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An inequality between symmetric function means of positive operators

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Dedicated to Professor B. Szőkefalvi-Nagy on his seventieth birthday

1. There are various methods of averaging of an n-tuple $\vec{A}=(A_1, \ldots, A_n)$ of bounded *positive* (semi-definite) operators on a Hilbert space. The most basic are the *arithmetic mean* $(A_1+\ldots+A_n)/n$ and the *harmonic mean* $n(A_1^{-1}+\ldots+A_n^{-1})^{-1}$ (provided all A_i are invertible). ANDERSON and TRAPP [2] called $(A_1^{-1}+\ldots+A_n^{-1})^{-1}$ the *parallel sum* of the n-tuple $\vec{A}$, and denoted it by $A_1: \ldots :A_n$, or in short $\prod_{i=1}^{n}: A_i$. Further they gave a variational description for parallel sum;

$$\left\langle x, \left(\prod_{i=1}^{n}: A_i\right) x\right\rangle = \inf\left\{\sum_{i=1}^{n} \langle x_i, A_i x_i\rangle \,\middle|\, x = \sum_{i=1}^{n} x_i\right\}, \tag{1}$$

where $\langle x, y\rangle$ denotes the inner product of the vectors x and y. Formula (1) was then used to define the parallel sum for a general n-tuple of positive operators.

For an n-tuple of positive numbers, $\vec{\alpha}=(\alpha_1, \ldots, \alpha_n)$, MARCUS and LOPES [5] defined symmetric function means (or Marcus—Lopes means) $E_{k,n}(\vec{\alpha})$ by

$$E_{k,n}(\vec{\alpha}) \equiv \frac{e_{k,n}(\vec{\alpha})}{e_{k-1,n}(\vec{\alpha})}, \quad k = 1, 2, \ldots, n \tag{2}$$

where $e_{k,n}(\vec{\alpha})$ is the normalized k-th elementary symmetric function of $\vec{\alpha}=(\alpha_1, \ldots, \alpha_n)$, that is,

$$e_{0,n}(\vec{\alpha}) \equiv 1 \quad \text{and} \quad e_{k,n}(\vec{\alpha}) \equiv \frac{\sum_{1\le i_1<\ldots<i_k\le n} \prod_{j=1}^{k} \alpha_{i_j}}{\binom{n}{k}}.$$

Using an equivalent version of definition (2), ANDERSON, MORLEY, and TRAPP [3] introduced two kinds of *symmetric function means* for an n-tuple $\vec{A}=(A_1, \ldots, A_n)$

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of positive operators;

$$\mathfrak{S}_{1,n}(\vec{A}) \equiv \left(\sum_{i=1}^{n} A_i\right)/n \quad \text{(arithmetic mean)},$$

$$\mathfrak{s}_{n,n}(\vec{A}) \equiv n\left(\prod_{i=1}^{n}{:}\ A_i\right) \quad \text{(harmonic mean)},$$

and

$$\mathfrak{S}_{k,n}(\vec{A}) \equiv \sum_{i=1}^{n}\left\{\left(\frac{1}{n-k+1}A_i\right) : \left(\frac{1}{k-1}\mathfrak{S}_{k-1,n-1}(\vec{A}_{(i)})\right)\right\}, \quad k = 2, \ldots, n$$

$$\mathfrak{s}_{k,n}(\vec{A}) \equiv \prod_{i=1}^{n}{:}\ \{kA_i + (n-k)\mathfrak{s}_{k,n-1}(\vec{A}_{(i)})\}, \quad k = 1, \ldots, n-1,$$

where $\vec{A}_{(i)}$ denotes the $(n-1)$-tuple $(A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n)$. By definition both $\mathfrak{S}_{k,n}(\vec{A})$ and $\mathfrak{s}_{k,n}(\vec{A})$ are invariant under permutations of $A_1, \ldots, A_n$, and the maps $\vec{A} \mapsto \mathfrak{S}_{k,n}(\vec{A})$ and $\vec{A} \mapsto \mathfrak{s}_{k,n}(\vec{A})$ are positively homogeneous and monotone with respect to coordinatewise ordering. If all A_i are invertible, then

$$\mathfrak{S}_{k,n}(\vec{A}^{-1})^{-1} = \mathfrak{s}_{n-k+1,n}(\vec{A}), \quad k = 1, \ldots, n,$$

where $\vec{A}^{-1} = (A_1^{-1}, \ldots, A_n^{-1})$. For any n-tuple $\vec{A}$

$$\mathfrak{s}_{1,n}(\vec{A}) = \mathfrak{S}_{1,n}(\vec{A}) \quad \text{and} \quad \mathfrak{S}_{n,n}(\vec{A}) = \mathfrak{s}_{n,n}(\vec{A}).$$

Besides the easily proved inequalities

$$n\left(\prod_{i=1}^{n}{:}\ A_i\right) \leqq \begin{Bmatrix}\mathfrak{S}_{k,n}(\vec{A}) \\ \mathfrak{s}_{k,n}(\vec{A})\end{Bmatrix} \leqq \left(\sum_{i=1}^{n} A_i\right)/n, \quad k = 1, \ldots, n$$

not much is known about the order relation among $\mathfrak{S}_{j,n}(\vec{A})$ and $\mathfrak{s}_{k,n}(\vec{A})$, $j, k=2, \ldots$ $\ldots, n-1$. If all A_i are scalars, that is, $\vec{A}=\vec{\alpha}$, then both $\mathfrak{S}_{k,n}(\vec{\alpha})$ and $\mathfrak{s}_{k,n}(\vec{\alpha})$ coincide with the Marcus—Lopes mean $E_{k,n}(\vec{\alpha})$. Therefore it follows via spectral theory that if $\vec{A}$ is a commuting n-tuple then

$$\mathfrak{S}_{k,n}(\vec{A}) = \mathfrak{s}_{k,n}(\vec{A}) \geqq \mathfrak{s}_{k+1,n}(\vec{A}) = \mathfrak{S}_{k+1,n}(\vec{A}), \quad k = 2, \ldots, n-2.$$

The equality $\mathfrak{S}_{k,n}(\vec{A}) = \mathfrak{s}_{k,n}(\vec{A})$, $k=2, \ldots, n-1$ is not valid in general for a non-commuting n-tuple.

ANDERSON, MORLEY, and TRAPP [3] asked if the inequalities

$$\mathfrak{S}_{k,n}(\vec{A}) \geqq \mathfrak{S}_{k+1,n}(\vec{A}), \quad k = 2, \ldots, n-2$$

(or equivalently

$$\mathfrak{s}_{k,n}(\vec{A}) \geqq \mathfrak{s}_{k+1,n}(\vec{A}), \quad k = 2, \ldots, n-2)$$

and

$$\mathfrak{S}_{k,n}(\vec{A}) \geqq \mathfrak{s}_{k,n}(\vec{A}), \quad k = 2, \ldots, n-1$$

are valid for every n-tuple $\vec{A}$. They mentioned, without proof, that in case $n=3$ the inequality $\mathfrak{S}_{2,3}(\vec{A}) \geqq \mathfrak{s}_{2,3}(\vec{A})$ could be derived via electrical network consideration.

The purpose of the present paper is to give a mathematical proof to the inequality $\mathfrak{S}_{2,3}(\vec{A}) \geqq \mathfrak{s}_{2,3}(\vec{A})$.

2. Our proof is based on a solution of an extremal problem, due to FLANDERS [4].

Lemma. *Given two set of vectors* $x_1, \ldots, x_m$ *and* $y_1, \ldots, y_n$, *define a functional* $\psi(A)$ *for an invertible positive operator* A *by*

$$\psi(A) \equiv \sum_{i=1}^{m} \langle x_i, Ax_i \rangle + \sum_{j=1}^{n} \langle y_j, A^{-1} y_j \rangle.$$

Then $\inf_A \psi(A) = 2\|[\langle x_i, y_j \rangle]\|_1$, *where* $\|[\langle x_i; y_j \rangle]\|_1$ *is the trace norm of the* $m \times n$ *matrix* $[\langle x_i, y_j \rangle]$.

See [1] and [4] for a proof.

Theorem. *For any triple* $\vec{A} = (A_1, A_2, A_3)$ *of positive operators*

$$\mathfrak{S}_{2,3}(\vec{A}) \geqq \mathfrak{s}_{2,3}(\vec{A}). \tag{3}$$

Proof. All A_i can be assumed invertible. Since $\mathfrak{s}_{2,3}(\vec{A}) = \mathfrak{S}_{2,3}(\vec{A}^{-1})^{-1}$, and

$$\langle x, \mathfrak{S}_{2,3}(\vec{A}^{-1})^{-1} x \rangle = \sup_y \frac{|\langle y, x \rangle|^2}{\langle y, \mathfrak{S}_{2,3}(\vec{A}^{-1}) y \rangle},$$

operator inequality (3) is equivalent to

$$\langle x, \mathfrak{S}_{2,3}(\vec{A}) x \rangle^{1/2} \cdot \langle y, \mathfrak{S}_{2,3}(\vec{A}^{-1}) y \rangle^{1/2} \geqq |\langle x, y \rangle| \quad \text{for all} \quad x, y,$$

which is equivalent, in view of the arithmetic-geometric means inequality, to

$$\langle x, \mathfrak{S}_{2,3}(\vec{A}) x \rangle + \langle y, \mathfrak{S}_{2,3}(\vec{A}^{-1}) y \rangle \geqq 2|\langle x, y \rangle| \quad \text{for all} \quad x, y. \tag{4}$$

Since

$$\mathfrak{S}_{2,3}(\vec{A}) = \frac{1}{2} \{A_1 : (A_2 + A_3) + A_2 : (A_3 + A_1) + A_3 : (A_1 + A_2)\},$$

formula (1) gives

$$\langle x, \mathfrak{S}_{2,3}(\vec{A}) x \rangle =$$
$$= \inf_{x_1, x_2, x_3} \frac{1}{2} \sum_{i=1}^{3} \{\langle x + x_i, A_i(x + x_i) \rangle + \langle x_{i+1}, A_i x_{i+1} \rangle + \langle x_{i+2}, A_i x_{i+2} \rangle\}, \tag{5}$$

where $x_j \equiv x_{j-3}$ for $j = 4, 5$, and similarly

$$\langle y, \mathfrak{S}_{2,3}(\vec{A}^{-1}) y \rangle =$$
$$= \inf_{y_1, y_2, y_3} \frac{1}{2} \sum_{i=3}^{3} \{\langle y + y_i, A_i^{-1}(y + y_i) \rangle + \langle y_{i+1}, A_i^{-1} y_{i+1} \rangle + \langle y_{i+2}, A_i^{-1} y_{i+2} \rangle\}, \tag{6}$$

where $y_j \equiv y_{j-3}$ for $i=4, 5$. Then Lemma yields, for fixed $x, x_1, x_2, x_3, y, y_1, y_2$, and y_3,

$$\begin{aligned}(7)\quad &\langle x+x_i, A_i(x+x_i)\rangle+\langle x_{i+1}, A_i x_{i+1}\rangle+\langle x_{i+2}, A_i x_{i+2}\rangle+\langle y+y_i, A_i^{-1}(y+y_i)\rangle+ \\ &+\langle y_{i+1}, A_i^{-1}y_{i+1}\rangle+\langle y_{i+2}, A_i^{-1}y_{i+2}\rangle \geqq 2\|S_i\|_1, \quad i=1,2,3,\end{aligned}$$

where

$$S_i \equiv \begin{bmatrix} \langle x+x_i, y+y_i\rangle & \langle x+x_i, y_{i+1}\rangle & \langle x+x_i, y_{i+2}\rangle \\ \langle x_{i+1}, y+y_i\rangle & \langle x_{i+1}, y_{i+1}\rangle & \langle x_{i+1}, y_{i+2}\rangle \\ \langle x_{i+2}, y+y_i\rangle & \langle x_{i+2}, y_{i+1}\rangle & \langle x_{i+2}, y_{i+2}\rangle \end{bmatrix}.$$

Now according to (5), (6) and (7), the inequality (4) will follow from

$$(8)\qquad \sum_{i=1}^{3} \|S_i\|_1 \geqq 2|\langle x, y\rangle|.$$

To see (8), consider a 3×3 Hermitian matrix

$$T = \begin{bmatrix} 2 & -1 & -1 \\ -1 & -1 & 2 \\ -1 & 2 & -1 \end{bmatrix}.$$

Since T has -3, 0 and 3 as its eigenvalues, $\|T\|_\infty$, the operator norm of T, is equal to 3. Easy computation shows $\sum_{i=1}^{3} \mathrm{tr}(S_i T) = 6\langle x, y\rangle$. Then

$$\sum_{i=1}^{3} \|S_i\|_1 = \frac{1}{3}\sum_{i=1}^{3} \|S_i\|_1 \cdot \|T\|_\infty \geqq \frac{1}{3}\left|\sum_{i=1}^{3} \mathrm{tr}(S_i T)\right| = 2|\langle x, y\rangle|.$$

This completes the proof.

The method in the above proof can be used to prove $\mathfrak{S}_{2,n}(\vec{A}) \geqq \mathfrak{s}_{n-1,n}(\vec{A})$ for every n-tuple $\vec{A}$. But the inequality $\mathfrak{S}_{k,n}(\vec{A}) \geqq \mathfrak{s}_{k,n}(\vec{A})$ stands still open.

Added in proof. In the revised version of [3] a different proof is presented.

References

[1] T. W. Anderson and I. Olkin, An extremal problem for positive definite matrices, *Linear and Multilinear Algebra,* **6** (1978), 257—262.

[2] W. N. Anderson and G. E. Trapp, Shorted operators. II, *SIAM J. Appl. Math.,* **28** (1975), 160—175.

[3] W. N. Anderson, T. D. Morely, and G. E. Trapp, Symmetric function means of positive operators, *Linear Alg. Appl.,* to appear.

[4] H. Flanders, An extremal problem in the space of positive definite matrices, *Linear and Multilinear Algebra,* **3** (1975), 33—39.

[5] M. Marcus and L. Lopes, Inequalities for symmetric functions and Hermitian matrices, *Canad. J. Math.,* **9** (1957), 305—312.

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