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A note on boundedly complete decomposition of a Banach space

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1. Introduction. Let E be a Banach space. A sequence (M_i) of subspaces of E is said to be a *decomposition* of E if each $x\in E$ can uniquely be expressed as $x=\sum_{i=1}^{\infty} x_i$, where $x_i\in M_i$ for each i, and convergence is with respect to the norm on E. The uniqueness implies the existence of (not necessarily continuous) associated projections P_i of E onto M_i such that $P_iP_j=\delta_{ij}P_j$, where $\delta_{ij}=0$ for $i\neq j$ and $\delta_{ij}=1$ for $i=j$, and we write $P_i(x)=x_i$. If each P_i is continuous, the decomposition is called a *Schauder decomposition* and we write it as (M_i, P_i). A decomposition (M_i) is called *boundedly complete* if the relation $\sup_{1\leqq n<\infty}\left\|\sum_{i=1}^{n} x_i\right\|<\infty$ implies that $\sum_{i=1}^{\infty} x_i$ converges, where $x_i\in M_i$ for each i.

The study of decomposition of a Banach space was initiated in the work of GRINBLYUM [3] and developed further in [2, 9, 10, 11, 12]. The purpose of the present note is to give certain sufficient conditions for a decomposition to be boundedly complete.

2. In this section, we state and prove a lemma, on which we rely heavily when proving our main results.

Lemma. *Let (M_i) be a Schauder decomposition of E. Then the following statements are equivalent:*

(A) *For each number $\lambda>0$ there exists a number $r_\lambda>0$ such that*

$$\left\|\sum_{i=1}^{n} x_i\right\| = 1, \quad \left\|\sum_{i=n+1}^{\infty} x_i\right\| \geqq \lambda \quad \textit{imply} \quad \left\|\sum_{i=1}^{\infty} x_i\right\| \geqq 1+r_\lambda$$

($x_i\in M_i$ for each i).

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(B) *For every* $\varepsilon>0$ *there exists a* $\delta>0$ *such that*

$$\left\|\sum_{i=1}^{n} x_i\right\| > 1-\delta, \quad \left\|\sum_{i=1}^{\infty} x_i\right\| = 1 \quad \textit{imply} \quad \left\|\sum_{i=n+1}^{\infty} x_i\right\| \leqq \varepsilon$$

($x_i \in M_i$ for each i).

Proof. (A)$\Rightarrow$(B). Suppose (A) holds and (B) is not true, then there exists an $\varepsilon>0$ such that for every $\delta>0$,

$$\left\|\sum_{i=1}^{n} x_i\right\| > 1-\delta, \quad \left\|\sum_{i=1}^{\infty} x_i\right\| = 1, \quad \left\|\sum_{i=n+1}^{\infty} x_i\right\| > \varepsilon \quad (x_i \in M_i).$$

Then, for $\lambda=\frac{\varepsilon}{K}$, where K is a constant appearing in Grinblyum's K-condition for Schauder decomposition (see [8], p. 93), there exists no $r_\lambda>0$ so as to satisfy (A). Indeed, let $r_\lambda>0$ be arbitrary, $\delta = r_\lambda/(1+r_\lambda)$ and $y_i = x_i \Big/ \left\|\sum_{j=1}^{n} x_j\right\|$. Then

$$\left\|\sum_{i=1}^{n} y_i\right\| = 1, \quad \left\|\sum_{i=n+1}^{\infty} y_i\right\| \geqq \varepsilon/K \left\|\sum_{j=1}^{\infty} x_j\right\| = \lambda,$$

$$\left\|\sum_{i=1}^{\infty} y_i\right\| = 1 \Big/ \left\|\sum_{j=1}^{n} x_j\right\| < \frac{1}{1-\delta} = 1+r_\lambda.$$

This is a contradiction and hence (A) implies (B).

(B)$\Rightarrow$(A). Assume that (A) is not true, i.e. there exists a $\lambda>0$ such that for every $r_\lambda>0$,

$$\left\|\sum_{i=1}^{n} x_i\right\| = 1, \quad \left\|\sum_{i=n+1}^{\infty} x_i\right\| \geqq \lambda, \quad \left\|\sum_{i=1}^{\infty} x_i\right\| < 1+r_\lambda.$$

Then, for $\varepsilon=\lambda(1-\eta)$ with $0<\eta<1$ arbitrary, there exists no $\delta>0$ so as to satisfy (B). Indeed, let $\delta>0$ be arbitrary with $\delta\leqq\eta$. Let $r_\lambda=\delta/(1-\delta)$ and $y_i=x_i\Big/\left\|\sum_{j=1}^{\infty} x_j\right\|$. Therefore

$$\left\|\sum_{i=1}^{n} y_i\right\| = 1 \Big/ \left\|\sum_{j=1}^{\infty} x_j\right\| > \frac{1}{1+r_\lambda} = 1-\delta, \quad \left\|\sum_{i=1}^{\infty} y_i\right\| = 1,$$

$$\left\|\sum_{i=n+1}^{\infty} y_i\right\| \geqq \lambda \Big/ \left\|\sum_{j=1}^{\infty} x_j\right\| > \frac{\lambda}{1+r_\lambda} \geqq \varepsilon,$$

which is a contradiction, hence (B) implies (A).

Note. The statements (A) and (B) in the lemma will be referred to as properties A and B, respectively.

3. Main results

Theorem 3.1. *Let (M_i) be a Schauder decomposition of E. If (M_i) satisfies property A (or B), then (M_i) is boundedly complete. The converse may not be true.*

Proof. Suppose $\sup_{1 \leq n < \infty} \left\| \sum_{i=1}^{n} x_i \right\| = \alpha < \infty$ with $x_i \in M_i$. Let $y_n = \sum_{i=1}^{n} x_i$, choose a sequence (n_k) of positive integers such that $\lim_{k \to \infty} \|y_{n_k}\| = \overline{\lim_{n \to \infty}} \|y_n\| = \beta$ (say). If $\beta = 0$, then $\sum_{i=1}^{\infty} x_i$ converges (to zero). If $\beta \neq 0$, we shall show that (y_{n_k}) is a Cauchy sequence. In fact, otherwise there would exist a $\delta > 0$ and subsequences $(y_{n_{k_j}})$, $(y_{n_{l_j}})$, of (y_{n_k}) with $n_{k_j} > n_{l_j}$ $(j = 1, 2, \ldots)$ such that

$$\|y_{n_{k_j}} - y_{n_{l_j}}\| \geqq \delta \quad (j = 1, 2, \ldots).$$

Then, since

$$\left\| \frac{y_{n_{k_j}} - y_{n_{l_j}}}{\|y_{n_{l_j}}\|} \right\| \geqq \frac{\delta}{\alpha} = \lambda > 0,$$

we would have by property A

$$\left\| \frac{y_{n_{l_j}}}{\|y_{n_{l_j}}\|} + \frac{y_{n_{k_j}} - y_{n_{l_j}}}{\|y_{n_{l_j}}\|} \right\| \geqq 1 + r_\lambda,$$

hence

$$\|y_{n_{k_j}}\| \geqq \|y_{n_{l_j}}\| (1 + r_\lambda).$$

Thus

$$\beta = \lim_{j \to \infty} \|y_{n_{k_j}}\| \geqq \lim_{j \to \infty} \|y_{n_{l_j}}\| (1 + r_\lambda) = \beta (1 + r_\lambda),$$

which is impossible since $\beta \neq 0$. Consequently, (y_{n_k}) is a Cauchy sequence. Hence $\lim_{k \to \infty} y_{n_k} = x \in E$. Therefore, (M_i) being a Schauder decomposition,

$$x = \lim_{k \to \infty} y_{n_k} = \lim_{k \to \infty} \sum_{i=1}^{n_k} x_i = \sum_{i=1}^{\infty} x_i.$$

This shows that $\sum_{i=1}^{\infty} x_i$ converges, whenever

$$\sup_{1 \leq n < \infty} \left\| \sum_{i=1}^{n} x_i \right\| < \infty.$$

For the converse, consider the following counter-example which would complete the proof of the theorem.

Example 3.2. Let $(\chi, \| \ \|)$ be a Banach space. Define

$$l_1(\chi) = \left\{ (x_i) : x_i \in \chi, \sum_{i=1}^{\infty} \|x_i\| < \infty \right\},$$

the norm on $l_1(\chi)$ being given by

$$\|(x_i)\|^* = \sum_{i=1}^{\infty} \|x_i\|.$$

Further, let us assume the Banach space χ to be such that the topological dual of the space $l_1(\chi)$ is its respective cross dual (see [6], Table 3.29, and [5]). Now, we observe that (N_i) with $N_i=\{\delta_i^{x_i}\colon x_i\in\chi\}$, where $\delta_i^{x_i}$ means the sequence $(0, 0, \ldots, x_i, 0, \ldots)$ i.e. the i-th entry in $\delta_i^{x_i}$ is x_i and all others are zero, forms a Schauder decomposition (see [4], p. 290, and [8], p. 95) of $l_1(\chi)$. Now, we define

$$\bar{N}_1 = \{\delta_1^{\frac{x}{2}}+\delta_2^{\frac{x}{2}}\colon x\in\chi\},\quad \bar{N}_2 = \{\delta_1^{-\frac{x}{2}}+\delta_2^{\frac{x}{2}}\colon x\in\chi\},\quad \bar{N}_i = N_i, \quad \text{for} \quad i \neq 1, 2.$$

Then $(\bar{N}_i)$ is a boundedly complete decomposition, but does not satisfy property A.

Remark. Properties A and B are not invariant under an isomorphism of the space E onto another space E_1. Hence they are not isomorphic properties since (N_i) forms a boundedly complete decomposition, equivalent to $(\bar{N}_i)$, of E which satisfies property A.

Definition 3.3. A Schauder decomposition (M_i) is said to be *monotone* if $\left\|\sum_{i=1}^{n} x_i\right\| \leqq \left\|\sum_{i=1}^{n+1} x_i\right\|$, for all n, where x_i is an arbitrary element of M_i.

Definition 3.4. A Banach space is *uniformly convex* if for every $\varepsilon>0$ there exists $\delta>0$ such that whenever $\|x\|, \|y\|\leqq 1$, and $\|x-y\|\geqq\varepsilon$, then $\|(x+y)/2\| \leqq$ $\leqq 1-\delta$.

Now, we give sufficient conditions for a decomposition to satisfy property A (or B).

Theorem 3.5. *If (M_i) is a monotone decomposition of a uniformly convex space E, then (M_i) satisfies property* B *(hence property* A*).*

Proof. Let property B be not true. Then for any given $\varepsilon>0$ and $\delta>0$ (in particular, we choose $\delta>0$ of definition 3.4), there exists a sequence (x_i), $x_i\in M_i$ such that

$$\left\|\sum_{i=1}^{\infty} x_i\right\| > 1-\delta, \quad \left\|\sum_{i=1}^{\infty} x_i\right\| = 1 \quad \text{and} \quad \left\|\sum_{i=n+1}^{\infty} x_i\right\| > \varepsilon.$$

Therefore, monotonicity of (M_i) implies

$$1-\delta < \left\|\sum_{i=1}^{n} x_i\right\| \leqq \left\|\sum_{i=1}^{\infty} x_i\right\| = 1.$$

Let $x=\sum_{i=1}^{n} x_i$ and $y=\sum_{i=1}^{\infty} x_i$. Then $\|x\|, \|y\| \leqq 1$, and

$$\|x-y\| = \left\| \sum_{i=n+1}^{\infty} x_i \right\| > \varepsilon,$$

and so $\|(x+y)/2\| \leqq 1-\delta$. Hence

$$1-\delta \geqq \|(x+y)/2\| = \left\| \sum_{i=1}^{n} x_i + \frac{1}{2} \sum_{i=n+1}^{\infty} x_i \right\| \geqq \left\| \sum_{i=1}^{n} x_i \right\| > 1-\delta,$$

which is a contradiction.

Corollary 3.6. *If (M_i) is a monotone decomposition of a uniformly convex space E, then (M_i) is boundedly complete.*

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