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The random martingale central limit theorem and weak law of large numbers with o-rates

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Dedicated to Professor Béla Szőkefalvi-Nagy on the occasion of his 70th birthday on 29 July 1983, in friendship and great respect

1. Introduction

Although the central limit theorem (CLT) for randomly indexed sums of random variables (r. vs.) has been quite a popular field of research in the past 30 years or so in the case of independent r.vs., the situation is quite different in the more difficult case of "dependent" r.vs. This convergence theorem has been equipped with large-O rates in a variety of papers (for the independent case see, e.g., [24], [26], [16], [23]; and for the dependent case [25], [11]) as well as with little-o rates, however much less so; see [4], [23] in the independent case or [6], [18], [8], [22], [10], [19], [20] in the (classical) non-random case.

On the other hand, the random weak law of large numbers (WLLN) seems hardly — with the exception of MOGYORÓDI [17] and CSÖRGŐ and RÉVÉSZ [13] — to have been considered before, even when the r.vs. are independent. For historical comments concerning random limit theorems without rates see [1], [12], [15], and with rates [11].

The purpose of this paper is to consider a comprehensive theorem on o-rates of convergence for normalized randomly indexed sums of not necessarily independent r. vs. which will include both the CLT and WLLN. The type of convergence to be considered will essentially be weak convergence. A particular type of "weak dependency" will be assumed, just as in [11], namely the situation of martingale difference sequences (MDS).

More concretely, this means the following: Let $(X_i)_{i\in \mathbf{N}}$ be a sequence of real valued r. vs. defined on a probability space $(\Omega, \mathscr{A}, P)$; and let $(\mathscr{F}_i)_{i\in \mathbf{P}}$ $(\mathbf{P}:=\mathbf{N}\cup\{0\})$ be an increasing sequence of sub-σ-algebras of $\mathscr{A}$ such that X_i is $\mathscr{F}_i$-measurable

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for each $i \in \mathbf{N}$. Then $(X_i, \mathscr{F}_i)_{i \in \mathbf{P}}, X_0 := 0$, is called a MDS if

(1.1) $$E[X_i | \mathscr{F}_{i-1}] = 0 \quad \text{a.s.} \quad (i \in \mathbf{N}).$$

Let us further recall the concept of a randomly indexed sum of r. vs. Let N_λ, $\lambda \in \mathbf{R}^+$, be an $\mathbf{N}$-valued r.v. defined on $(\Omega, \mathscr{A}, P)$ that is independent of the r.v. X_i, $i \in \mathbf{N}$ for each $\lambda \in \mathbf{R}^+$, and let $N_\lambda \to \infty$ in probability for $\lambda \to \infty$. The normalized random sums to be considered in this paper are of the form

(1.2) $$T_{N_\lambda} := \varphi(N_\lambda) S_{N_\lambda}$$

where $S_{N_\lambda} := \sum_{i=1}^{N_\lambda}$, and where $\varphi: \mathbf{N} \to \mathbf{R}^+$ is a positive, normalizing function. The weak convergence concerns the o-rate with which $E[f(T_{N_\lambda})] - E[f(Z)]$ tends to zero for $\lambda \to \infty$. Here the limiting r.v. Z is assumed to be φ-decomposable. This means that for each $n \in \mathbf{N}$ there exist independent r.vs. Z_i, $Z_i = Z_{i,n}$, $1 \leq i \leq n$, such that the distribution P_Z of Z can be represented as

(1.3) $$P_Z = P_{\varphi(n) \sum_{i=1}^{n} Z_i}.$$

With these preparations the general theorem of this paper may be stated roughly as follows: If $(X_i | \mathscr{F}_i)_{i \in \mathbf{P}}$ is a MDS, Z a φ-decomposable r.v. with zero mean such that the r-th absolute moments of the r.vs. X_i, $i \in \mathbf{N}$, and the decomposition components Z_i, $i \in \mathbf{N}$, are finite for some $r \in \mathbf{N}$, and both sequences $(X_i)_{i \in \mathbf{N}}$, $(Z_i)_{i \in \mathbf{N}}$ satisfy a generalized, random Lindeberg condition of order r (see (2.6)) and are related by

$$E\left[\left(\sum_{i=1}^{N_\lambda} E[|X_i|^r] + E[|Z_i|^r]\right)^{-1} (\varphi(N_\lambda))^{j-r} \sum_{i=1}^{N_\lambda} |E[X_i^j | \mathscr{F}_{i-1}] - E[Z_i^j]|\right] \to 0$$

for $\lambda \to \infty$ and each $1 \leq j \leq r$, then

$$|E[f(T_{N_\lambda})] - E[f(Z)]| = o_f\left\{E\left[(\varphi(N_\lambda))^r \sum_{i=1}^{N_\lambda} E[|X_i|^r] + E[|Z_i|^r]\right]\right\}$$

for $\lambda \to \infty$ for all $f \in C_B^r(\mathbf{R})$ (see definition (2.1)) provided an additional boundedness condition (see (3.4)) is assumed.

By specializing the limiting r.v. Z and the normalizing function φ the random sum CLT as well as the random WLLN, both equipped with o-rates, will be deduced as particular cases of this general theorem.

The results of this paper generalize those known in the area in several respects. It contains those of BUTZER and HAHN [7], [8] for the case of independent r.vs. and classical (non-random) sums since a sequence of independent r.vs. with zero means builds a MDS. It also includes a result of A. K. BASU [3] on the CLT for "dependent" r.vs. as well as of Z. RYCHLIK and D. SZYNAL [23] on the random CLT with o-rates for independent r.vs. The fact that the moments of X_i and Z_i coincide up to the order r, a condition needed in [23] and which would correspond to condition (3.7) of this paper, is now replaced by the weaker hypothesis (3.5).

Concerning the proofs, they are based upon a modification of the Lindeberg—Trotter operator approach tailored to the situation of not necessarily independent r.vs. as well as of randomly indexed r.vs. X_i which are independent of the index variable N_λ, $\lambda\in\mathbf{R}^+$, as already applied in BUTZER—SCHULZ [11]. This time the proofs are more difficult than for the large-O theorems of [11] not so much because of their length but since they use further basic concepts of probability such as the random Lindeberg condition. So in this sense the equipment of convergence assertions with little-o rates is a more typical generalization than that with large-O rates.

Section 2 is concerned with questions of notation as well as with the definitions of generalized Lindeberg and Liapounov conditions of given order and connection between these and the Feller condition in the case of random sums. Section 3 is devoted to the general theorem of the paper stated above, and Sections 4 and 5 to the random CLT and WLLN, respectively.

2. Notations; Generalized random Lindeberg and related conditions

In the following, $C_B=C_B(\mathbf{R})$ will denote the class of all real valued, bounded, uniformly continuous functions defined on the reals $\mathbf{R}$, endowed with norm $\|f\|_{C_B}:=\sup_{x\in\mathbf{R}}|f(x)|$. For $r\in\mathbf{P}=\{0,1,2,\ldots\}$ we set

$$C_B^0:=C_B,\quad C_B^r:=\{f\in C_B;\ f',f'',\ldots,f^{(r)}\in C_B\},\tag{2.1}$$

the semi-norm on C_B^r given by $|g|_{C_B^r}:=\|g^{(r)}\|_{C_B}$. Lipschitz classes of index $r\in\mathbf{N}$ and order α, $0<\alpha\leqq r$, will also be needed. These are defined for $f\in C_B$ by

$$\mathrm{Lip}\,(\alpha;\ r;\ C_B):=\{f\in C_B;\ \omega_r(t;\ f;\ C_B)\leqq L_f t^\alpha,\quad t>0\},$$

where L_f is the Lipschitz constant, and

$$\omega_r(t;\ f;\ C_B):=\sup_{|h|\leqq t}\left\|\sum_{k=1}^{r}(-1)^{r-k}\binom{r}{k}f(\cdot+kh)\right\|_{C_B}$$

denotes the r-th modulus of continuity.

The concept of φ-decomposability, defined in (1.3), can be extended to randomly indexed r.vs. since the range of the index r.v. N_λ is a subset of $\mathbf{N}$. In fact, for any decomposable r.v. Z one has by (1.3)

$$P_Z=P_{\varphi(N_\lambda)\sum_{i=1}^{N_\lambda}Z_i}\quad(\lambda\in\mathbf{R}^+).\tag{2.2}$$

If the decomposition r.vs. Z_i; $i \in \mathbf{N}$, are independent of N_λ for each $\lambda \in \mathbf{R}^+$, which will be assumed in the sequel, the usual rules for conditional expectations yield

$$P_Z = \sum_{n=1}^{\infty} p_n P_{\varphi(n)\sum_{i=1}^{n} Z_i} \quad (2.3)$$

where $p_n = p_n(\lambda) := P\{\omega;\, N_\lambda(\omega) = n\}$. This implies that for the expectation of Z,

$$E(Z) = \sum_{n=1}^{\infty} p_n E\left[\varphi(n) \sum_{i=1}^{n} Z_i\right].$$

Another relation that will often be used in the following is

$$E[f(Z)] = E\left[f\left(\varphi(N_\lambda) \sum_{i=1}^{N_\lambda} Z_i\right)\right] = \sum_{n=1}^{\infty} p_n E\left[f\left(\varphi(n) \sum_{i=1}^{n} Z_i\right)\right] \quad (f \in C_B^r), \quad (2.4)$$

valid in view of (2.2) and (2.3); and analogously for the r.vs. T_{N_λ} (recall (1.2)), namely

$$E[f(T_{N_\lambda})] = \sum_{n=1}^{\infty} p_n E[f(T_n)]. \quad (2.5)$$

The following generalization of the well-known Lindeberg-condition will play an important role in the proofs of this paper.

Definition 1. Let $(X_i)_{i \in \mathbf{N}}$ be a sequence of real valued r.vs. having finite moments of order s, $0 < s < \infty$. Then $(X_i)_{i \in \mathbf{N}}$ is said to satisfy the generalized random Lindeberg condition of order s, if for every $\delta > 0$,

$$L_{N_\lambda}^s(\delta) := E\left[\frac{\sum_{i=1}^{N_\lambda} \int_{|x| \geq \delta/\varphi(N_\lambda)} |x|^s \, dF_{X_i}(x)}{\sum_{i=1}^{N_\lambda} E[|X_i|^s]}\right] \to 0 \quad \text{for} \quad \lambda \to \infty. \quad (2.6)$$

In case $r=2$ and $\varphi(N_\lambda) := s_{N_\lambda}^{-1}$, where $s_{N_\lambda} := \left(\sum_{i=1}^{N_\lambda} E[X_i^2]\right)^{1/2}$, one obtains the usual random Lindeberg condition (cf. RYCHLIK [21]).

If the parameter λ is a positive integer n and if, for every n, the r.v. N_λ takes the value n with probability one, and if $\varphi(n) := s_n$, then (2.6) reduces to the Lindeberg condition of order s, introduced in [6], a definition which has in the meantime been taken over and used effectively by PRAKASA RAO [18], RYCHLIK and SZYNAL [22], [23] and BASU [3]. The reader should recall that there are various (different) generalizations of the Lindeberg and Liapounov conditions (see, e.g., BROWN [5], BASU [2], [3]).

The following lemma relates Lindeberg conditions of different orders. It will be shown that under an additional assumption a Lindeberg condition of higher order implies one of lower order.

Lemma 1. *If the generalized random Lindeberg condition of order* $r+\varepsilon$, $r\in\mathbf{N}$, $0<\varepsilon\leqq 1$, *is satisfied, then that of order* r *holds provided there exist constants* $M, \lambda_0\in\mathbf{R}^+$ *such that*

$$\left|\frac{\sum_{i=1}^{N_\lambda} E[|X_i|^{r+\varepsilon}]}{(\varphi(N_\lambda))^{-\varepsilon}\sum_{i=1}^{N_\lambda} E[|X_i|^r]}\right| \leqq M \quad a.s. \quad (\lambda\geqq\lambda_0). \tag{2.7}$$

Proof. Because $|x|\geqq\delta/\varphi(N_\lambda)$ implies $|x|^{r+\varepsilon}\geqq|x|^r(\delta/\varphi(N_\lambda))^\varepsilon$, one has for arbitrary $\varepsilon>0$ according to (2.7)

$$L_{N_\lambda}^{r+\varepsilon}(\delta)\geqq E\left[\frac{\delta^\varepsilon\sum_{i=1}^{N_\lambda}\int_{|x|\geqq\delta/\varphi(N_\lambda)}|x|^r\,dF_{X_i}(x)}{(\varphi(N_\lambda))^\varepsilon\sum_{i=1}^{N_\lambda}E[|X_i|^{r+\varepsilon}]}\right]\geqq$$

$$\geqq E\left[\frac{\delta^\varepsilon\sum_{i=1}^{N_\lambda}\int_{|x|\geqq\delta/\varphi(N_\lambda)}|x|^r\,dF_{X_i}(x)}{M\sum_{i=1}^{N_\lambda}E[|X_i|^r]}\right]=\frac{\delta^\varepsilon}{M}L_{N_\lambda}^r(\delta).$$

The Liapounov condition of order r, introduced in [8], can also be extended to the situation of random sums just in the same manner as the Lindeberg condition.

Definition 2. Let $(X_i)_{i\in\mathbf{N}}$ be a sequence of real valued r.vs. for which the r-th order moment $(0<r<\infty)$ is finite. Then $(X_i)_{i\in\mathbf{N}}$ is said to satisfy the generalized Liapounov condition of order r, if there exists an $\varepsilon>0$ such that

$$\lim_{\lambda\to\infty} E\left[\frac{\sum_{i=1}^{N_\lambda}E[|X_i|^{r+\varepsilon}]}{(\varphi(N_\lambda))^{-\varepsilon}\sum_{i=1}^{N_\lambda}E[|X_i|^r]}\right]=0.$$

Just as in the classical case (cf. [8]) the following lemma holds.

Lemma 2. *If a sequence* $(X_i)_{i\in\mathbf{N}}$ *satisfies the generalized random Liapounov condition of order* r, *then it also satisfies the random Lindeberg condition of order* r.

Proof. Since $|x|\geqq\delta/\varphi(N_\lambda)$ implies $|x|^{r+\varepsilon}\geqq|x|^r(\delta/\varphi(N_\lambda))^\varepsilon$ for each $\varepsilon>0$, one has

$$L_{N_\lambda}^r(\delta)\leqq E\left[\frac{\sum_{i=1}^{N_\lambda}\int_{|x|\geqq\delta/\varphi(N_\lambda)}|x|^{r+\varepsilon}\,dF_{X_i}(x)}{\delta^\varepsilon(\varphi(N_\lambda))^{-\varepsilon}\sum_{i=1}^{N_\lambda}E[|X_i|^r]}\right]\leqq E\left[\frac{\sum_{i=1}^{N_\lambda}E[|X_i|^{r+\varepsilon}]}{\delta^\varepsilon(\varphi(N_\lambda))^{-\varepsilon}\sum_{i=1}^{N_\lambda}E[|X_i|^r]}\right].$$

Since $\delta>0$ is arbitrary, the assertion follows.

It was Z. Rychlik [21] who extended the Feller condition to the situation of random sums. It states

Definition 3. A sequence of real valued r.vs. $(X_i)_{i\in\mathbf{N}}$ with $0<E[X_i^2]<\infty$ is said to satisfy a random Feller condition, if

$$\lim_{\lambda\to\infty} E\left[\max_{1\le i\le N_\lambda} \frac{E[X_i^2]}{s_{N_\lambda}^2}\right] = 0. \tag{2.8}$$

The well-known connection between the Lindeberg and Feller conditions remains also valid in the random case.

Lemma 3. *If a sequence* $(X_i)_{i\in\mathbf{N}}$ *of r.vs. with* $0<E[X_i^2]<\infty$ *satisfies condition* (2.6) *for* $s=2$ *and* $\varphi(N_\lambda)=s_{N_\lambda}^{-1}$, *then* (2.8) *is satisfied.*

Proof. For arbitrary $\delta>0$ and $1\le i\le N_\lambda$ one has

$$E[X_i^2] = \int_{\mathbf{R}} x^2\, dF_{X_i}(x) \le \delta^2 s_{N_\lambda}^2 + \sum_{i=1}^{N_\lambda} \int_{|x|\ge\delta s_{N_\lambda}} x^2\, dF_{X_i}(x) \quad \text{a.s.}$$

This implies that

$$\max_{1\le i\le N_\lambda} \frac{E[X_i^2]}{s_{N_\lambda}^2} \le s_{N_\lambda}^{-2} \sum_{i=1}^{N_\lambda} \int_{|x|\ge\delta s_{N_\lambda}} x^2\, dF_{X_i}(x) \quad \text{a.s.}$$

Taking expectations of both sides yields the assertion.

3. General convergence theorem for MDS with o-rates

The following main approximation theorem for MDS for random sums with "little-o" rates will be established by the Lindeberg—Trotter operator-theoretic approach as tailored to the situation for MDS in [14], this time however modified to the instance of o-rates. For this purpose, additional assumptions are necessary, namely a generalized random Lindeberg condition of order r which is needed not only for the r.vs. X_i, $i\in\mathbf{N}$, but also for the decomposition components Z_i, $i\in\mathbf{N}$, as well as a type of boundedness condition upon the higher order moments of X_i and Z_i (cf. (3.4)) in association with the φ-function.

Theorem 1. *Let* $(X_i, \mathscr{F}_i)_{i\in\mathbf{P}}$ *be a MDS, Z a φ-decomposable r.v. with* $E[Z]=0$ *such that*

$$\zeta_{r,i} := E[|X_i|^r] < \infty, \quad \xi_{r,i} := E[|Z_i|^r] < \infty \quad (i\in\mathbf{N}) \tag{3.1/2}$$

for some $r\in\mathbf{N}$. *Set*

$$M(n) := \sum_{i=1}^{n} (\zeta_{r,i}+\xi_{r,i}) \quad (n\in\mathbf{N}). \tag{3.3}$$

Further assume that

$$(3.4)\qquad (\varphi(N_\lambda))^r M(N_\lambda) = O\{E[(\varphi(N_\lambda))^r M(N_\lambda)]\} \quad a.s. \quad (\lambda\to\infty).$$

a) *If the sequences of r.vs.* $(X_i)_{i\in N}$ *as well as of decomposition components* $(Z_i)_{i\in N}$ *satisfy the generalized random Lindeberg condition* (2.6) *of order* r *and further the condition*

$$(3.5)\qquad E\left[(M(N_\lambda))^{-1}(\varphi(N_\lambda))^{j-r}\sum_{i=1}^{N_\lambda}|E[X_i^j|\mathscr{F}_{i-1}]-E[Z_i^j]|\right] = o(1)$$

for $\lambda\to\infty$ *for each* $1\leq j\leq r$, *one has for each* $f\in C_B^r$,

$$(3.6)\qquad |E[f(T_{N_\lambda})]-E[f(Z)]| = o_f\{E[(\varphi(N_\lambda))^r M(N_\lambda)]\} \quad (\lambda\to\infty).$$

If instead of (3.5) *the stronger condition*

$$(3.7)\qquad E[X_i^j|\mathscr{F}_{i-1}] = E[Z_i^j] \quad a.s. \quad (i\in\mathrm{N},\quad 1\leq j\leq r)$$

is satisfied, then the estimate (3.6) *again holds.*

b) *If the r.vs.* X_i *as well as* Z_i, $i\in\mathrm{N}$, *are identically distributed such that assumption* (3.7) *holds, and if the normalizing function* φ *satisfies the conditions*

$$(3.8)\qquad \varphi(N_\lambda) = o(1) \quad a.s. \quad (\lambda\to\infty),$$

$$(3.9)\qquad \varphi(N_\lambda) = o(E[\varphi(N_\lambda)]) \quad a.s. \quad (\lambda\to\infty),$$

then $f\in C_B^r$ *implies*

$$|E[f(T_{N_\lambda})]-E[f(Z)]| = o_f\{E[(\varphi(N_\lambda))^r N_\lambda(\zeta_{r,1}+\xi_{r,1})]\} \quad (\lambda\to\infty).$$

Proof. a) Setting $R_{n,i}:=\sum_{k=1}^{i-1}X_k+\sum_{k=i+1}^{n}Z_k$, $1\leq k\leq n$, $n\in\mathrm{N}$, an application of Taylor's formula up to the order r to both $f(\varphi(n)R_{n,i}+\varphi(n)X_i)$ and $f(\varphi(n)R_{n,i}+$ $+\varphi(n)Z_i)$ for $f\in C_B^r$ yields

$$f(T_n)-f\left(\varphi(n)\sum_{i=1}^n Z_i\right) = \sum_{i=1}^n\sum_{j=1}^r\frac{(\varphi(n))^j}{j!}\{f^{(j)}(\varphi(n)R_{n,i})X_i^j-f^{(j)}(\varphi(n)R_{n,i})Z_i^j\}+$$

$$+\sum_{i=1}^n\frac{1}{(r-1)!}\int_0^1(1-t)^{r-1}\{f^{(r)}(\varphi(n)R_{n,i}+t\varphi(n)X_i)(\varphi(n)X_i)^r-$$

$$-f^{(r)}(\varphi(n)R_{n,i})(\varphi(n)X_i)^r+f^{(r)}(\varphi(n)R_{n,i}+t\varphi(n)Z_i)(\varphi(n)Z_i)^r-$$

$$-f^{(r)}(\varphi(n)R_{n,i})(\varphi(n)Z_i)^r\}\,dt.$$

If one divides both sides of this equation by $(\varphi(n))^r M(n)$, and then takes the expectations of both sides, one deduces

$$(3.10)\qquad \left|(\varphi(n))^{-r}(M(n))^{-1}\left\{E\left[f(T_n)-f\left(\varphi(n)\sum_{i=1}^{n} Z_i\right)\right]\right\}\right| \leqq$$

$$\leqq (M(n))^{-1}\left\{\left|E\left[\sum_{i=1}^{n}\sum_{j=1}^{r}\frac{(\varphi(n))^{j-r}}{j!}\{f^{(j)}(\varphi(n)R_{n,i})X_i^j-f^{(j)}(\varphi(n)R_{n,i})Z_i^j\}\right]\right|+\right.$$

$$+E\left[\sum_{i=1}^{n}\frac{1}{(r-1)!}\int_0^1(1-t)^{r-1}\{|f^{(r)}(\varphi(n)R_{n,i}+t\varphi(n)X_i)X_i^r-f^{(r)}(\varphi(n)R_{n,i})X_i^r|+\right.$$

$$\left.\left.+|f^{(r)}(\varphi(n)R_{n,i}+t\varphi(n)Z_i)Z_i^r-f^{(r)}(\varphi(n)R_{n,i})Z_i^r|\}\,dt\right]\right\}.$$

Since $f\in C_B^r$, $f^{(r)}$ is uniformly continuous on $\mathbf{R}$, i.e., to each $\varepsilon>0$ there exists a $\delta=\delta(\varepsilon)>0$ such that

$$(3.11)\qquad |f^{(r)}(\varphi(n)R_{n,i}+t\varphi(n)X_i)-f^{(r)}(\varphi(n)R_{n,i})|<\varepsilon \quad (i\in\mathbf{N})$$

if $|t\varphi(n)X_i|<\delta$, thus if $|X_i|<\delta/\varphi(n)$ since $0\leqq t\leqq 1$. Likewise one has an estimate corresponding to (3.11) when X_i is replaced by Z_i.

However, the $\zeta_{r,i}$ and $\xi_{r,i}$ are finite by hypothesis. So

$$(3.12)\qquad E[|\{f^{(r)}(\varphi(n)R_{n,i}+t\varphi(n)X_i)-f^{(r)}(\varphi(n)R_{n,i})\}X_i^r|]=$$

$$=E[|f^{(r)}(\varphi(n)R_{n,i}+t\varphi(n)X_i)-f^{(r)}(\varphi(n)R_{n,i})||X_i|^r\{\mathbf{1}_{|X_i|<\delta/\varphi(n)}+\mathbf{1}_{|X_i|\geqq\delta/\varphi(n)}\}]\leqq$$

$$\leqq \varepsilon\zeta_{r,i}+2|f|_{C_B^r}\int_{|x|\geqq\delta/\varphi(n)}|x|^r\,dF_{X_i}(x).$$

Here $\mathbf{1}_A$ denotes the indicator function of the set $A\subset\Omega$. Analogously one obtains an estimate corresponding to (3.12) when X_i is replaced by Z_i and $\zeta_{r,i}$ by $\xi_{r,i}$.

By applying the same arguments concerning conditional expectations as were used in the proof of the associated "large-O" theorem ([11, Theorem 1a)]), one has for $1\leqq j\leqq r$

(3.13)

$$\left|\sum_{i=1}^{n}E[f^{(j)}(\varphi(n)R_{n,i})(X_i^j-Z_i^j)]\right|=\left|\sum_{i=1}^{n}E[f^{(j)}(\varphi(n)R_{n,i})(E[X_i^j|\mathcal{F}_{i-1}]-E[Z_i^j])]\right|\leqq$$

$$\leqq E\left[\sum_{i=1}^{n}\{|f^j|_{C_B}|E[X_i^j|\mathcal{F}_{i-1}]-E[Z_i^j]|\}\right].$$

Let us now form the inequality (3.10), this time the sum $M(n)$ weighted with the probabilities p_n of (2.3). On account of (2.4), (2.5) and the inequalities (3.13), (3.12) and its counterpart for Z_i, this yields

$$\left|E\left[(\varphi(N_\lambda))^{-r}(M(N_\lambda))^{-1}\left\{f(T_{N_\lambda})-f\left(\varphi(N_\lambda)\sum_{i=1}^{N_\lambda}Z_i\right)\right\}\right]\right| \leqq \tag{3.14}$$

$$\leqq E\left[(M(N_\lambda))^{-1}\sum_{i=1}^{N_\lambda}\sum_{j=1}^{r}\left\{\frac{(\varphi(N_\lambda))^{j-r}}{j!}|f^{(j)}|_{C_B}|E[X_i^j|\mathscr{F}_{i-1}]-E[Z_i^j]|\right\}\right]+\varepsilon+$$

$$+2|f|_{C_B^r}\left\{(M(N_\lambda))^{-1}\left(E\left[\sum_{i=1}^{N_\lambda}\int_{|x|\geqq\delta/\varphi(N_\lambda)}|x|^r\,dF_{X_i}(x)\right]+E\left[\sum_{i=1}^{N_\lambda}\int_{|x|\geqq\delta/\varphi(N_\lambda)}|x|^r\,dF_{Z_i}(x)\right]\right)\right\}.$$

In view of the Lindeberg conditions for the r.vs. X_i and Z_i as well as (3.5) the right side of the foregoing inequality can be made arbitrarily small for $\lambda\to\infty$.

Now on account of condition (3.4) there exist $c_1, \lambda_0\in\mathbf{R}^+$ such that

$$c_1|(\varphi(N_\lambda))^r(M(N_\lambda))|^{-1}\geqq|E[(\varphi(N_\lambda))^r M(N_\lambda)]|^{-1}\quad\text{a.s.}$$

for each $\lambda>\lambda_0$. Since the left side of (3.14) vanishes for $\lambda\to\infty$, this implies that

$$\left|E\left[f(T_{N_\lambda})-f\left(\varphi(N_\lambda)\sum_{i=1}^{N_\lambda}Z_i\right)\right]\right|=o_f\{E[(\varphi(N_\lambda))^r M(N_\lambda)]\}\quad(\lambda\to\infty).$$

Because of (2.2), this gives the desired estimate (3.6).

It is obvious that (3.7) is sufficient for (3.5) to hold.

b) The proof of part b) follows from a) provided one can show that assumption (3.9) implies the random Lindeberg conditions for the X_i and Z_i for $i\in\mathbf{N}$. Since the X_i are now identically distributed, the Lindeberg condition for X_i reduces to

$$\lim_{\lambda\to\infty}E\left[\int_{|x|\geqq\delta/\varphi(N_\lambda)}|x|^r\,dF_{X_1}(x)\right]=0\quad(\delta>0). \tag{3.15}$$

Because of condition (3.9), (3.15) is satisfied if

$$\lim_{\lambda\to\infty}\int_{|x|\geqq\delta/E[\varphi(N_\lambda)]}|x|^r d\,F_{X_1}(x)=0\quad(\delta>0). \tag{3.16}$$

But in view of assumption (3.8) one has $E[\varphi(N_\lambda)]=o(1)$ for $\lambda\to\infty$. Therefore the range of integration in (3.16) approaches the empty set for $\lambda\to\infty$, and so the Lindeberg condition for X_i follows from the absolute continuity property of the Lebesgue integral. Since one can show in the same way that the assumptions of part a) are satisfied for the decomposition components Z_i, the proof of the theorem is complete.

Remark 1. Concerning the possible fulfilment of assumption (3.4), the left side of (3.4) is constant a.s. and so trivially true for usual sums, thus for $\lambda=n\in\mathbf{N}$ when the r.vs. N_n take on the value n with probability 1. A sufficient condition for the validity of (3.7) and so also for (3.5) in the case of identically distributed r.vs. $(X_i)_{i\in N}$ is the requirement $E[X_1^j]=E[Z_1^j]$, $1\leqq j\leqq r$, since then $E[X_1^j|\mathscr{F}_0]=E[X_1^j]$, where $\mathscr{F}_0:=\{\Phi, \Omega\}$.

Remark 2. It is not possible to deduce o-estimates for the *strong* convergence of the r.vs. T_{N_λ} towards the r.v. Z comparable to that in [11] with the help of the modification of a lemma of V. M. ZOLOTAREV [27] given in [11]. For the application of this lemma includes an estimate of the metric $\sup_{t\in R}|F_{T_{N_\lambda}}(t)-F_Z(t)|$ from above by $\sup_{f\in D}|E[f(T_{N_\lambda})]-E[f(Z)]|$, where $D:=\{f\in C_B^r;\ |f^{(r)}|\leqq 1\}$. But since the uniform continuity of $f^{(r)}$ is used in the proof of Theorem 1, in order to deduce a reasonable estimate the latter supremum would have to be taken over a class of functions the r-th derivatives of which are equicontinuous. In this respect one should also recall [9] concerned with connections between the rates of weak and strong convergence in the particular case of the CLT.

4. The random CLT for MDS with o-rates

We now wish to apply our general Theorem 1 to a concrete limiting r.v. Z, namely to the Gaussian distributed r.v. X^* with mean zero and variance 1. However, the resulting random CLT is not a direct application of Theorem 1 since here the random Feller condition is only needed for the r.vs. X_i, $i\in\mathbf{N}$. Together with the random Lindeberg condition for the sequence $(X_i)_{i\in\mathbf{N}}$ it implies just the random Lindeberg condition for the r.vs. Z_i. Furthermore, it is not necessary to assume in part b) of the theorem condition (4.3) which corresponds to the requirement (3.4). The special form of the normalizing function $\varphi(n)$ now makes it possible to deduce (4.3) from (4.6).

Theorem 2. *Let $(X_i, \mathscr{F}_i)_{i\in\mathbf{P}}$ be a MDS such that* (3.1) *holds for some $r\in\mathbf{N}$, $r\geqq 2$, let X^* be a Gaussian distributed r.v. with mean zero and variance* 1, *and let $(a_i)_{i\in\mathbf{N}}$ be any sequence of positive reals with $A_{N_\lambda}:=\left(\sum_{i=1}^{N_\lambda} a_i^2\right)^{1/2}$.*

a) *Assume that the sequence $(X_i)_{i\in\mathbf{N}}$ satisfies the random Lindeberg condition* (2.6) *of order r with $\varphi(N_\lambda):=A_{N_\lambda}^{-1}$, as well as a random Feller-type condition, namely*

$$\lim_{\lambda\to\infty} E\left[\max_{1\leqq i\leqq N_\lambda}\frac{a_i}{A_{N_\lambda}}\right]=0. \tag{4.1}$$

If additionally

$$E\left[\frac{A_{N_\lambda}^{r-j}\sum_{i=1}^{N_\lambda}|E[X_i^j\,|\mathscr{F}_{i-1}]-a_i^jE[X^{*j}]|}{\sum_{i=1}^{N_\lambda}(\zeta_{r,i}+a_i^rE[|X^*|^r])}\right]=o(1)\quad(1\leqq j\leqq r) \tag{4.2}$$

for $\lambda\to\infty$, as well as

$$A_{N_\lambda}^{-r}\sum_{i=1}^{N_\lambda}\zeta_{r,i}=O\left(E\left[A_{N_\lambda}^{-r}\sum_{i=1}^{N_\lambda}\zeta_{r,i}\right]\right)\quad a.s.\quad(\lambda\to\infty), \tag{4.3}$$

then one has for each $f \in C_B^r$,

(4.4)

$$|E[f(A_{N_\lambda}^{-r} S_{N_\lambda})] - E[f(X^*)]| = o_f\left\{E\left[A_{N_\lambda}^{-r} \sum_{i=1}^{N_\lambda} (\zeta_{r,i} + a_i^r E[|X^*|^r])\right]\right\} \quad (\lambda \to \infty).$$

If instead of (4.2) *the stronger condition*

(4.5) $$E[X_i^j | \mathcal{F}_{i-1}] = a_i^j E[X^{*j}] \quad (i \in \mathbf{N}, \quad 1 \leqq j \leqq r)$$

is satisfied, then the estimate (4.4) *again holds.*

b) *If the r.vs. are identically distributed,* $a_i = 1$, $i \in \mathbf{N}$, *and if condition* (4.5) *as well as*

(4.6) $$N_\lambda^{-1/2} = O\{E[N_\lambda^{-1/2}]\} \quad a.s. \quad (\lambda \to \infty)$$

hold, then $f \in C_B^r$ *implies for* $\lambda \to \infty$,

(4.7) $$|E[f(S_{N_\lambda}/\sqrt{N_\lambda})] - E[f(X^*)]| = o_f\{E[N_\lambda^{(2-r)/2}(\zeta_{r,1} + E[|X^*|^r])]\}.$$

Proof. a) The r.v. X^* is φ-decomposable for each $n \in \mathbf{N}$ into n independent, normally distributed r.vs. Z_i, $1 \leqq i \leqq n$, namely $Z_i = a_i X^*$. Moreover, one can ensure as in [11, Theorem 1] that the Z_i, $i \in \mathbf{N}$, N_λ, $\lambda \in \mathbf{R}^+$, as well as the sub-σ-algebras $\mathcal{F}_i$, $i \in \mathbf{N}$, are all independent. So X^* can be decomposed in the form (2.3). Since $E[Z_i^j] = a_i^j E[X^{*j}]$ for $i \in \mathbf{N}$, assumptions (3.4) and (3.5) are satisfied on account of (4.3) and (4.2). Furthermore, the random Lindeberg condition for the X_i and the Feller-type condition (4.1) yield the random Lindeberg condition for the Z_i (cf. [23]). So Theorem 1 may be applied since the moments (3.1/2) exist here, too.

b) Setting $Z_i := X^*$, $i \in \mathbf{N}$, and $\varphi(N_\lambda) := N_\lambda^{-1/2}$ in Theorem 1b), then assumptions (3.7) and (3.9) reduce exactly to conditions (4.5) and (4.6), whereas condition (3.8) is satisfied because $N_\lambda \to \infty$ for $\lambda \to \infty$. It just remains to show that condition (4.6) suffices for the requirement

(4.8) $$N_\lambda^{(2-r)/2} = O(E[N_\lambda^{(2-r)/2}]) \quad \text{a.s.} \quad (\lambda \to \infty),$$

namely for (3.4) with $\varphi(N_\lambda) = N_\lambda^{-1/2}$. In case $r = 2$ there is nothing to prove, and (4.6) coincides with (4.8) for $r = 3$. For $r \geqq 4$ one has

$$E[N_\lambda^{-1/2}] \leqq (E[N_\lambda^{-(r-2)/2}])^{1/(r-2)}$$

by Hölders inequality. This yields that (4.6) follows from (4.8). So assertion (4.7) is a consequence of Theorem 1b).

5. The random WLLN for MDS with o-rates

The final application of Theorem 1 will be the WLLN with o-error bounds for random sums in a version adapted to the applicability of this theorem. Thus instead of being concerned with the usual *stochastic* convergence of the r.vs. T_{N_λ} towards zero, namely of

$$\lim_{\lambda\to\infty} P(\{|T_{N_\lambda}| \geqq \varepsilon\}) = 0 \quad (\varepsilon > 0), \tag{5.1}$$

we plan to estimate the o-rate with which T_{N_λ} converges *weakly* to the degenerate limiting r.v. X_0; namely of

$$\lim_{\lambda\to\infty} |E[f(T_{N_\lambda})] - E[f(X_0)]| = 0 \quad (f\in C_B^r), \tag{5.2}$$

any $r\in \mathbf{N}$. As a matter of fact, the convergence definitions (5.1) and (5.2) are equivalent. Indeed, (5.1) implies (5.2), by standard arguments, and the converse holds since the limiting r.v. X_0 is a constant a.s.

Since $E[f(X_0)] = \int_{\mathbf{R}} f(x)\,dP_{X_0}(x) = f(0)$ for all $f\in C_B^r$, the following formulation of the WLLN with o-rates is feasible.

Theorem 3. *Let* $(X_i, \mathscr{F}_i)_{i\in \mathbf{P}}$ *be a MDS, and let* $r\in\mathbf{N}$.

a) *If the sequence* $(X_i)_{i\in\mathbf{N}}$ *satisfies* (3.1) *as well as the random Lindeberg condition* (2.6) *of order* r, *and if*

$$E\left[\frac{\varphi(N_\lambda)^{j-r} \sum_{i=1}^{N_\lambda} |E[X_i^j|\mathscr{F}_{i-1}]|}{\sum_{i=1}^{N_\lambda} \zeta_{r,i}}\right] = o(1) \quad (\lambda\to\infty,\ 1\leqq j\leqq r) \tag{5.3}$$

for $\lambda\to\infty$; *as well as*

$$(\varphi(N_\lambda))^r \sum_{i=1}^{N_\lambda} \zeta_{r,i} = O\left\{E\left[(\varphi(N_\lambda))^r \sum_{i=1}^{N_\lambda} \zeta_{r,i}\right]\right\} \quad (\lambda\to\infty), \tag{5.4}$$

then one has for each $f\in C_B^r$;

$$|E[f(T_{N_\lambda})] - E[f(X_0)]| = o_f\left\{E\left[(\varphi(N_\lambda))^r \sum_{i=1}^{N_\lambda} \zeta_{r,i}\right]\right\} \quad (\lambda\to\infty).$$

b) *If the sequence* $(X_i)_{i\in\mathbf{N}}$ *satisfies the random Lindeberg condition of order* 1 *with* $\varphi(N_\lambda) := N_\lambda^{-1}$, *as well as* (5.4) *for* $r=1$, *and if*

$$\sum_{i=1}^{N_\lambda} E[|X_i|] = O(N_\lambda) \quad a.s. \quad (\lambda\to\infty), \tag{5.5}$$

then

$$\lim_{\lambda\to\infty} E[f(S_{N_\lambda}/N_\lambda)] = f(0). \tag{5.6}$$

c) *If the r.vs.* X_i, $i\in\mathbf{N}$, *are identically distributed,* $\zeta_1 < \infty$ *and*

$$N_\lambda^{-1} = O\{E[N_\lambda^{-1}]\} \quad (\lambda\to\infty), \tag{5.7}$$

then the random WLLN in the form (5.6) *again holds.*

Proof. a) If one chooses the decomposition components Z_i such that $P_{Z_i}=P_{X_0}$ for all $i\in N$, then $P_{X_0}=\sum_{n=1}^{\infty} p_n P_{\varphi(n)\Sigma_{i=1}^n Z_i}$, and part a) follows from Theorem 1 a) since here the sum $M(n)$, defined in (3.3), reduces to $\sum_{i=1}^{n} \zeta_{r,i}$, and therefore conditions (5.3) and (5.4) are special cases of assumptions (3.5) and (3.4).

b) Part b) follows from a) with $r=1$ and $\varphi(N_\lambda)=N_\lambda^{-1}$, because condition (5.5) implies assumption (3.1), whereas (5.3) is fulfilled because of the definition (1.1) of a MDS.

c) Setting $\varphi(N_\lambda):=N_\lambda^{-1}$, part c) turns out to be a special case of Theorem 1 b) if one considers that condition (3.7) is fulfilled for $r=1$ because of (1.1) and that (3.9) reduces to assumption (5.7).

It is an open question whether the convergence assertions (5.1) and (5.2) are still equivalent to another under suitable conditions if they are equipped with rates. This is generally not the case in the corresponding situation for the CLT, see again BUTZER—HAHN [9].

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