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# The modulus of variation of a function and the Banach indicatrix

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*Dedicated to Professor Béla Szőkefalvi-Nagy on his 70th birthday*

It is well known that the notion of variation of a function was introduced by C. JORDAN in 1881 in the paper [12], devoted to the convergence of Fourier series. In 1924 N. WIENER [22] generalized this notion and introduced the notion of $P$-variation. Finally, L. YOUNG [23] introduced the notion of $\Phi$-variation of a function.

Definition 1 (see [23]). Let $\Phi$ be a strictly increasing continuous function on $[0, \infty)$ and $\Phi(0)=0$. $f$ will be said to have bounded $\Phi$-variation on $[a, b]$, or $f\in V_\Phi$, if

$$v_\Phi(f) = \sup_{\Pi} \sum_{k=1}^{n} \Phi(|f(x_k)-f(x_{k-1})|) < \infty,$$

where $\Pi=\{a\leqq x_0<x_1<\ldots<x_n\leqq b\}$ is an arbitrary partition.

If $\Phi(u)=u$, then $V_\Phi$ coincides with the Jordan class $V$ and when $\Phi(u)=u^p$, $p>1$, it coincides with the Wiener class $V_p$. In 1973 Z. A. CHANTURIA [5] introduced the notion of the modulus of variation of a function.

Definition 2. Let $f$ be bounded on $[a, b]$. The modulus of variation of the function $f$ is the function $\upsilon(n,f)$ defined by $\upsilon(0,f)=0$ and for $n\geqq 1$,

$$\upsilon(n,f) = \sup_{\Pi_n} \sum_{k=0}^{n-1} |f(x_{2k+1})-f(x_{2k})|,$$

where $\Pi_n$ is an arbitrary system of disjoint intervals $(x_{2k}, x_{2k+1})$, $k=0, 1, \ldots n-1$, of the interval $[a, b]$.

The modulus of variation $\upsilon(n, f)$ is non-decreasing and convex upwards ([5], [19]). Such a function will be called a modulus of variation. If a modulus of variation $\upsilon(n)$ is given then the class of functions $f$, given on $[a, b]$, for which

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$v(n, f)=O(v(n))$ when $n\to\infty$, will be denoted by $V[v]$. It is known that if $\Phi$ is convex and $\Phi(u)\not\sim u$ *) on $[0, \delta]$ then $V_\Phi\subset V[n\Phi^{-1}(1/n)]$ is a strict inclusion ([5], [8]).

In 1925 S. BANACH [3] introduced the function $N(y,f)$ for continuous functions $f$: for every $y\in(-\infty, +\infty)$, $N(y,f)$ is equal to the number (finite or infinite) of solutions of equations $f(x)=y$. Following I. P. NATANSON [16] (p. 112) $N(y,f)$ will be called the Banach indicatrix. BANACH [3] proved that a continuous function $f$ belongs to $V$ if and only if $N(y,f)$ is summable on $[m(f), M(f)]$, where $m(f)=\inf_{x\in[a,b]} f(x)$ and $M(f)=\sup_{x\in[a,b]} f(x)$.

S. M. LOZINSKI [14] generalized the notion of the Banach indicatrix for bounded functions which have only discontinuities of the first kind. Denote this class by $W(a,b)$. S. M. LOZINSKI [13] showed that the Banach theorem is valid without assuming the continuity of $f$.

One can obtain the class $W(a,b)$ from $C(a,b)$ by a monotone transformation of the argument, as it follows from the following theorem of O. D. TSERETELI [20] (p. 42) and [21] (p. 131): Let $f\in W(a,b)$. Then there exist functions $\chi$ and $F$ satisfying the following conditions: $\chi$ increases on $[a,b]$, $F$ is continuous on $[\chi(a), \chi(b)]$ and $f(x)=F(\chi(x))$.

The definition of Lozinski is equivalent to the following

Definition 3. Let $f\in W(a,b)$. The Banach indicatrix of $f$ is defined by $N(y,f):=N(y, F)$, where $F$ is determined by the relation $f(x)=F(\chi(x))$.

Since the variation of a function does not vary for monotone transformations of the argument, thus by virtue of Tsereteli's theorem, Lozinski's result is a consequence of Banach's theorem.

T. ZEREKIDZE [24] proved the analogue of Banach's theorem for the classes $V_p$: If $f\in W(a,b)$ and $p>1$ then from the condition

$$\int_{-\infty}^{+\infty} [N(y,f)]^{1/p}\,dy<\infty$$

it follows that $f\in V_p$. The converse does not hold.

The purpose of the present paper is to study the relationship between the degree of summability of the Banach indicatrix and the modulus of variation of the function in question. The results obtained are then applied to some problems of the theory of Fourier series.

Let $\Omega$ be an increasing concave function on $[0, +\infty)$, $\Omega(0)=0$, $\lim_{x\to\infty}\Omega(x)=\infty$, $\lim_{x\to\infty}\frac{\Omega(x)}{x}=0$. The following theorem holds.

*) $\Phi(u)\sim\Psi(u)$ on $[a,b]$ if there exist positive constants $A$ and $B$ such that $A\Phi(u)<\Psi(u)<B\Phi(u)$, when $u\in[a,b]$.

Theorem 1. *If* $f \in W(a, b)$ *and*

$$\int_{m(f)}^{M(f)} \Omega(N(y, f))\, dy < \infty, \tag{1}$$

*the modulus of variation* $\upsilon(n, f)$ *of* $f$ *satisfies the following relation*

$$\sum_{n=1}^{\infty} [2\Omega(n) - \Omega(n-1) - \Omega(n+1)]\upsilon(n, f) < \infty. \tag{2}$$

The proof is based on the following lemma.

Lemma 1. *If* $f \in W(a, b)$, *then*

$$\upsilon(n, f) \leqq 3 \int_{m(f)}^{M(f)} N_n(y, f)\, dy,$$

*where*

$$N_n(y, f) = \begin{cases} N(y, f) & \text{when } N(y, f) \leqq n, \\ n & \text{when } N(y, f) > n. \end{cases}$$

Proof. By virtue of Tsereteli's theorem it suffices to prove the lemma for $f \in C(a, b)$. By the definition of the modulus of variation of a function, for any $\varepsilon > 0$ one can find $2n$ points $\{x_k^{(\varepsilon)}\}_{k=0}^{2n-1}$ such that

$$a \leqq x_0^{(\varepsilon)} < x_1^{(\varepsilon)} \leqq \ldots \leqq x_{2n-2}^{(\varepsilon)} < x_{2n-1}^{(\varepsilon)} \leqq b$$

and

$$\upsilon(n, f) \leqq \sum_{k=0}^{n-1} |f(x_{2k+1}^{(\varepsilon)}) - f(x_{2k}^{(\varepsilon)})| + \varepsilon.$$

Introduce the function

$$g_n(x) = \begin{cases} f(x_k^{(\varepsilon)}) & \text{when } x = x_k^{(\varepsilon)}, \quad k = 0, 1, \ldots, 2n-1, \\ f(x_0^{(\varepsilon)}) & \text{when } x = a, \\ f(x_{2n-1}^{(\varepsilon)}) & \text{when } x = b, \text{ and} \\ \text{linear for all other} & x \in [a, b]. \end{cases}$$

Let

$$m_k = \min\{f(x_k^{(\varepsilon)}),\ f(x_{k+1}^{(\varepsilon)})\}, \quad M_k = \max\{f(x_k^{(\varepsilon)}),\ f(x_{k+1}^{(\varepsilon)})\}.$$

Then on any segment $[x_k^{(\varepsilon)}, x_{k+1}^{(\varepsilon)}]$ the equation $g_n(x) = y$, $y \in [m_k, M_k]$, has a unique solution, whereas the equation $f(x) = y$ has at least one solution, i.e., for any $y$, $N(y, g_n) \leqq N(y, f)$. On the other hand, $N(y, g_n) \leqq 2n + 1$. Therefore

$$N(y, g_n) \leqq \min\{N(y, f), 2n+1\} \leqq 3 \min\{N(y, f), n\} = 3N_n(y, f). \tag{3}$$

Let us estimate the variation of the function $g_n$. We have

$$\upsilon(g_n) \geqq \sum_{k=1}^{2n-1} |f(x_{k+1}^{(\varepsilon)}) - f(x_k^{(\varepsilon)})| \geqq \upsilon(n, f) - \varepsilon,$$

whence by virtue of Banach's theorem and relation (3),

$$\upsilon(n, f) \leqq \upsilon(g_n)+\varepsilon = \int_{m(g_n)}^{M(g_n)} N(y, g_n)\, dy+\varepsilon \leqq 3 \int_{m(f)}^{M(f)} N_n(y, f)\, dy+\varepsilon.$$

Since $\varepsilon$ is arbitrary, the lemma is proved.

Proof of Theorem 1. Introduce the notations

$$\sigma(n) = [\Omega(n)-\Omega(n-1)]\,n, \quad e_n = \{y;\ N(y, f) = n\},$$

$$E_n = \bigcup_{k=1}^{n} e_k = \{y;\ 1 \leqq N(y, f) \leqq n\}, \quad E_n' = \bigcup_{k=n+1}^{\infty} e_k = \{y;\ N(y, f) > n\}.$$

It is easy to see that by the properties of $\Omega$ we have

1) $$\sigma(n) \leqq \Omega(n), \quad n = 1, 2, \ldots,$$

2) $$\frac{\sigma(n)}{n} \geqq \frac{\sigma(n+1)}{n+1}, \quad n = 1, 2, \ldots.$$

Using these relations and Abel's transformation, we get

$$\int_{m(f)}^{M(f)} \Omega(N(y, f))\, dy \geqq \int_{m(f)}^{M(f)} \sigma(N(y, f))\, dy =$$

$$= \sum_{n=1}^{\infty} \int_{e_n} \frac{\sigma(N(y, f))}{N(y, f)} N(y, f)\, dy = \sum_{n=1}^{\infty} \frac{\sigma(n)}{n} \int_{e_n} N(y, f)\, dy \geqq \tag{4}$$

$$\geqq \sum_{k=1}^{n-1} \left(\frac{\sigma(k)}{k} - \frac{\sigma(k+1)}{k+1}\right) \int_{E_k} N(y, f)\, dy + \frac{\sigma(n)}{n} \int_{E_n} N(y, f)\, dy.$$

In virtue of Lemma 1,

$$\upsilon(n, f) \leqq 3 \int_{m(f)}^{M(f)} N_n(y, f)\, dy = 3 \int_{E_n} N(y, f)\, dy + 3n\mu E_n'.$$

From here and (4) it follows that

$$\int_{m(f)}^{M(f)} \Omega(N(y, f))\, dy \geqq$$

$$\geqq \sum_{k=1}^{n-1} \left(\frac{\sigma(k)}{k} - \frac{\sigma(k+1)}{k+1}\right) \left(\frac{1}{3}\upsilon(k, f) - k\mu E_k'\right) + \frac{\sigma(n)}{n} \int_{E_n} N(y, f)\, dy =$$

$$= \frac{1}{3} \sum_{k=1}^{n-1} \left(\frac{\sigma(k)}{k} - \frac{\sigma(k+1)}{k+1}\right) \upsilon(k, f) - \sum_{k=1}^{n} \frac{\sigma(k)}{k} (k\mu E_k' - (k-1)\mu E_{k-1}') + \frac{\sigma(n)}{n} n\mu E_n +$$

$$+ \frac{\sigma(n)}{n} \int_{E_n} N(y, f)\, dy \geqq$$

$$\geqq \frac{1}{3} \sum_{k=1}^{n-1} \left(\frac{\sigma(k)}{k} - \frac{\sigma(k+1)}{k+1}\right) \upsilon(k, f) - \sum_{k=1}^{n} \frac{\sigma(k)}{k} (k\mu E_k' - (k-1) E_{k-1}') + \frac{\sigma(n)}{3n} \upsilon(n, f).$$

But since

$$k\mu E'_k-(k-1)\mu E'_{k-1}=\mu E'_{k-1}-k\mu e_k$$

thus

$$\int_{m(f)}^{M(f)}\Omega(N(y,f))\,dy\geqq\frac{1}{3}\sum_{k=1}^{n}[2\Omega(k)-\Omega(k+1)-\Omega(k-1)]\upsilon(k,f)+$$

$$+\sum_{k=1}^{n}\sigma(k)\mu e_k-\sum_{k=1}^{n}\frac{\sigma(k)}{k}\mu E'_{k-1}.$$

From the latter relation it follows that

$$\sum_{k=1}^{n}[2\Omega(k)-\Omega(k+1)-\Omega(k-1)]\upsilon(k,f)\leqq 3\int_{m(f)}^{M(f)}\Omega(N(y,f))dy+3\sum_{k=1}^{n}\frac{\sigma(k)}{k}\mu E'_{k-1}\leqq$$

$$\leqq 3\int_{m(f)}^{M(f)}\Omega(N(y,f))dy+3\sum_{j=1}^{\infty}\mu e_k\left(\sum_{k=1}^{\infty}\frac{\sigma(j)}{j}\right)=$$

$$=3\int_{m(f)}^{M(f)}\Omega(N(y,f))dy+3\sum_{k=1}^{\infty}\mu e_k\Omega(k)\leqq 6\int_{m(f)}^{M(f)}\Omega(N(y,f))\,dy<\infty.$$

Theorem 1 is proved.

We give some corollaries of Theorem 1.

Corollary 1. *If $f\in W(a,b)$ and for $\alpha>0$,*

$$\int_{m(f)}^{M(f)}\ln^{\alpha}(N(y,f)\,dy<\infty, \tag{5}$$

*then*

$$\sum_{n=1}^{\infty}\frac{\ln^{\alpha-1}(n+1)}{n^2}\upsilon(n,f)<\infty.$$

Proof. Like before, we may assume that $f\in C(a,b)$. Then for $y\in[m(f),M(f)]$, $N(y,f)\geqq 1$, therefore (5) is equivalent to

$$\int_{m(f)}^{M(f)}\ln^{\alpha}(1+N(y,f))\,dy<\infty.$$

Take now $\Omega(x)=\ln^{\alpha}(1+x)$. Then

$$2\Omega(n)-\Omega(n+1)-\Omega(n-1)>c\,\frac{\ln^{\alpha-1}(n+1)}{n^2}$$

whence by virtue of Theorem 1 we obtain the statement of Corollary 1.

Corollary 2. *If $f\in W(a,b)$ and for $p>1$,*

$$\int_{m(f)}^{M(f)}N^{1/p}(y,f)\,dy<\infty,$$

*then*

$$\sum_{n=1}^{\infty} \frac{v(n, f)}{n^{2-1/p}} < \infty.$$

In fact, for the proof it is sufficient to take $\Omega(x)=x^{1/p}$.

Theorem 1 cannot be converted since Theorem 2 holds.

Theorem 2. *Let $\Omega$ satisfy the above conditions. Then there exists a function $f_0 \in C(a, b)$ for which* (2) *is valid, but* (1) *is not fulfilled.*

Proof. Let us show first that there exist an increasing sequence of integers $\{\mu_k\}_{k=0}^{\infty}$ and a sequence of positive numbers $\{b_k\}_{k=1}^{\infty}$ such that

(6)
$$\sum_{k=1}^{\infty} [\Omega(\mu_k) - \Omega(\mu_{k-1})] b_k < \infty$$

and

(7)
$$\sum_{k=1}^{\infty} \Omega(\mu_k - \mu_{k-1}) b_k = \infty.$$

Let

$$\mu_{2k+1} = 2\left[\frac{\Omega^{-1}(2k+1)}{2}\right] - 1, \quad \mu_{2k} = 2\left[\frac{\Omega^{-1}(2k)}{2}\right], \quad \alpha_k = \frac{\Omega^{-1}(k)}{2} - \left[\frac{\Omega^{-1}(k)}{2}\right].$$

Since the function $\Omega^{-1}$ is convex and can be represented as

$$\Omega^{-1}(x) = \int_0^x P(t)\, dt,$$

where $P(t)\uparrow$ on $[0, +\infty)$, and since

$$\lim_{x\to\infty} \frac{\Omega(x)}{x} = 0,$$

we have

$$\lim_{t\to\infty} P(t) = \infty.$$

Taking into account the above facts we have

$$\mu_{k+1} - \mu_k = \Omega^{-1}(k+1) - \Omega^{-1}(k) - 2\alpha_{k+1} - 2\alpha_k - 1 \geqq \int_k^{k+1} P(t)\, dt - 5,$$

i.e., $\mu_{k+1} - \mu_k \to \infty$ when $k \to \infty$; thus $\{\mu_k\}_{k=0}^{\infty}$ increases, beginning with some number $k_0$. It will be assumed without loss of generality that $k_0 = 0$.

Since $\Omega$ is convex upwards, $\Omega(x-y) \geqq \Omega(x) - \Omega(y)$ for $x > y > 0$, hence

(8)
$$\Omega(\mu_{k+1}) - \Omega(\mu_k) \leqq \Omega(\Omega^{-1}(k+1) - 2\alpha_{k+1}) - \Omega(\Omega^{-1}(k) - 2\alpha_k - 1) \leqq$$
$$\leqq \Omega(\Omega^{-1}(k+1)) - \Omega(\Omega^{-1}(k) - 3) \leqq k+1 - \Omega(\Omega^{-1}(k)) + \Omega(3) = 1 + \Omega(3).$$

Denote $\lambda_k=\Omega(\mu_k-\mu_{k-1})$. It is obvious that $\lambda_k\to\infty$, $k\to\infty$.

We shall divide the set of natural numbers $\mathbf{N}$ into subsets $\mathbf{N}=\bigcap_{k=1}^{\infty}\mathbf{N}_k$ in the following way: $\mathbf{N}_1=\{1\}$. If the sets $\mathbf{N}_1, \dots, \mathbf{N}_k$ are already constructed, then $\mathbf{N}_{k+1}$ is constructed as follows: Let $\tilde{\mathbf{N}}_k=\mathbf{N}\setminus\bigcup_{i=1}^{k}\mathbf{N}_i$ and $\Lambda_k=\min_{i\in\tilde{\mathbf{N}}_k}\lambda_i$; then

$$\mathbf{N}'_{k+1}=\{n;\ \Lambda_k\leqq\lambda_n<2\Lambda_k\}.$$

If $|\mathbf{N}'_{k+1}|\geqq|\mathbf{N}_k|$ then we put $\mathbf{N}_{k+1}=\mathbf{N}'_{k+1}$ and if $|\mathbf{N}'_{k+1}|<|\mathbf{N}_k|$, then to the set $\mathbf{N}'_{k+1}$ we add $|\mathbf{N}_k|-|\mathbf{N}'_{k+1}|$ natural numbers successively, beginning with the maximal term of the set $|\mathbf{N}'_{k+1}|$. The obtained set will be $\mathbf{N}_{k+1}$.

In virtue of the construction,

$$(9)\qquad 1)\ \ |\mathbf{N}_{k+1}|\geqq|\mathbf{N}_k|\ \ \text{and}\ \ 2)\ \ \Lambda_{k+1}\geqq 2\Lambda_k.$$

Suppose

$$b_m=\frac{1}{|\mathbf{N}_k|\Lambda_k},\quad \text{when}\quad m\in N_k.$$

It is clear that by virtue of (9), $b_m\geqq b_{m+1}$. Then we have

$$\sum_{m=1}^{\infty}b_m=\sum_{k=1}^{\infty}\sum_{m\in\mathbf{N}_k}\frac{1}{|\mathbf{N}_k|\Lambda_k}=\sum_{k=1}^{\infty}\frac{1}{\Lambda_k}\leqq\Lambda_1\sum_{k=1}^{\infty}\frac{1}{2^k}<\infty,$$

whence, applying (8), we get

$$\sum_{k=1}^{\infty}[\Omega(\mu_k)-\Omega(\mu_{k-1})]b_k<\infty.$$

On the other hand,

$$\sum_{m=1}^{\infty}\lambda_m b_m=\sum_{k=1}^{\infty}\sum_{m\in\mathbf{N}_k}\lambda_m b_m\geqq\sum_{k=1}^{\infty}\Lambda_k\frac{1}{\Lambda_k}=\infty.$$

Thus, the required sequences are constructed.

Let $\sum_{j=k}^{\infty}b_j=y_k$ and $m_k=\mu_k-\mu_{k-1}$. Note that $m_k$ is an odd number. Let us construct the function $f_0$ in the following way: divide the segment $[1/2^k, 1/2^{k-1}]$ into $m_k$ parts by means of points

$$\{x_i^{(k)}\}_{i=1}^{m_k+1},\quad 1/2^k=x_1^{(k)}<x_2^{(k)}<\dots<x_{m_k}^{(k)}<x_{m_k+1}^{(k)}=1/2^{k-1}$$

and let

$$f_0(x)=\begin{cases}\frac{1}{2}\{(y_k+y_{k+1})+(-1)^i(y_k-y_{k+1})\} & \text{when}\quad x=x_i^{(k)},\ i=1,\dots,m_k+1,\\ \text{linear when}\quad x\in[x_i^{(k)},x_{i+1}^{(k)}], & \\ 0\ \text{when}\quad x=0. & \end{cases}$$

Since $y_k \to 0$ when $k \to \infty$, thus $f_0$ is continuous on $[0, 1]$. Further, $N(y, f_0) = m_k$ when $y \in (y_{k+1}; y_k)$, hence, using (7), we get

$$\int_{m(f)}^{M(f)} \Omega(N(y, f_0))\,dy = \sum_{k=1}^{\infty} \int_{y_{k+1}}^{y_k} \Omega(N(y, f_0))\,dy = \sum_{k=1}^{\infty} b_k \Omega(m_k) =$$

$$= \sum_{k=1}^{\infty} b_k \Omega(\mu_k - \mu_{k-1}) = \infty.$$

Next we show that

$$\sum_{n=1}^{\infty} [2\Omega(n) - \Omega(n+1) - \Omega(n-1)]\, v(n, f_0) < \infty. \tag{10}$$

Consider two auxiliary functions

$$f_1(x) = \begin{cases} y_{k+1} & \text{when } \; x \in [1/2^k, 1/2^{k-1}),\; k = 1, 2, \ldots, \\ 0 & \text{when } \; x = 0, \end{cases}$$

and $f_2 = f_0 - f_1$. Then, it is obvious that

$$v(n, f_0) \leqq v(n, f_1) + v(n, f_2). \tag{11}$$

In virtue of the monotonicity of the function $f_1$, for all $n$,

$$v(n, f_1) = y_2 = \sum_{i=2}^{\infty} b_i. \tag{12}$$

Let us estimate now the modulus of variation of the function $f_2$. For a natural $n$ we choose the number $k$ such that

$$\mu_{k-1} = \sum_{i=1}^{k-1} m_i < n \leqq \sum_{i=1}^{k} m_i = \mu_k.$$

Then $v(n, f_2) - v(n-1, f_2) = b_k$ whence, according to (7),

$$\begin{aligned} &\sum_{k=1}^{\infty} [\Omega(n) - \Omega(n-1)][v(n, f_2) - v(n-1, f_2)] = \\ &= \sum_{k=1}^{\infty} \sum_{n=\mu_{k-1}+1}^{\mu_k} [\Omega(n) - \Omega(n-1)]\, b_k = \sum_{k=1}^{\infty} [\Omega(\mu_k) - \Omega(\mu_{k-1})]\, b_k = B < \infty. \end{aligned} \tag{13}$$

Using the relations (11), (12), and (13) we have

$$\sum_{k=1}^{n} [2\Omega(k) - \Omega(k+1) - \Omega(k-1)]\, v(k, f_0) \leqq \sum_{k=1}^{n} [2\Omega(k) - \Omega(k+1) - \Omega(k-1)]\, v(k, f_1) +$$

$$+ \sum_{k=1}^{n} [\Omega(k) - \Omega(k-1)][v(k, f_2) - v(k-1, f_2)] + [\Omega(n) - \Omega(n+1)]\, v(n, f_2) \leqq$$

$$\leqq y_2[\Omega(1) + \Omega(n) - \Omega(n+1)] + B \leqq \Omega(1) y_2 + B,$$

whence the validity of relation (10) follows. Theorem 2 is proved.

The following theorem states the dependence between the degree of summability of the Banach indicatrix and the classes $V_\Phi$.

Theorem 3. *Let* $\Phi$ *be a continuous increasing convex function on* $[0, \infty)$, $\Phi(0)=0$, $\lim_{u\to 0}\frac{\Phi(u)}{u}=0$, *and let*

(14)
$$\Omega(x) = \begin{cases} \int_{1/x}^{1} \frac{1}{t\Phi^{-1}(t)}\,dt & \text{when } x\in[1, \infty), \\ 0 & \text{when } x\in[0, 1). \end{cases}$$

*If* $f\in W(a, b)$ *and* (1) *is fulfilled, then* $f\in V_\Phi$.

For the proof of this theorem two lemmas are needed.

Lemma 2 (see [11], p. 111 or [19], p. 160). *Let* $0\leqq a_n\downarrow$, $0\leqq b_n\downarrow$, *and let the relations* $\sum_{i=1}^{k} a_i \leqq \sum_{i=1}^{k} b_i$ *be true for* $k=1, 2, \ldots, m$. *Then for convex functions* $\Phi$ *the inequality*

$$\sum_{i=1}^{m} \Phi(a_i) \leqq \sum_{i=1}^{m} \Phi(b_i)$$

*holds.*

Lemma 3. *Let* $0<a_n\downarrow$ *and let* $\Phi$ *be a convex increasing function on* $[0, \infty)$ *and* $\Phi(u)>0$ *for* $u>0$. *Then*

(15)
$$\sum_{n=1}^{\infty} a_n \frac{1}{n\Phi^{-1}(1/n)} < \infty \quad \Rightarrow \quad \sum_{n=1}^{\infty} \Phi(a_n) < \infty.$$

Proof. Since $\Phi$ is convex, therefore $\Phi(u)/u$ increases, and hence $u/\Phi^{-1}(u)$ also increases, i.e., the sequence $\left\{\frac{1}{n\Phi^{-1}(1/n)}\right\}$ decreases. Starting from this, by virtue of Cauchy's theorem on numerical series, the convergence of the first series under (15) is equivalent to that of the series

(16)
$$\sum_{n=1}^{\infty} a_{2^n} \frac{1}{\Phi^{-1}(1/2^n)}.$$

From the convergence of series (16) it follows that there exists a natural number $n_0$ such that $a_{2^n}<\Phi^{-1}(1/2^n)$ for $n>n_0$. Since $u/\Phi(u)\downarrow$, from the latter inequality we obtain

$$\frac{a_{2^n}}{\Phi(a_{2^n})} \geqq \frac{\Phi^{-1}(1/2^n)}{\Phi(\Phi^{-1}(1/2^n))} = 2^n\Phi^{-1}(1/2^n) \quad \text{when} \quad n > n_0,$$

or

$$2^n\Phi(a_{2^n}) \leqq a_{2^n}\frac{1}{\Phi^{-1}(1/2^n)}, \quad n > n_0.$$

From this relation and from the convergence of series (16) we obtain that

$$\sum_{n=1}^{\infty} 2^n \Phi(a_{2^n}) \quad \text{and} \quad \sum_{n=1}^{\infty} \Phi(a_n)$$

converge. The lemma is proved.

Proof of Theorem 3. Let us show first that $\Omega$ satisfies the conditions of Theorem 1. In fact, we have

1) $$\lim_{x\to\infty} \frac{\Omega(x)}{x} = \lim_{x\to\infty} \frac{1}{x} \int_{1/x}^{1} \frac{1}{t\Phi^{-1}(t)}\,dt = \lim_{x\to\infty} \frac{1}{x} \int_{1}^{x} \frac{1}{t\Phi^{-1}(1/t)}\,dt =$$

$$= \lim_{x\to\infty} \frac{1}{x\Phi^{-1}(1/x)} = 0,$$

2) $$\lim_{x\to\infty} \Omega(x) = \lim \int_{1/x}^{1} \frac{dt}{t\Phi^{-1}(t)} \geqq \lim_{x\to\infty} \int_{1/x}^{2/x} \frac{dt}{t\Phi^{-1}(t)} \geqq \lim_{x\to\infty} \frac{1}{x} \cdot \frac{1}{(2/x)\Phi^{-1}(2/x)} = \infty,$$

3) the function $\Omega$ is convex upwards, since $\Omega'(x) = \frac{1}{x\Phi^{-1}(1/x)}$ is a decreasing function.

Since all the conditions of Theorem 1 are fulfilled, thus (2) is also satisfied. We will show that (2) implies the relation

(17) $$\sum_{n=1}^{\infty} [\Omega(n) - \Omega(n-1)][\upsilon(n, f) - \upsilon(n-1, f)] < \infty.$$

To this end it is sufficient to prove that

(18) $$\lim_{n\to\infty} [\Omega(n) - \Omega(n-1)]\upsilon(n, f) = 0.$$

By virtue of the convergence of series (2), for any $\varepsilon > 0$ one can find an $n$ such that for any $m > n$ the relation

$$\sum_{k=n}^{m} [2\Omega(k) - \Omega(k+1) - \Omega(k-1)]\upsilon(k, f) < \varepsilon$$

holds, whence, by virtue of the monotonicity of $\upsilon(n, f)$ and the fact that $\Omega(m+1) - \Omega(m) \to 0$, $m \to \infty$, we get

$$\varepsilon > \upsilon(n, f) \sum_{k=n}^{m} [2\Omega(k) - \Omega(k+1) - \Omega(k-1)] =$$

$$= \upsilon(n, f)[\Omega(n) - \Omega(n-1) + \Omega(m) - \Omega(m+1)] \geqq \frac{1}{2} \upsilon(n, f)[\Omega(n) - \Omega(n-1)].$$

Thus (18) is proved and it proves also (17).

But then, since

$$\Omega(n)-\Omega(n-1) = \int_{n-1}^{n} \frac{dt}{t\Phi^{-1}(1/t)} \geqq \frac{1}{n\Phi^{-1}(1/n)},$$

we have

$$\sum_{n=1}^{\infty} \frac{1}{n\Phi^{-1}(1/n)} [v(n, f)-v(n-1, f)] < \infty,$$

and this, by virtue of Lemma 3, gives

$$\sum_{n=1}^{\infty} \Phi\big(v(n, f)-v(n-1, f)\big) < \infty. \tag{19}$$

We may now show that $f \in V_\Phi$. Let us take an arbitrary partition $\Pi = \{a \leqq x_0 < x_1 < \ldots < x_m \leqq b\}$; without loss of generality it may be assumed that

$$|f(x_k)-f(x_{k-1})| \geqq |f(x_{k+1})-f(x_k)|.$$

For every $n=1, 2, \ldots, m$ we have

$$\sum_{k=1}^{n} |f(x_k)-f(x_{k-1})| \leqq v(n, f) = \sum_{k=1}^{n} \big(v(k, f)-v(k-1, f)\big).$$

Therefore, if we take $a_k = |f(x_k)-f(x_{k-1})|$ and $b_k = v(k, f)-v(k-1, f)$, and apply Lemma 2 and relation (19), we have

$$\sum_{k=1}^{m} \Phi\big(|f(x_k)-f(x_{k-1})|\big) \leqq \sum_{k=1}^{m} \Phi\big(v(k, f)-v(k-1, f)\big) \leqq$$

$$\leqq \sum_{k=1}^{\infty} \Phi\big(v(k, f)-v(k-1, f)\big) < \infty.$$

Thus, as it was required, we proved that $f \in V_\Phi$.

Corollary 3. [24] *Let $f \in W(a, b)$, and for $p > 1$,*

$$\int_{m(f)}^{M(f)} [N(y, f)]^{1/p}\, dy < \infty.$$

*Then $f \in V_p$.*

Corollary 4. *Let $f \in W(a, b)$, and for $\alpha > 1$,*

$$\int_{m(f)}^{M(f)} \ln^\alpha\big(N(y, f)+1\big) < \infty.$$

*Then $f \in V_\Phi$, where $\Phi(x) = \exp(-x^{1/(1-\alpha)})$ in $(0, \delta)$; $\delta > 0$.*

We shall show that Theorem 3 cannot be converted.

Theorem 4. *Let the function* $\Phi$ *satisfy the conditions of Theorem* 3, *and let* $\Omega$ *be defined by* (14). *Then there exists a function* $f_0 \in V_\Phi$ *which does not satisfy relation* (1).

Proof. In virtue of Theorem 2 there exists a function $f_0$ which satisfies relation (2) and does not satisfy relation (1). But the previous theorem shows that from (2) it follows $f_0 \in V_\Phi$.

The results obtained will be applied to some problems of the theory of Fourier series.

1. By the well-known Jordan theorem, if a $2\pi$-periodic continuous function $f$ has bounded variation, then its Fourier series $\sigma(f)$ converges uniformly ([12]). This theorem was generalized by WIENER [22] for the class $C \cap V_2$, by MARCINKIEWICZ [15] (p. 40) for the class $C \cap V_p$, by L. YOUNG [23] for the class $C \cap V_\Phi$, where $\Phi(u) = \exp(-u^{-\alpha})$, $0 < \alpha < 1/2$. SALEM [18] obtained the most general condition on $\Phi$, providing the uniform convergence of Fourier series of the class $C \cap V_\Phi$, which reads as follows: Let $\Phi$ be a convex increasing function, and let $\Psi$ be a function, complementary in the sense of Young*) to the function $\Phi$; if $f \in C \cap V_\Phi$ and

$$\sum_{n=1}^{\infty} \Psi\left(\frac{1}{n}\right) < \infty, \tag{20}$$

then $\sigma(f)$ converges uniformly.

K. I. OSKOLKOV [17] proved that (20) is equivalent to the condition

$$\int_0^1 \ln \frac{1}{\Phi(u)}\, du < \infty.$$

A. M. GARSIA and S. SAWYER [9] proved that if $f \in C(0, 2\pi)$ and

$$\int_{m(f)}^{M(f)} \ln N(y, f)\, dy < \infty, \tag{21}$$

then $\sigma(f)$ converges uniformly.

From Corollary 1 it follows that if (21) is satisfied then

$$\sum_{k=1}^{\infty} \frac{v(k, f)}{k^2} < \infty. \tag{22}$$

But if $f \in C(0, 2\pi)$ and (22) holds true, then as it was proved in [6], the Fourier series of the function $f$ converges uniformly, i.e., the theorem of Garsia—Sawyer is the result of Corollary 2 from [6].

*) $\Psi(u) = \max\limits_{v \geq 0} \{uv - \Phi(v)\}$.

2. V. O. Asatiani [1] obtained the analogue of condition (22) for $(C, -\alpha)$-summability $(0<\alpha<1)$ of Fourier series. He proved that if $f \in C(0, 2\pi) \cap V[\upsilon]$, and for $0<\alpha<1$,

$$\sum_{k=1}^{\infty} \frac{\upsilon(k)}{k^{2-\alpha}} < \infty,$$

then $\sigma(f)$ is uniformly $(C, -\alpha)$-summable to $f$. From this result and Corollary 2 we have

Theorem 5. *Let* $f \in C(0, 2\pi)$ *and assume that for* $0<\alpha<1$,

$$\int_{m(f)}^{M(f)} N^{\alpha}(y, f)\, dy < \infty.$$

*Then* $\sigma(f)$ *is uniformly* $(C, -\alpha)$*-summable to* $f$.

3. Wiener's criterion on the continuity of functions of bounded variation is well known: Let

$$\min\{f(x-0),\ f(x+0)\} \leqq f(x) \leqq \max\{f(x-0),\ f(x+0)\} \tag{23}$$

for any $x$, and let $a_k$ and $b_k$ be the Fourier coefficients of the function $f$, $\varrho_k = \sqrt{a_k^2 + b_k^2}$. If $f \in V[0, 2\pi]$ then for $f$ to be continuous, each of the following conditions is necessary and sufficient:

$$\sum_{k=1}^{n} k^2 \varrho_k^2 = o(n), \tag{24}$$

$$\sum_{k=1}^{n} k \varrho_k = o(n). \tag{25}$$

S. M. Lozinski [14] showed that instead of conditions (24) or (25) one may take

$$\sum_{k=1}^{n} \varrho_k = o(\ln n), \tag{26}$$

$$\sum_{k=n}^{\infty} \varrho_k^2 = o\left(\frac{1}{n}\right). \tag{27}$$

B. I. Golubov [10] applied these results to the classes $V_p$ when $1<p<2$, and showed that for the classes $V_p$ with $p \geqq 2$ a similar theorem does not hold. Z. A. Chanturia [7] (see also [8]) proved a theorem containing all of the previous results: If $f$ satisfies condition (23), and its modulus of variation satisfies the condition

$$\sum_{n=1}^{\infty} \frac{\upsilon^2(n, f)}{n^2} < \infty, \tag{28}$$

then for the function $f$ to be continuous, each of conditions (24)—(27) is necessary and sufficient.

From the theorems of Wiener and Banach—Lozinski it follows that if $f \in W(0, 2\pi)$ satisfies condition (23) and its Banach indicatrix is summable then each of conditions (24)—(27) is necessary and sufficient for the continuity of the function $f$. We shall now prove a theorem which is much stronger and is in certain sense best possible.

Theorem 6. *If $f \in W(0, 2\pi)$ satisfies condition* (23) *and its Banach indicatrix satisfies the condition*

$$\int_{m(f)}^{M(f)} N^{1/2}(y, f)\, dy < \infty, \tag{29}$$

*then each of conditions* (24)—(27) *is necessary and sufficient for the continuity of $f$.*

Proof. It suffices to prove that (29) implies (28). By virtue of Corollary 2, (29) yields

$$\sum_{n=1}^{\infty} \frac{\upsilon(n, f)}{n^{3/2}} < \infty. \tag{30}$$

Since the general term of the last series decreases monotonically, we have

$$\frac{\upsilon(n, f)}{n^{3/2}} = o\left(\frac{1}{n}\right),$$

or $\upsilon(n, f) \leqq cn^{1/2}$. Therefore

$$\frac{\upsilon^2(n, f)}{n^2} \leqq c \frac{\upsilon(n, f)}{n^{3/2}}.$$

The latter inequality and the convergence of series (30) imply the convergence of (28), which was to be proved.

We shall now show that Theorem 6 is, in a certain sense, best possible, namely, if we take an integral class wider than (29) then Theorem 6 does not hold; more exactly, the following statement is true.

Theorem 7. *Let $\Omega$ be a convex upwards increasing function. If*

$$\lim_{u \to \infty} \frac{\Omega(u)}{\sqrt{u}} = 0, \tag{31}$$

*then there exists a function $f_0 \in C(0, 2\pi)$ for which*

$$\int_{m(f)}^{M(f)} \Omega(N(y, f))\, dy < \infty,$$

*but* (24) *and* (27) *do not hold.*

Proof. By virtue of (31) we may choose an increasing sequence of even natural numbers $\{n_k\}_{k=1}^{\infty}$ such that $n_{k+1}/n_k \geqq q > 1$ and

$$\sum_{k=1}^{\infty} \frac{\Omega(n_k)}{\sqrt{n_k}} < \infty. \tag{32}$$

Let $c_k = \sum_{i=k}^{\infty} \frac{1}{\sqrt{n_i}}$; then

$$c_k^2(n_k - n_{k-1}) = \left(\sum_{i=k}^{\infty} \frac{1}{\sqrt{n_i}}\right)^2 (n_k - n_{k-1}) \geqq \frac{1}{n_k}(n_k - n_{k-1}) = 1 - \frac{n_{k-1}}{n_k} \geqq 1 - \frac{1}{q} > 0,$$

i.e.,

$$\sum_{k=1}^{\infty} c_k^2(n_k - n_{k-1}) = \infty. \tag{33}$$

Take now $n_0 = 0$ and choose the sequence $\{B_n\}_{n=1}^{\infty}$ with $B_n = c_k$, when $n_{k-1}/2 < n \leqq n_k/2$. It is clear that $B_n \downarrow 0$, and in virtue of (33),

$$\sum_{n=1}^{\infty} B_n^2 = \infty. \tag{34}$$

Following the scheme of [8] we construct the function $f_0$ as follows:

$$f_0(x) = \begin{cases} B_k & \text{when } x \in I_{2k+1}, \quad k = 1, 2, \dots, \\ 0 & \text{when } x \in I_{2k}, \quad x \in \left[\frac{\pi^2}{2}, 2\pi\right] \cup \left[0, \frac{3}{2}\right], \\ \text{linear for all other } x & \text{from } [0, 2\pi], \end{cases}$$

where $I_k$ is a specially chosen sequence of segments such that $I_k$ lies to the right of $I_{k-1}$.

The fact that if (34) is fulfilled then $f_0$ does not satisfy conditions (24) and (27), but

$$\int_{m(f)}^{M(f)} \Omega(N(y, f_0))\, dy < \infty,$$

is proved in [8]. In fact, using (32) we have

$$\int_{m(f)}^{M(f)} \Omega(N(y, f_0))\, dy = \sum_{k=1}^{\infty} \int_{c_{k+}}^{c_k} \Omega(N(y, f_0))\, dy =$$

$$= \sum_{k=1}^{\infty} (c_k - c_{k+1}) \Omega(n_k) = \sum_{k=1}^{\infty} \frac{\Omega(n_k)}{\sqrt{n_k}} < \infty.$$

Theorem 7 is proved.

It should be noted, finally, that some of the results of the present paper were published without proof in [2].

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