\in C^n$ a vector. Take two trees $q=\omega_1(q_1, \ldots, q_l)$ and $q'=\omega_2(q_1, \ldots, q_l)$ such that $\omega_1, \omega_2\in\Omega_l$, $\omega_1\neq\omega_2$ and $q, q'\in S(n)$. Then we have

$$q(\mathbf{c}) = \big(\omega_1^1(q_1^1(\mathrm{pr}_1(\mathbf{c}_1), \ldots, \mathrm{pr}_1(\mathbf{c}_n)), \ldots, q_l^1(\mathrm{pr}_1(\mathbf{c}_1), \ldots, \mathrm{pr}_1(\mathbf{c}_n))), \ldots$$
$$\ldots, \omega_1^r(q_1^r(\mathrm{pr}_r(\mathbf{c}_1), \ldots, \mathrm{pr}_r(\mathbf{c}_n)), \ldots, q_l^r(\mathrm{pr}_r(\mathbf{c}_1), \ldots, \mathrm{pr}_r(\mathbf{c}_n)))\big)$$

and

$$q'(\mathbf{c}) = \big(\omega_2^1(q_1^1(\mathrm{pr}_1(\mathbf{c}_1), \ldots, \mathrm{pr}_1(\mathbf{c}_n)), \ldots, q_l^1(\mathrm{pr}_1(\mathbf{c}_1), \ldots, \mathrm{pr}_1(\mathbf{c}_n))), \ldots$$
$$\ldots, \omega_2^r(q_1^r(\mathrm{pr}_r(\mathbf{c}_1), \ldots, \mathrm{pr}_r(\mathbf{c}_n)), \ldots, q_l^r(\mathrm{pr}_r(\mathbf{c}_1), \ldots, \mathrm{pr}_r(\mathbf{c}_n)))\big),$$

where $q_i^j=\psi^j(\mathbf{c}_1, \dots, \mathbf{c}_n, q_i)$ $(i=1, \dots, l;\ j=1, \dots, r)$, $(\omega_1^1, \dots, \omega_1^r)=\psi_l(q_1(\mathbf{c}), \dots$ $\dots, q_l(\mathbf{c}), \omega_1)$ and $(\omega_2^1, \dots, \omega_2^r)=\psi_l(q_1(\mathbf{c}), \dots, q_l(\mathbf{c}), \omega_2)$. By our remark following the definition of $\psi^i(\mathbf{a}, p)$, the inclusions $\omega_i^j(q_1^j, \dots, q_l^j)\in S(n)$ hold for all $i\,(=1, 2)$ and $j\,(=1, \dots, r)$. Therefore, $q(\mathbf{c})=q'(\mathbf{c})$, i.e., $(\mathscr{C}, \mathbf{c})$ is not k-free. Since $(\mathscr{C}, \mathbf{c})$ was chosen arbitrarily, by Theorem 1 and Statement 3, this contradicts the assumption that K is m-complete with respect to the product, ending the proof of Theorem 2.

We shall show that if a system of algebras satisfies the conclusions of Theorem 2 then it is m-complete with respect to the loop-free product. To this two lemmas are needed.

In the next lemma Σ will be a fixed ranked alphabet of rank type R such that for every $l\in R$, Σ_l is a two-element set: $\Sigma_l=\{\sigma_l, \sigma_l'\}$.

Lemma 3. *Let K be a system of algebras with rank type R satisfying the conclusions of Theorem 2. Then for, arbitrary $m, n\geqq 0$, every m-free system $(\mathscr{A}, \mathbf{a})$ $(\mathscr{A}=(A, \Sigma), \mathbf{a}\in A^n)$ can be represented m-isomorphically by an α_0-product of algebras from K.*

Proof. We proceed by induction on m.

Let $m=0$. It follows from our assumptions that in K there is an algebra with at least two elements. Moreover, if $0\in R$ then this algebra can be chosen in such a way that it has at least two distinct 0-ary operations. One can easily show that a quasi-direct power of this algebra 0-isomorphically represents $(\mathscr{A}, \mathbf{a})$.

Now suppose that Lemma 3 has been proved for every $k\leqq m$. Let $(\mathscr{B}, \mathbf{b})$ be an m-free system, where $\mathscr{B}=(B, \Sigma)$ and $\mathbf{b}=(b_1, \dots, b_n)\in B^n$. Take the index set $I=\{(p, q)\mid p, q\in F_\Sigma(X_n),\ p\neq q,\ h(p)=m+1,\ h(q)\leqq m+1\}$. Consider a pair $(p, q)\in I$, and let $p=\delta_l(p_1, \dots, p_l)$ where δ_l is σ_l or σ_l'. Then by our assumptions, there is a $\mathscr{C}^{(p)}=(C^{(p)}, \Sigma^{(p)})$ in K, an n-dimensional vector $\mathbf{c}=(c_1, \dots, c_n)$ with components from $C^{(p)}$; a $p'=\omega(p_1', \dots, p_l')$ $(\omega\in\Sigma_l^{(p)})$ and an $\omega'\in\Sigma_l^{(p)}$ such that $p'\in[p]$, and $\omega(p_1'(\mathbf{c}), \dots, p_l'(\mathbf{c}))\neq\omega'(p_1'(\mathbf{c}), \dots, p_l'(\mathbf{c}))$. Define an α_0-product $\mathscr{A}^{(p,q)}=$ $=(A^{(p,q)}, \Sigma)=\psi^{(p,q)}(\mathscr{B}, \mathscr{C}^{(p)}, \Sigma)$ in the following way: take an arbitrary node d of p different from its root. Let $t=\delta_r(t_1, \dots, t_r)$ be the subtree of p at d, and $t'=\omega_r(t_1', \dots, t_r')$ the subtree of p' at d. Then $\psi_r^{(p,q)(2)}(t_1^{\mathscr{B}}(\mathbf{b}), \dots, t_r^{\mathscr{B}}(\mathbf{b}), \delta_r)=\omega_r$. In all other cases, except $\psi_l^{(p,q)(2)}(p_1^{\mathscr{B}}(\mathbf{b}), \dots, p_l^{\mathscr{B}}(\mathbf{b}), \delta_l)$, $\psi_s^{(p,q)(2)}$ $(s\in R)$ is given arbitrarily in accordance with the definition of the α_0-product. Moreover, $\psi_s^{(p,q)(1)}$ is the identity mapping on Σ_s for every $s\in R$. Finally, let

$$\psi_l^{(p,q)(2)}(p_1^{\mathscr{B}}(\mathbf{b}), \dots, p_l^{\mathscr{B}}(\mathbf{b}), \delta_l)=\begin{cases}\omega & \text{if } q(\mathbf{a}^{(p,q)})=(b, c) \\ & \quad\text{and } c\neq\omega(p_1'(\mathbf{c}), \dots, p_l'(\mathbf{c})), \\ \omega' & \text{otherwise,}\end{cases}$$

where $\mathbf{a}^{(p,q)}=((b_1, c_1), \ldots, (b_n, c_n))$. ($q(\mathbf{a}^{(p,q)})$ is defined since $p\neq q$ and $(\mathscr{B}, \mathbf{b})$ is m-free.)

By the m-freeness of $(\mathscr{B}, \mathbf{b})$ and the choice of $(\mathscr{C}^{(p)}, \mathbf{c})$, $\mathscr{A}^{(p,q)}$ has the following properties:

(i) if t and t' are distinct trees from $F^m_\Sigma(X_n)$ then $t(\mathbf{a}^{(p,q)})\neq t'(\mathbf{a}^{(p,q)})$ since they differ at least in their first components, and

(ii) $p(\mathbf{a}^{(p,q)})\neq q(\mathbf{a}^{(p,q)})$ since they differ at least in their second components.

Afterwards form the direct product $\mathscr{D}=(D, \Sigma)=\prod(\mathscr{A}^{(p,q)}|(p, q)\in I)$ and the vector $\mathbf{d}\in D^n$ with $\mathrm{pr}_j(\mathbf{d})=(\mathrm{pr}_j(\mathbf{a}^{(p,q)})|(p, q)\in I)$ $(j=1, \ldots, n)$. Obviously, the system $(\mathscr{D}, \mathbf{d})$ is $(m+1)$-free. Since the quasi-direct power is a special α_0-product and the formation of α_0-products is associative this, by Statement 3, ends the proof of Lemma 3.

Lemma 4. *Let Σ be a ranked alphabet of rank type R such that for every $l\in R$, $|\Sigma_l|\geqq 2$. Moreover fix an $l\in R$ and take the ranked alphabet Σ^l with $\Sigma^l_l=\Sigma_l\cup\{\bar\sigma\}$ and $\Sigma^l_k=\Sigma_k$ if $k\neq l$. If for certain $m, n\geqq 0$ and class K of algebras an m-free system $(\mathscr{A}, \mathbf{a})$ with $\mathscr{A}=(A, \Sigma)$ and $\mathbf{a}\in A^n$ can be represented m-isomorphically by an α_0-product of algebras from K then every m-free system $(\mathscr{B}, \mathbf{b})$ with $\mathscr{B}=(B, \Sigma^l)$ and $\mathbf{b}\in B^n$ can be represented m-isomorphically by an α_0-product of algebras from K.*

Proof. Let $(\mathscr{A}, \mathbf{a})$ be an m-free system with $\mathscr{A}=(A, \Sigma)$ and $\mathbf{a}=(a^{(1)}, \ldots, a^{(n)})\in A^n$ which can be represented m-isomorphically by an α_0-product of algebras from K. Take two different fixed elements $\sigma_1, \sigma_2\in\Sigma_l$. Define two (one-factor) α_0-products $\mathscr{A}_1=(A, \Sigma^l)=\psi(\mathscr{A}, \Sigma^l)$ and $\mathscr{A}_2=(A, \Sigma^l)=\bar\psi(\mathscr{A}, \Sigma^l)$ in the following way:

(i) $$\psi^{(1)}_k(\sigma)=\bar\psi^{(1)}_k(\sigma)=\sigma \quad (\sigma\in\Sigma_k,\ k\neq l),$$

(ii) $$\psi^{(1)}_l(\sigma)=\begin{cases}\sigma & \text{if } \sigma\neq\bar\sigma,\\ \sigma_1 & \text{if } \sigma=\bar\sigma\end{cases}$$

and

(iii) $$\bar\psi^{(1)}_l(\sigma)=\begin{cases}\sigma & \text{if } \sigma\neq\bar\sigma,\\ \sigma_2 & \text{if } \sigma=\bar\sigma.\end{cases}$$

One can easily see that in $\mathscr{A}_1$ the operator $\bar\sigma$ is realized as σ_1 in $\mathscr{A}$, and in $\mathscr{A}_2$ the operator $\bar\sigma$ has the same effect as σ_2 in $\mathscr{A}$. Moreover, all other operators have the same realizations in $\mathscr{A}$, $\mathscr{A}_1$ and $\mathscr{A}_2$.

For every $p\in F_{\Sigma^l}(X_n)$ let $p_1=\psi^1(a^{(1)}, \ldots, a^{(n)}, p)$ and $p_2=\bar\psi^1(a^{(1)}, \ldots, a^{(n)}, p)$, that is p_i $(i=1, 2)$ is obtained by replacing every occurrence of the label $\bar\sigma$ in p by σ_i. Obviously $p^{\mathscr{A}_1}(\mathbf{a})=p_1^{\mathscr{A}}(\mathbf{a})$ and $p^{\mathscr{A}_2}(\mathbf{a})=p_2^{\mathscr{A}}(\mathbf{a})$.

We show that the system $(\mathscr{B}, \mathbf{b})$, where $\mathscr{B}$ is the direct product $\mathscr{A}_1\times\mathscr{A}_2$ and $\mathbf{b}=((a^{(1)}, a^{(1)}), \ldots, (a^{(n)}, a^{(n)}))$, is m-free. Since the direct product is a special α_0-product and the formation of the α_0-product is associative this, by Statement 3, will complete the proof of Lemma 4.

Take two different trees $p, q \in F_{\Sigma'}^m(X_n)$, and let us distinguish the following three cases.

(1) None of the nodes of p and q is labelled by $\bar{\sigma}$. Then $p^{\mathscr{B}}(\mathbf{b}) = (p^{\mathscr{A}}(\mathbf{a}), p^{\mathscr{A}}(\mathbf{a}))$ and $q^{\mathscr{B}}(\mathbf{b}) = (q^{\mathscr{A}}(\mathbf{a}), q^{\mathscr{A}}(\mathbf{a}))$ differ in both of their components.

(2) One of p and q, say p, has a node labelled by $\bar{\sigma}$ and none of the nodes of q is labelled by $\bar{\sigma}$. If $p_1 = q (= q_1 = q_2)$ then $p_2 \neq q_2$ since $p_1 \neq p_2$. Thus, $p(\mathbf{b})$ and $q(\mathbf{b})$ differ at least in one of their components.

(3) Both p and q have nodes labelled by $\bar{\sigma}$. If $p_1 = q_1$ then $p_2 \neq q_2$ since $p \neq q$. Again $p(\mathbf{b})$ and $q(\mathbf{b})$ differ at least in one of their components.

Now we are ready to state and prove

Theorem 5. *A system of algebras is m-complete with respect to the product if and only if it is m-complete with respect to the α_0-product.*

Proof. Obviously, if a system of algebras is m-complete with respect to the α_0-product then it is m-complete with respect to the product.

Conversely, let K be a system of algebras of rank type R which is m-complete with respect to the product. Then, by Lemma 3, for arbitrary $m, n \geqq 0$ and Σ of rank type R with $|\Sigma_l| = 2$ $(l \in R)$ every m-free system $(\mathscr{A}, \mathbf{a})$ $(\mathscr{A} = (A, \Sigma),\ \mathbf{a} \in A^n)$ can be represented m-isomorphically by an α_0-product of algebras from K. From this, by a repeated application of Lemma 4, we get that the previous statement is valid for arbitrary ranked alphabet Σ of rank type R if $|\Sigma_l| \geqq 2$ $(l \in R)$. Moreover, if we omit an operation in an algebra belonging to an m-free system then the resulting system is m-free, too. Therefore, by Theorem 1, K is m-complete with respect to the α_0-product, which ends the proof of Theorem 5.

From the above theorem we directly get

Corollary 6. *For arbitrary $i, j \geqq 0$ the α_i-product is metrically equivalent to the α_j-product.*

Since there exists a one-element system of algebras which is isomorphically complete with respect to the product ([5], [8]) and for arbitrary $m, n \geqq 0$ and ranked alphabet Σ there is an m-free system $(\mathscr{A}, \mathbf{a})$ $(\mathscr{A} = (A, \Sigma),\ \mathbf{a} \in A^n)$ such that $\mathscr{A}$ is finite, we have

Corollary 7. *There exists a one-element system of algebras which is m-complete with respect to the α_0-product.*

Finally, we give an m-complete system consisting of two algebras which is not isomorphically complete.

Let R be a rank type with $0 \in R$ and Σ the ranked alphabet of rank type R fixed for Lemma 3. Consider the Σ-algebras $\mathscr{A} = (\{a_1, a_2\}, \Sigma)$ and $\mathscr{B} = (\{b_1, b_2\}, \Sigma)$

where

$$\sigma_0^{\mathscr{A}} = a_1, \sigma_0'^{\mathscr{A}} = a_2,$$

$$\sigma_l^{\mathscr{A}}(c_1, ..., c_l) = \sigma_l'^{\mathscr{A}}(c_1, ..., c_l) = a_1 \quad (l > 0;\ c_1, ..., c_l \in A)$$

and

$$\sigma_0^{\mathscr{B}} = \sigma_0'^{\mathscr{B}} = b_1,$$

$$\sigma_l^{\mathscr{B}}(c_1, ..., c_l) = b_1 \quad (l > 0;\ c_1, ..., c_l \in B),$$

$$\sigma_l'^{\mathscr{B}}(c_1, ..., c_l) = \begin{cases} b_2 & \text{if } c_1 = ... = c_l = b_1, \\ b_1 & \text{otherwise} \end{cases} \quad (l > 0;\ c_1, ..., c_l \in B).$$

The system $K = \{\mathscr{A}, \mathscr{B}\}$ obviously satisfies the conclusions of Theorem 2 (by $\mathscr{A}$ for the only nontrivial shape of height 0 and by $\mathscr{B}$ if the given shape is higher than 0). Therefore, K is m-complete with respect to the α_0-product. Moreover K is not isomorphically complete since for arbitrary $l \in R$ with $l > 0$, none of the equations $\sigma_l^{\mathscr{A}}(a_2, ..., a_2) = a_2$, $\sigma_l'^{\mathscr{A}}(a_2, ..., a_2) = a_2$, $\sigma_l^{\mathscr{B}}(b_2, ...; b_2) = b_2$ and $\sigma_l'^{\mathscr{B}}(b_2, ..., b_2) = b_2$ holds.

It follows from Theorem 1 in [2] that if a finite system of automata is m-complete with respect to the α_0-product then it always contains an automaton forming a simple system which is m-complete with respect to the α_0-product. One can easily see that neither $\{\mathscr{A}\}$ nor $\{\mathscr{B}\}$ is m-complete with respect to the α_0-product, showing that the existence even of a 0-ary operator (in addition to unary operators) alters the conditions of m-completeness.

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BOLYAI INSTITUTE
ARADI VÉRTANÚK TERE 1
6720 SZEGED, HUNGARY