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# Semigroups of continuous functions

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*To Professor Béla Szőkefalvi-Nagy, to our Master, to my Friend*

**0. Introduction.** Let $X$ be a topological space and $C(X)$ denote the set of all continuous, real-valued functions defined on $X$. $C(X)$ is a ring under pointwise addition and multiplication of functions. A classical theorem [2] states that the isomorphy of the rings $C(X)$ and $C(Y)$ implies the homeomorphy of $X$ and $Y$ provided $X$ and $Y$ are compact Hausdorff spaces. Somewhat surprisingly, A. N. Milgram [7] has shown that the same is true if one replaces the isomorphy of the rings $C(X)$ and $C(Y)$ by the isomorphy of the multiplicative semigroups of $C(X)$ and $C(Y)$.

Another generalization was furnished by E. Hewitt [5]; he replaced the condition for $X$ and $Y$ to be compact by that of being *realcompact* (but kept the ring isomorphy of $C(X)$ and $C(Y)$). As to the concept of a realcompact space, let us recall the following definitions.

In a topological space $X$, denote

(1) $$Z(f) = \{x \in X\colon\ f(x) = 0\}$$

for $f \in C(X)$,

(2) $$Z(X) = \{Z(f)\colon\ f \in C(X)\}.$$

A subset $\mathfrak{z} \subset Z(X)$ is said to be a *z-filter* iff

(3.a) $$\emptyset \neq \mathfrak{z} \neq Z(X),$$

(3.b) $$Z_1 \in \mathfrak{z},\quad Z_2 \in Z(X),\quad Z_1 \subset Z_2 \quad \text{implies} \quad Z_2 \in \mathfrak{z}.$$

(3.c) $$Z_1,\ Z_2 \in \mathfrak{z} \quad \text{implies} \quad Z_1 \cap Z_2 \in \mathfrak{z}.$$

A $z$-filter $\mathfrak{z}$ is said to be *fixed* iff $\bigcap \mathfrak{z} \neq \emptyset$, *maximal* iff $\mathfrak{z}' = \mathfrak{z}$ for every $z$-filter $\mathfrak{z}' \supset \mathfrak{z}$, and *real* iff $Z_n \in \mathfrak{z}$ $(n \in \mathbf{N})$ implies $\bigcap_1^\infty Z_n \in \mathfrak{z}$. Now $X$ is said to be *realcompact* iff it is a Tychonoff space such that every real maximal $z$-filter is fixed.

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It is a natural question whether these two generalizations can be unified. In fact, the paper [8] contains the following statement:

Theorem A. *If $X$ and $Y$ are realcompact spaces such that the (multiplicative) semigroups $C(X)$ and $C(Y)$ are isomorphic then $X$ and $Y$ are homeomorphic.*

However, the proof in [8] of this statement is rather long, goes through arguments concerning the *lattices* $C(X)$ and $C(Y)$, and seems to contain some gaps. Therefore it is desirable to have a short proof operating directly with the semigroup structure of $C(X)$ and $C(Y)$. This is desirable also because, as it was shown in [4], Theorem A implies

Theorem B. *If $X$ and $Y$ are arbitrary topological spaces, then the isomorphy of the semigroups $C(X)$ and $C(Y)$ implies the isomorphy of the rings $C(X)$ and $C(Y)$.*

The proof of Theorem B is based on Theorem C below. In order to formulate it, we have to recall one more definition. Let $X$ be a Tychonoff space, and denote by $\upsilon X$ the set of all real maximal $z$-filters in $X$, equipped with the topology for which the sets

$$B(Z) = \{\mathfrak{z} \in \upsilon X \colon Z \in \mathfrak{z}\} \quad (Z \in Z(X)) \tag{4}$$

constitute a closed base; $\upsilon X$ is realcompact and is called the *Hewitt realcompactification* of $X$ (see the monograph [3] for more details).

Theorem C. *If $X$ and $Y$ are Tychonoff spaces such that the semigroups $C(X)$ and $C(Y)$ are isomorphic then $\upsilon X$ and $\upsilon Y$ are homeomorphic.*

Theorem C contains Theorem A because $\upsilon X$ is homeomorphic to $X$ if $X$ is realcompact.

One of the purposes of the present paper is to present a method furnishing a simple proof of Theorem C. However, our method furnishes essentially more. Firstly, we can consider, instead of real-valued functions, functions with values in suitable topological semigroups. Secondly (which is more important), the condition of semigroup isomorphy can be replaced by an essentially weaker condition.

**1. $d$-mappings and $d$-ideals.** Let $S$ be a semigroup. For $f, g \in S$, we introduce the notation $g \rhd f$ iff $f$ is a right divisor of $g$, i.e., iff there is $h \in S$ such that $g = hf$. The relation $\rhd$ is transitive; it is reflexive (i.e. a preordering) if $S$ contains a left unity element.

If $S_1$ and $S_2$ are semigroups with the respective relations $\rhd_1$ and $\rhd_2$, we say that a mapping $\varphi \colon S_1 \to S_2$ is a *d-mapping* iff $f, g \in S_1, g \rhd_1 f$ implies $\varphi(g) \rhd_2 \varphi(f)$. A bijective mapping $\varphi \colon S_1 \to S_2$ such that both $\varphi$ and $\varphi^{-1}$ are $d$-mappings will be called a *d-isomorphism*; $S_1$ and $S_2$ are said to be *d-isomorphic* iff there exists a $d$-isomorphism from $S_1$ onto $S_2$. If $S_1$ and $S_2$ are semigroup isomorphic then they are clearly $d$-isomorphic but the converse is false; e.g., two groups of the same cardinality are always $d$-isomorphic (because $g \rhd f$ holds for any two elements $f, g$ of a group $S$).

A subset $D$ of a semigroup $S$ will be a called a *d-ideal* iff

(1.1) $$\emptyset \neq D \neq S,$$

(1.2) $$f \in D, \quad g \in S, \quad g \rhd f \quad \text{implies} \quad g \in D,$$

(1.3) $f, g \in D$ implies the existence of $h \in D$ such that $f \rhd h$, $g \rhd h$.

This is a special case of the general Definition 1.2 in [6]. A $d$-ideal is (by (1.2)) a left semigroup ideal.

Lemma 1. *If the semigroup $S$ contains a right unity element $e$, and $e \rhd f$, then $f$ cannot belong to any d-ideal $D$.*

Proof. Clearly $g \rhd e$ for every $g \in S$, hence $f \in D$ would imply $D = S$. □

A $d$-ideal $D$ is said to be *maximal* iff $D' = D$ holds for every $d$-ideal $D' \supset D$. By the Kuratowski—Zorn lemma, in a semigroup with right unity element, every $d$-ideal is contained in a maximal $d$-ideal. For a $d$-isomorphism $\varphi: S_1 \to S_2$ and $D \subset S_1$, $\varphi(D)$ is a (maximal) $d$-ideal in $S_2$ iff $D$ is a (maximal) $d$-ideal in $S_1$.

**2. Quasi-real semigroups.** Let $\mathbf{R}$ denote the real line, $\mathbf{R}^+$ the subset $(0, +\infty)$, and $\mathbf{R}_0^+$ the subset $[0, +\infty)$. Both $\mathbf{R}^+$ and $\mathbf{R}_0^+$ are semigroups (the first one even a group) under the multiplication of real numbers, and also topological spaces as subspaces of $\mathbf{R}$ equipped with the usual topology.

A set $\mathbf{S}$ will be called a *quasi-real semigroup* iff

(2.1) $\mathbf{S}$ is a semigroup;

(2.2) $\mathbf{S}$ contains $\mathbf{R}_0^+$ as a subsemigroup;

(2.3) $0 \in \mathbf{R}_0^+$ is a zero element in $\mathbf{S}$ (i.e., $0 \cdot a = a \cdot 0 = 0$ for $a \in \mathbf{S}$);

(2.4) $1 \in \mathbf{R}_0^+$ is a unity element in $\mathbf{S}$ (i.e., $1 \cdot a = a \cdot 1 = a$ for $a \in \mathbf{S}$);

(2.5) For $a \in \mathbf{S}$, $a \neq 0$, there is $b \in \mathbf{S}$ such that $a \cdot b = b \cdot a = 1$ (such a $b$ is clearly unique and will be denoted by $1/a$);

(2.6) $\mathbf{S}$ is a topological space;

(2.7) $\mathbf{R}_0^+$ is a subspace of $\mathbf{S}$;

(2.8) The mappings $(a, b) \mapsto a \cdot b$ and $a \mapsto 1/a$ are continuous from $\mathbf{S} \times \mathbf{S}$ into $\mathbf{S}$ and $\mathbf{S} - \{0\}$ into $\mathbf{S}$, respectively;

(2.9) There is a continuous mapping $a \mapsto |a|$ from $\mathbf{S}$ into $\mathbf{R}_0^+$ such that $|a \cdot b| = |a| \cdot |b|$, $|a| = a$ for $a \in \mathbf{R}_0^+$;

(2.10) The sets $V_\varepsilon = \{x \in \mathbf{S}: |x| < \varepsilon\}$ $(\varepsilon > 0)$ constitute a neighbourhood base of 0 in $\mathbf{S}$.

By (2.5) and (2.9), $|a| = 0$ iff $a = 0$.

As examples of quasi-real semigroups, we can mention the semigroups $\mathbf{R}_0^+$; $\mathbf{R}$, $\mathbf{C}$ (=the complex numbers) with the usual multiplication, topology, and absolute value, further many subsemigroups of $\mathbf{C}$, e.g., those composed of the numbers with arguments $2\pi r$ where $r \in \mathbf{Q}$, or $r = m/n$ where $n \in \mathbf{N}$ is fixed and $m \in \mathbf{Z}$. These examples are commutative; a non-commutative one is furnished by the real quaternions with the usual multiplication, absolute value and the topology inherited from $\mathbf{R}^4$.

We obtain further examples from

Theorem 1. *Let* $\mathbf{G}$ *be a topological group that contains* $\mathbf{R}^+$ *as a (topological) subgroup; suppose there is a continuous homomorphism* $\alpha\colon \mathbf{G} \to \mathbf{R}^+$ *such that* $\alpha(a) = a$ *for* $a \in \mathbf{R}^+$. *Let* $\mathbf{S} = \mathbf{G} \cup \{\omega\}$ *where* $\omega \notin \mathbf{G}$, *and define*

$$a \cdot \omega = \omega \cdot a = \omega \quad (a \in \mathbf{G}), \quad \omega \cdot \omega = \omega, \quad \alpha(\omega) = 0.$$

*Equip* $\mathbf{S}$ *with a topology in the manner that* $\mathbf{G}$ *be a subspace of* $\mathbf{S}$ *and the sets* $U_\varepsilon \cup \{\omega\}$, *where*

$$U_\varepsilon = \{x \in \mathbf{G}\colon \alpha(x) < \varepsilon\} \quad (\varepsilon > 0),$$

*constitute a neighbourhood base of* $\omega$. *After having identified* $\omega$ *with the real number* 0, $\mathbf{S}$ *will be a quasi-real semigroup (with* $|x| = \alpha(x)$*).*

*Conversely, every quasi-real semigroup can be obtained from a topological group* $\mathbf{G}$ *with the help of this construction.*

Proof. $\mathbf{S}$ fulfils (2.1)—(2.5) with the identification of $\omega$ and 0. The continuity of $\alpha$ implies that every $U_\varepsilon$ is open in $\mathbf{G}$; therefore there is a topology on $\mathbf{S}$ such that $\mathbf{G}$ is a subspace of $\mathbf{S}$ and the sets $U_\varepsilon \cup \{\omega\}$ constitute a neighbourhood base of $\omega$ (see e.g. [1], (6.1.2)). Such a topology is unique because $\mathbf{G}$ is necessarily open in $\mathbf{S}$; indeed, if $\omega$ belonged to every neighbourhood (in $\mathbf{S}$) of a point $a \in \mathbf{G}$, then the filter base $\{U_\varepsilon\colon \varepsilon > 0\}$ would converge to $a$ in $\mathbf{G}$, which is in contradiction with the fact that $\left\{x \in \mathbf{G}\colon \alpha(x) > \frac{\alpha(a)}{2}\right\}$ is a neighbourhood of $a$. For this topology (and $|x| = \alpha(x)$), (2.6)—(2.10) are evidently true.

Conversely, if $\mathbf{S}$ is a quasi-real semigroup, define $\mathbf{G} = \mathbf{S} - \{0\}$. By (2.1)—(2.5), $\mathbf{G}$ is a group containing $\mathbf{R}^+$ as a subgroup; by (2.6)—(2.8), it is a topological group, and $\mathbf{R}^+$ is a topological subgroup of $\mathbf{G}$. By (2.9), $\alpha(x) = |x|$ defines a continuous homomorphism $\alpha\colon \mathbf{G} \to \mathbf{R}^+$, and, by (2.10), all requirements are fulfilled for $\omega = 0$. □

E.g., let $\mathbf{G}$ be the set of all non-singular, real, quadratic matrices of order $m$ (for a given $m \in \mathbf{N}$) with matrix multiplication and the topology inherited from $\mathbf{R}^{m^2}$. The diagonal matrices with all elements in the diagonal equal to the same $c > 0$ constitute a topological subgroup isomorphic to $\mathbf{R}^+$; after having identified this matrix with $c$, define $\alpha(M)=|\det M|^{1/m}$ in order to obtain a group $\mathbf{G}$ satisfying the hypotheses of Theorem 1.

Many examples can be obtained from

Theorem 2. *Let* $\mathbf{T}$ *be an arbitrary topological group with unity element* $e$. *Then the direct product* $\mathbf{G}=\mathbf{T}\times\mathbf{R}^+$ *satisfies the hypotheses of Theorem* 1 *provided the elements* $(e, y)$ *are identified with* $y>0$ *and* $\alpha(x; y)=y$. □

Observe that Theorem 1 furnishes examples that are not contained in Theorem 2. E.g., let $\mathbf{G}$ be the multiplicative group of all non-singular, real, quadratic matrices of order 2 with the topology inherited from $\mathbf{R}^4$. Identify the matrix

$$\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \quad (x>0)$$

with the number $x$, and define $\alpha(M)=|\det M|$. If $\mathbf{G}$ were of the form $\mathbf{T}\times\mathbf{R}^+$ then $\mathbf{T}$ would be isomorphic to the subgroup of $\mathbf{G}$ consisting of the elements $M$ such that $\alpha(M)=1$. However, this is impossible because, e.g.,

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

**3. *d*-ideals of $S(X)$.** Let $X$ be a topological space, $\mathbf{S}$ a quasi-real semigroup, and denote by $S(X)$ the set of all continuous functions from $X$ into $\mathbf{S}$. $S(X)$ is a semigroup under pointwise multiplication of functions. Our purpose is to show that the $d$-ideals of the semigroup $S(X)$ are connected to the $z$-filters in $X$ in the same manner as the ideals of the ring $C(X)$ are (see [3]).

For $f\in S(X)$, define

(3.1) $$Z(f) = \{x\in X: f(x) = 0\},$$

(3.2) $$|f|(x) = |f(x)| \quad (x\in X).$$

Lemma 2. $f\in S(X)$ *implies* $|f|\in C(X)$. *Conversely,* $g\in C(X)$, $g\geqq 0$ *implies* $g\in S(X)$. □

Lemma 3. *For* $f\in S(X)$, *we have* $Z(f)=Z(|f|)$; *consequently*

$$\{Z(f): f\in S(X)\} = Z(X). \quad □$$

Lemma 4. $Z(fg)=Z(f)\cup Z(g)$ *for* $f, g\in S(X)$. □

Lemma 5. *If* $D$ *is a d-ideal in* $S(X)$, *then*

(3.3) $$Z(D) = \{Z(f): f\in D\}$$

*is a z-filter in* $X$.

Proof. By Lemma 3, $Z(D)\subset Z(X)$. $D\neq\emptyset$ implies $Z(D)\neq\emptyset$. On the other hand, since the constant function 1 is a unity element in $S(X)$, and $f\in S(X), Z(f)=\emptyset$ implies $1=\frac{1}{f}\cdot f$ for $\frac{1}{f}\in S(X)$, where, of course,

$$\frac{1}{f}(x)=\frac{1}{f(x)} \quad (x\in X), \tag{3.4}$$

$f\in D$ is impossible by Lemma 1. Therefore $\emptyset\notin Z(D)$.

If $Z_1\in Z(D)$, $Z_2\in Z(X)$, $Z_1\subset Z_2$, say $Z_1=Z(f), f\in D$, $Z_2=Z(g)$, $g\in S(X)$ (cf. Lemma 3), then, by Lemma 4, $gf\in D$ implies $Z_2=Z_2\cup Z_1=Z(gf)\in Z(D)$.

Now let $Z_1, Z_2\in Z(D)$, say $Z_1=Z(f)$, $Z_2=Z(g)$, $f, g\in D$. By (1.3), there is $h\in D$ such that $f\rhd h$, $g\rhd h$. By Lemma 4, $Z(f)\supset Z(h)$, $Z(g)\supset Z(h)$, hence $Z_1\cap Z_2\supset Z(h)\in Z(D)$. Thus $Z_1\cap Z_2\in Z(D)$ because $Z(X)$ is a lattice ([3], 1.10) so that $Z_1\cap Z_2\in Z(X)$. □

Lemma 6. *If* $\mathfrak{z}$ *is a z-filter in* $X$, *then*

$$Z^{-1}(\mathfrak{z})=\{f\in S(X)\colon Z(f)\in\mathfrak{z}\} \tag{3.5}$$

*is a d-ideal in* $S(X)$.

Proof. $\emptyset\notin\mathfrak{z}$ implies $1\notin Z^{-1}(\mathfrak{z})$, and $\mathfrak{z}\neq\emptyset$ implies $Z^{-1}(\mathfrak{z})\neq\emptyset$ by Lemma 3. If $f\in Z^{-1}(\mathfrak{z})$, $g\in S(X)$, $g\rhd f$, then $Z(g)\supset Z(f)$ by Lemma 4 so that $Z(g)\in\mathfrak{z}$, $g\in Z^{-1}(\mathfrak{z})$.

Now let $f, g\in Z^{-1}(\mathfrak{z})$. Define

$$h(x)=(|f(x)|+|g(x)|)^{1/2} \quad (x\in X).$$

Then $h\in S(X)$ by Lemma 2, and $Z(h)=Z(f)\cap Z(g)$ implies $h\in Z^{-1}(\mathfrak{z})$. We show $f\rhd h$.

For this purpose, define

$$k(x)=\begin{cases}0 & \text{if } x\in Z(f),\\ f(x)\cdot\dfrac{1}{h(x)} & \text{if } x\in X-Z(f).\end{cases}$$

Then $k\in S(X)$. In fact, $k$ is obviously continuous at the points of $X-Z(f)$. The equality

$$|k(x)|=\frac{|f(x)|^{1/2}}{(|f(x)|+|g(x)|)^{1/2}}\cdot|f(x)|^{1/2}$$

shows by (2.10) that the same holds at the points of $Z(f)$. Finally $f=kh$ is obvious.

We prove $g\rhd h$ similarly. □

Lemma 7. *If $D$ is a d-ideal in $S(X)$, $\mathfrak{z}$ a z-filter in $X$, then*

$$Z^{-1}(Z(D)) \supset D, \quad Z(Z^{-1}(\mathfrak{z})) = \mathfrak{z}. \quad \square \tag{3.6}$$

Lemma 8. *If $D$ is a maximal d-ideal, then $Z(D)$ is a maximal z-filter, and $D=Z^{-1}(Z(D))$.*

Proof. For a $z$-filter $\mathfrak{z}'\supset Z(D)$, we have by (3.6) $Z^{-1}(\mathfrak{z}')\supset Z^{-1}(Z(D))\supset D$, hence $Z^{-1}(\mathfrak{z}')=Z^{-1}(Z(D))=D$, and $\mathfrak{z}'=Z(Z^{-1}(\mathfrak{z}'))=Z(D)$. $\square$

Lemma 9. *If $\mathfrak{z}$ is a maximal z-filter, then $Z^{-1}(\mathfrak{z})$ is a maximal d-ideal.*

Proof. For a $d$-ideal $D'\supset Z^{-1}(\mathfrak{z})$, we have by (3.6) that $Z(D')\supset Z(Z^{-1}(\mathfrak{z}))=\mathfrak{z}$, hence $Z(D')=\mathfrak{z}$, and $D'\supset Z^{-1}(\mathfrak{z})=Z^{-1}(X(D'))\supset D'$ so that $D'=Z^{-1}(\mathfrak{z})$. $\square$

Lemma 10. *The formulas*

$$\mathfrak{z} = Z(D), \quad D = Z^{-1}(\mathfrak{z}) \tag{3.7}$$

*establish a bijection from the set of all maximal d-ideals $D$ in $S(X)$ onto the set of all maximal z-filters $\mathfrak{z}$ in $X$.* $\square$

**4. Construction of $\upsilon X$.** Let $X$ be a Tychonoff space. Our purpose is to show that $\upsilon X$ or, more precisely, a space homeomorphic to $\upsilon X$ can be constructed as soon as we know the relation $\rhd$ in $S(X)$ (not necessarily the semigroup structure of $S(X)$).

In fact, the knowledge of this relation permits us to determine all $d$-ideals, hence all maximal $d$-ideals in $S(X)$; thus we have, by Lemma 10, a set from which a bijection goes onto the set of all maximal $z$-filters in $X$. In order to know $\upsilon X$ as a set, we have to select those maximal $d$-ideals $D$ for which $Z(D)$ is a *real* $z$-filter.

Lemma 11. *If $f, g\in S(X)$, then $Z(f)\subset Z(g)$ holds iff $g$ belongs to every maximal d-ideal containing $f$.*

Proof. If $D$ is a maximal $d$-ideal, $f\in D$, and $Z(f)\subset Z(g)$, then $Z(f)\in Z(D)$, hence $Z(g)\in Z(D)$ by Lemma 5, and $g\in D$ by Lemma 8.

Conversely, if $x\in Z(f)-Z(g)$, then $\mathfrak{z}=\{Z\in Z(X): x\in Z\}$ is a maximal $z$-filter ([3], 3.18) such that $Z(f)\in\mathfrak{z}$; $Z(g)\notin\mathfrak{z}$, hence $Z^{-1}(\mathfrak{z})$ is a maximal $d$-ideal (by Lemma 9) such that $f\in Z^{-1}(\mathfrak{z})$, $g\notin Z^{-1}(\mathfrak{z})$. $\square$

Lemma 12. *For a maximal d-ideal $D$, $Z(D)$ is a real maximal z-filter iff $f_n\in D$ $(n\in\mathbf{N})$ implies the existence of $g\in D$ such that $Z(g)\subset Z(f_n)$ for $n\in\mathbf{N}$.*

Proof. If $Z(D)$ is a real $z$-filter, and $f_n\in D$ for $n\in\mathbf{N}$, then

$$Z_0 = \bigcap_1^\infty Z(f_n)\in Z(D),$$

hence $Z_0=Z(g)$ for some $g\in D$. Conversely, suppose $f_n\in D$, $g\in D$, $Z(g)\subset Z(f_n)$ for every $n\in\mathbf{N}$. Then $Z_0$ defined as above belongs to $Z(X)$ ([3], 1.14), and $Z(g)\subset Z_0$ implies $Z_0\in Z(D)$ by Lemma 5. □

By Lemmas 11 and 12, the knowledge of $\rhd$ permits to determine those maximal $d$-ideals $D$ for which $Z(D)\in \upsilon X$. For $f\in S(X)$, $Z=Z(f)$, the set $B(Z)$ defined by (4) is composed of all $Z(D)\in \upsilon X$ for which $f\in D$ (Lemma 8). Hence we obtain a space homeomorphic to $\upsilon X$ by defining the points to be those maximal $d$-ideals $D$ that fulfil the condition formulated in Lemma 12, and by choosing for a closed base the system of the sets $B(f)$ consisting of those points $D$ for which $f\in D$ $(f\in S(X))$.

**5. Main results.** We get as an immediate consequence of the argument above:

Theorem 3. *Let $X$ and $Y$ be Tychonoff spaces, $\mathbf{S}_1$ and $\mathbf{S}_2$ quasi-real semigroups. Define $S_1(X)$ and $S_2(Y)$ to be the semigroups of all continuous functions $f: X\to\mathbf{S}_1$ and $g: Y\to\mathbf{S}_2$, respectively. If $S_1(X)$ and $S_2(Y)$ are d-isomorphic, then $X$ and $Y$ are homeomorphic. In particular, $X$ and $Y$ are homeomorphic provided they are realcompact.* □

We obtain Theorem C as a corollary because $\mathbf{R}$ is a quasi-real semigroup and semigroup isomorphy implies $d$-isomorphy. One can, of course, prove this theorem directly, without making use of the definitions and results in Section 2; the statements concerning $\mathbf{S}$ quoted in Section 3 are obvious in the case $\mathbf{S}=\mathbf{R}$.

Moreover, the argument applied in [4] leads to the following sharper form of Theorem B:

Theorem 4. *For arbitrary topological spaces $X$ and $Y$, if the multiplicative semigroups $C(X)$ and $C(Y)$ are d-isomorphic, then the rings $C(X)$ and $C(Y)$ are isomorphic.* □

**6. The case $\mathbf{S}=\mathbf{R}$.** If $\mathbf{S}=\mathbf{R}$ then $S(X)=C(X)$. If we agree in calling $d$-ideals of a ring $A$ the $d$-ideals of the multiplicative semigroup of $A$, Lemmas 8 and 9 imply, according to [3], 2.5:

Theorem 5. *The maximal d-ideals of the ring $C(X)$ coincide with the maximal ideals.* □

It is a natural question whether there is some connection between $d$-ideals and ideals of $C(X)$ in general.

Lemma 13. *Every d-ideal $D$ of a ring $A$ is a left ideal in $A$.*

Proof. It suffices to prove that $f, g\in D$ implies $f-g\in D$. Now there is $h\in D$ such that $f=f_1h$, $g=g_1h$ for some $f_1, g_1\in A$, hence $f-g=(f_1-g_1)h\in D$. □

In particular, every $d$-ideal of the (commutative) ring $C(X)$ is an ideal. The converse is not true in general. In fact, let $X=\mathbf{R}$,

$$(6.1)\qquad f_0(x)=\max(x,0),\quad g_0(x)=\min(x,0)\quad (x\in X),$$

and let $I$ be the ideal generated by $\{f_0, g_0\}$, i.e.,

$$(6.2)\qquad I=\{ff_0+gg_0\colon f,\ g\in C(X)\}.$$

Suppose $h\in I, f_0\rhd h, g_0\rhd h$. Then

$$(6.3)\qquad f_0=f_1h,\quad g_0=g_1h,\quad f_1,\ g_1\in C(X),$$

hence $Z(h)\subset Z(f_0)\cap Z(g_0)=\{0\}$. Consequently

$$(6.4)\qquad (-\infty,0)\subset Z(f_1),\quad (0,+\infty)\subset Z(g_1).$$

Select $f, g\in C(X)$ such that $h=ff_0+gg_0$; then (by (6.3))

$$(6.5)\qquad h=(ff_1+gg_1)h$$

so that

$$(6.6)\qquad f(x)f_1(x)+g(x)g_1(x)=1$$

for $x\neq 0$ and, by continuity, for $x=0$, too. The first member of the left-hand side of (6.6) vanishes for $x<0$, the second one for $x>0$ (see (6.4)), hence both vanish for $x=0$: a contradiction.

The ideal $I$ in the preceding example was generated by a subset of cardinality 2. For 1 instead of 2, we have the following obvious

Lemma 14. *Every proper left ideal generated by an element of a ring $A$ with unity element is a d-ideal.* □

For another result in the same direction, let us recall that an ideal $I$ of $C(X)$ is said to be a *z-ideal* iff $I=Z^{-1}(Z(I))$ (with a notation analogous to (3.3) and (3.5)).

Lemma 15. *Every proper z-ideal of the ring $C(X)$ is a d-ideal.*

Proof. By [3], 2.3, $Z(I)$ is a $z$-filter for every proper ideal $I$ of $C(X)$, hence Lemma 6 furnishes the statement. □

On the other hand, a $d$-ideal of $C(X)$ need not be a $z$-ideal. Again for $X=\mathbf{R}$, the ideal $I$ generated by $\{h_0\}$, where $h_0(x)=x$ for $x\in X$, is a $d$-ideal by Lemma 14, but fails to be a $z$-ideal ([3], 2.4).

We can summarize our results as follows:

Theorem 6. *We have the following implications in $C(X)$:*

*proper z-ideal $\Rightarrow$ d-ideal $\Rightarrow$ proper ideal,*

*and none of them can be reversed in general.* □

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