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Polynomials over groups and a theorem of Fejér and Riesz

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1. Introduction

A theorem of L. Fejér and F. Riesz asserts that every non-negative trigonometric polynomial is the absolute square of another trigonometric polynomial.[1]) In this note we show that the theorem does not hold in several variables.

We discovered this in the course of seeking a theorem of Fejér—Riesz type as a statement about polynomials over groups. The idea of polynomials over a group G is adequately expressed, it seems to us, by the discrete complex group algebra $\mathbf{C}[G]$ of G. This algebra is the set of complex valued functions on G of finite support, endowed with functional addition and convolution multiplication. It has an involution and a scalar product which fulfill the H^* axiom of AMBROSE [3]. In such an algebra one can define the *positivity dual* $'A$ of a subset $A \subset \mathbf{C}[G]$, this being the set of elements having non-negative scalar product with all elements of A. We consider the subset $\mathrm{S}(\mathbf{C}[G])$ of *hermitian squares* ff^* of elements of $\mathbf{C}[G]$ and, if G is abelian, the subset $\mathbf{P}(\mathbf{C}[G])$ of *positive* elements, these being the elements with non-negative Fourier transform. Interpreting these subsets for the group $\mathbf{Z}$ of integers one sees that the Fejér—Riesz theorem is equivalent to the relation $\mathbf{P}(\mathbf{C}[\mathbf{Z}])=\mathrm{S}(\mathbf{C}[\mathbf{Z}])$. Accordingly we say, for any discrete abelian group G, that the *extended* Fejér—Riesz theorem holds for G if $\mathbf{P}(\mathbf{C}[G])=\mathrm{S}(\mathbf{C}[G])$, "positive equals square".

For the class of discrete abelian groups we find, by positivity-duality and harmonic analysis, that $\mathrm{S}\subset\mathbf{P}={}'\mathbf{P}={}'\mathrm{S}$, so that $\mathbf{P}=\mathrm{S}$ if and only if $'\mathrm{S}=\mathrm{S}$, which is to say that a necessary and sufficient condition for the truth of the extended Fejér—Riesz theorem over a discrete abelian group is the self-duality of its set of hermitian squares.

For any finite group, abelian or not, we find by pure algebra (the Wedderburn theorem) that always $'\mathrm{S}=\mathrm{S}$, and therefore in particular that S is a cone. If one de-

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1) According to Fejér's account of the matter, he conjectured the result and Riesz gave the first proof. Fejér gives that proof in his paper [1] of 1916. A more accessible reference is Pólya and Szegő [2], Sechster Abschnitt, Problem 40.

fines **P** for finite groups in terms of a natural operator-valued Fourier transform one gets also the Fejér—Riesz relation $\mathbf{P}=\mathbf{S}$.

In the investigation of the extent of validity of the self-duality $'\mathbf{S}=\mathbf{S}$, or equivalently in the abelian case the Fejér—Riesz relation $\mathbf{P}=\mathbf{S}$, the next case after $\mathbf{Z}$ to check is $\mathbf{Z}\oplus\mathbf{Z}$. Here we find, by essentially algebraic means, a class of counterexamples to the result, and this shows also that the Fejér—Riesz theorem does not hold for trigonometric polynomials in several variables. By the same methods we also find that $\mathbf{S}(\mathbf{C}[\mathbf{Z}\oplus\mathbf{Z}])$ is not a cone.

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2. Discrete group algebra

Let G be a group, thought of as multiplicative. By the *discrete complex group algebra* $\mathbf{C}[G]$ of G one means the set of all complex valued functions $f\colon G\to\mathbf{C}$ of finite support, endowed with functional addition and convolution as multiplication. We write $f\in\mathbf{C}[G]$ as $f=\Sigma f(x)x$, regarding $f(x)\in\mathbf{C}$ as the coordinates of f relative to the elements of G as a Hamel base of $\mathbf{C}[G]$. Then $f+g=\Sigma\,(f(x)+g(x))x$, $fg=$ $=(\Sigma f(x)x)(\Sigma g(y)y)=\Sigma(\Sigma f(xy^{-1})g(y))x$, all sums being automatically finite. With the scalar product $\langle f,g\rangle=\Sigma f(x)\overline{g(x)}$ the algebra is a pre-Hilbert space. It admits the involution $f\to f^*=\Sigma\overline{f(x^{-1})}x$. This involution and the two products (scalar and algebra) are related by the H^* law of AMBROSE [3],

$$(1)\qquad \langle fg,h\rangle=\langle g,f^*h\rangle=\langle f,hg^*\rangle$$

where $f,g,h\in\mathbf{C}[G]$. This may be proved at once by checking it on group elements viewed as members of the algebra, observing that $x^*=x^{-1}$ for $x\in G$. $\mathbf{C}[G]$ is not a normed algebra unless G is the trivial group.

By the *positivity dual* $'A$ of a subset $A\subset\mathbf{C}[G]$ one means $\{g\in\mathbf{C}[g]\colon \langle g,f\rangle\geqq 0$ for all $f\in A\}$. The positivity dual resembles in its simplest properties the commutor of a set of elements, and the notation "prime before" is intended to suggest the resemblance. In particular we have

$$(2)\qquad A\subset B \text{ implies } 'B\subset 'A.$$

We denote by $\mathbf{S}(\mathbf{C}[G])$ the set $\{ff^*\colon f\in\mathbf{C}[G]\}$ of *hermitian squares* in $\mathbf{C}[G]$. We have

$$(3)\qquad \mathbf{S}(\mathbf{C}[G])\subset{}'\mathbf{S}(\mathbf{C}[G])$$

by (1), as follows. If $f,g\in\mathbf{S}$, say $f=uu^*$, $g=vv^*$, then $\langle f,g\rangle=\langle uu^*,vv^*\rangle=$ $=\langle u^*,u^*vv^*\rangle=\langle u^*v,u^*v\rangle\geqq 0$.

A linear operator $T\in L(H)$ on a pre-Hilbert space H is *operator-positive*, in symbols $T\geqq 0$, if $\langle T\varphi, \varphi\rangle\geqq 0$ for all $\varphi\in H$. We denote by λ_* the *left regular representation* of $\mathbf{C}[G]$ (thus $\lambda_*(f)g=fg$ for $f, g\in\mathbf{C}[G]$), by $\Lambda(\mathbf{C}[G])$ the set $\{f\colon \lambda_*(f)\geqq 0\}$ of elements of $\mathbf{C}[G]$ which go over into positive operators in the left regular representation, and by $\mathbf{PD}(\mathbf{C}[G])$ the set of *positive definite* elements of $\mathbf{C}[G]$.[2]) We have

$$\mathbf{PD}(\mathbf{C}[G]) = {}'\mathbf{S}(\mathbf{C}[G]) = \Lambda(\mathbf{C}[G]), \tag{4}$$

as follows. For any $f, g\in\mathbf{C}[G]$, $\Sigma\Sigma f(xy^{-1})\overline{g(x)}g(y) = \Sigma\Sigma f(t)\overline{g(ty)}g(y) = \Sigma f(t)\Sigma g(ty)\overline{g(y)}=\langle f, gg^*\rangle$; and since the generic $g\in\mathbf{C}[G]$, a function of finite support, determines the generic finite subset $\{c_j\}\subset\mathbf{C}$, the equation proves that $\mathbf{PD}={}'\mathbf{S}$. Since also $\langle\lambda_*(f)g, g\rangle=\langle f, gg^*\rangle$, we have $f\in{}'\mathbf{S}$ if and only if $\lambda_*(f)\geqq 0$, or ${}'\mathbf{S}=\Lambda$, and the proof is complete.[3])

3. Discrete abelian groups

For a discrete abelian group G the following facts are well known [5]. The set of characters of G forms a compact group $\hat{G}$; each $f\in\mathbf{C}[G]$ has a Fourier transform $\hat{f}\colon G\to\mathbf{C}$ defined as $\hat{f}(\chi)=\Sigma\overline{\chi(x)}f(x)$, $\chi\in\hat{G}$; $\hat{f}$ is continuous on $\hat{G}$; $\widehat{f^*}=\overline{\hat{f}}$; $\widehat{fg}=\hat{f}\hat{g}$; and $\langle f, g\rangle=\int_G \hat{f}(\chi)\overline{\hat{g}(\chi)}\,d\chi$ for all $f, g\in\mathbf{C}[G]$.

We call *positive* those elements $f\in\mathbf{C}[G]$ such that $\hat{f}\geqq 0$, and we denote by $\mathbf{P}(\mathbf{C}[G])$ the set $\{f\colon \hat{f}\geqq 0\}$ of positive elements of $\mathbf{C}[G]$. The relation (3) has for discrete abelian groups the following refinement:

$$\mathbf{S}(\mathbf{C}[G])\subset\mathbf{P}(\mathbf{C}[G])\subset{}'\mathbf{P}(\mathbf{C}[G])\subset{}'\mathbf{S}(\mathbf{C}[G]). \tag{5}$$

For if $f=gg^*\in\mathbf{S}$ then $\hat{f}=\hat{g}\overline{\hat{g}}\geqq 0$, so $f\in\mathbf{P}$; and if $f\in\mathbf{P}$ then for any $g\in\mathbf{P}$ we have $\langle f, g\rangle=\int_G \hat{f}\overline{\hat{g}}=\int_G \hat{f}\hat{g}\geqq 0$ so $f\in{}'\mathbf{P}$. Thus $\mathbf{S}\subset\mathbf{P}\subset{}'\mathbf{P}$. And $\mathbf{S}\subset\mathbf{P}$ entails ${}'\mathbf{P}\subset{}'\mathbf{S}$ by the duality relation (2).

Theorem 1. *For any discrete abelian group G we have* $\mathbf{P}(\mathbf{C}[G])={}'\mathbf{S}(\mathbf{C}[G])$.

Proof. We treat G as a locally compact abelian group. For such groups it is a consequence of the L^1 inversion theorem that an integrable positive definite func-

[2]) We adhere to the usual sense of this term: $f\in\mathbf{C}[G]$ is positive definite if $\Sigma\Sigma f(x_i x_j^{-1})c_i\bar{c}_j\geqq 0$ for all finite subsets $\{x_i\}\subset G$ and $\{c_i\}\subset\mathbf{C}$.

[3]) The relation $\mathbf{PD}={}'\mathbf{S}$ has a general form valid over locally compact groups. See [4], #13.4.4, page 256.

tion has a non-negative Fourier transform.[4] This may be expressed (in an abbreviated notation) as follows:

$$L^1 \cap \mathbf{PD} \subset \mathbf{P}. \tag{6}$$

Since in $\mathbf{C}[G]$ all elements have finite support we have in fact $\mathbf{PD}(\mathbf{C}[G]) \subset \mathbf{P}(\mathbf{C}[G])$; in combination with (4) this yields the relation $'\mathbf{S}(\mathbf{C}[G]) \subset \mathbf{P}(\mathbf{C}[G])$; and this together with (5) yields the asserted equality, completing the proof.

Taken together with (4), Theorem 1 asserts that in $\mathbf{C}[G]$ for any discrete abelian group G the set of positive definite elements is equal to the set of elements with non-negative Fourier transform, $\mathbf{PD}(\mathbf{C}[G]) = \mathbf{P}(\mathbf{C}[G])$. This is not true over an arbitrary locally compact abelian group. Indeed, without conditions the statement may be vacuous (for example, over the group $\mathbf{R}$ of real numbers the function $\exp(ix)$ is positive definite but has no Fourier transform as a function).[5]

Let us interprete our apparatus for the group $\mathbf{Z}$. The characters are the maps $\chi_t\colon \mathbf{Z} \to \mathbf{T}^1$ defined as $\chi_t(n) = \exp(int)$, where $t \in [0, 2\pi]$ and $\mathbf{T}^1 = \{z \in \mathbf{C}\colon |z| = 1\}$. The element $f = \Sigma f(n) n \in \mathbf{C}[\mathbf{Z}]$ [6] has involute $f^* = \Sigma \overline{f(-n)} n$ and Fourier transform $\hat{f}(\chi_t) = \Sigma f(n) \overline{\chi_t(n)} = \Sigma f(n) \exp(-int)$, a trigonometric polynomial. We have $\widehat{ff^*} = |\Sigma f(n) \exp(-int)|^2$. The theorem of Fejér and Riesz [1] is thus equivalent to the relation $\mathbf{P}(\mathbf{C}[\mathbf{Z}]) = \mathbf{S}(\mathbf{C}[\mathbf{Z}])$. Accordingly we say, for any discrete abelian group G, that the *extended* Fejér—Riesz theorem holds for G if $\mathbf{P}(\mathbf{C}[G]) = \mathbf{S}(\mathbf{C}[G])$. We are now in position to characterize those discrete abelian groups for which the extended Fejér—Riesz theorem holds.

Theorem 2. *For any discrete abelian group G the Fejér—Riesz relation $\mathbf{P}(\mathbf{C}[G]) = \mathbf{S}(\mathbf{C}[G])$ holds if and only if the set of hermitian squares in $\mathbf{C}[G]$ is self dual,* $\mathbf{S}(\mathbf{C}[G]) = {}'\mathbf{S}(\mathbf{C}[G])$.

Proof. By Theorem 1 the sequence (5) has the further refinement

$$\mathbf{S}(\mathbf{C}[G]) \subset \mathbf{P}(\mathbf{C}[G]) = {}'\mathbf{P}(\mathbf{C}[G]) = {}'\mathbf{S}(\mathbf{C}[G]), \tag{7}$$

whence $\mathbf{S} = \mathbf{P}$ if and only if $\mathbf{S} = {}'\mathbf{S}$, q.e.d.

The question of what groups fulfill this condition is essentially open. What little we know about it will be presented in Section 5.

[4] [5] Corollaire 1, page 92.

[5] With conditions a variety of statements can be made; for instance (in abbreviated notation) $L^1 \cap L^2 \cap \mathbf{PD} = L^1 \cap L^2 \cap \mathbf{P}$. That the left side is included in the right side comes immediately from (6); the opposite inclusion follows from the Plancherel theorem together with [4], # 13.4.4, page 256.

[6] The summation is that of $\mathbf{C}[\mathbf{Z}]$, not that of $\mathbf{Z}$.

4. Finite groups

Our discussion of discrete abelian groups did not yield an extended Fejér—Riesz theorem for that class and in particular left open the question whether the hermitian squares form a cone. In contrast to this we find for finite groups, abelian or not, that the set of hermitian squares is a cone, and moreover in the strong sense of being self dual.[7]) We have for this result a proof belonging to pure algebra.

Theorem 3. *For any finite group G we have* $\mathbf{S}(\mathbf{C}[G])={}'\mathbf{S}(\mathbf{C}[G])$.

Proof. It is well known that $\mathbf{C}[G]$ is the finite algebra-direct sum of minimal two-sided ideals, each of which is principal, generated by a uniquely determined idempotent, and that these idempotents are pairwise mutually annihilating, central, and hermitian. That is to say, if u_j are the generators of the minimal two-sided ideals $\mathbf{C}[G]u_j$ then $u_j^2=u_j^*=u_j$, $u_ju_k=\delta_{jk}u_k$, and $u_jf=fu_j$ for all $f\in\mathbf{C}[G]$. Each $f\in\mathbf{C}[G]$ has the unique decomposition $f=\Sigma fu_j$ into its components $fu_j\in\mathbf{C}[G]u_j$, whence $ff^*=\Sigma fu_j\Sigma f^*u_k=\Sigma ff^*u_n$, which is to say $\mathbf{S}(\mathbf{C}[G])\subset\oplus_j\mathbf{S}(\mathbf{C}[G]u_j)$. Conversely, if $h=\Sigma g_jg_j^*$ with $g_j\in\mathbf{C}[G]u_j$ then $g_j=g_ju_j$ and $h=\Sigma(g_ju_j)^*g_ju_j=\Sigma g_j^*u_j\cdot\Sigma g_ku_k=$ $=gg^*\in\mathbf{S}(\mathbf{C}[G])$, where $g=\Sigma g_ju_j$. Therefore

$$\mathbf{S}(\mathbf{C}[G]) = \oplus_j\mathbf{S}(\mathbf{C}[G]u_j). \tag{8}$$

Let $f\in{}'\mathbf{S}(\mathbf{C}[G])$ be given. Since for any $g\in\mathbf{C}[G]$ we have $g^*gu_j\in\mathbf{S}(\mathbf{C}[G])$ $\big($because $g^*gu_j=g^*gu_ju_j^*=(gu_j)^*gu_j\big)$, we have $\langle f, g^*gu_j\rangle\geqq 0$ for all j. But $\langle f,g^*gu_j\rangle=$ $=\langle fu_j,g^*gu_j\rangle$, so $fu_j\in{}'\mathbf{S}(\mathbf{C}[G]u_j)$, whence ${}'\mathbf{S}(\mathbf{C}[G])\subset\oplus_j{}'\mathbf{S}(\mathbf{C}[G]u_j)$. Conversely, if $h=\Sigma g_j$ with $g_j\in{}'\mathbf{S}(\mathbf{C}[G]u_j)$ then for any $f=\Sigma f^*fu_j\in\mathbf{S}(\mathbf{C}[G])$ we have $\langle h,f\rangle=$ $=\Sigma\langle g_j,f^*fu_j\rangle\geqq 0$, $h\in{}'\mathbf{S}(\mathbf{C}[G])$, so that

$${}'\mathbf{S}(\mathbf{C}[G]) = \oplus_j{}'\mathbf{S}(\mathbf{C}[G]u_j). \tag{9}$$

Since the minimal ideals $\mathbf{C}[G]u_j$ are simple as rings each one is algebra-isomorphic (by the Wedderburn theorem) to the full algebra $L(H_j)$ of all linear transformations on a finite dimensional Hilberst space H_j. The left regular representation λ_* is faithful, and so maps each ideal $\mathbf{C}[G]u_j$ algebra-isomorphically onto a subalgebra of $L(\mathbf{C}[G])$. Since $\mathbf{C}[G]u_j$ is a full ring, so therefore is $\lambda_*(\mathbf{C}[G]u_j)$.

For H finite dimensional the algebra $L(H)$, though not a group algebra, has the involution $T\to T^*$ defined by the operator adjoint, and it has the scalar product $\langle T, S\rangle=\operatorname{trace}(TS^*)$, the so-called *trace inner product*, which trivially fulfills the Ambrose law (1). We may therefore define positivity duality and the sets $\mathbf{S}$, $'\mathbf{S}$ over $L(H)$.

[7]) Evidently the positivity dual of any set is a cone.

Lemma. *For any finite dimensional Hilbert space H we have* $\mathbf{S}(L(H))= {}'\mathbf{S}(L(H))$.

Proof. We have $\mathbf{S}(L(H))\subset {}'\mathbf{S}(L(H))$ by (3) since that result depends only upon (1). For the opposite inclusion let $c\in {}'\mathbf{S}(L(H))$ be given. Pick arbitrarily a unit vector $v\in H$, extend the set $\{v\}$ to an orthonormal basis $\{e^1=v, e^2, \dots, e^d\}$ of H, and let p be the orthogonal projection onto the subspace spanned by e^1. Then $\langle cv, v\rangle_H=\langle ce^1, e^1\rangle_H=\Sigma\langle cpe^i, e^i\rangle_H=\mathrm{trace}\,(cp)=\langle c, p\rangle=\langle c, pp^*\rangle\geqq 0$, whence c is a positive hermitian operator. If b is its positive square root then $c=bb^*\in\mathbf{S}(L(H))$ and the lemma is proved.

Returning to the proof of Theorem 3, we claim that the isomorphism λ_* is an essentially H^*-map in the sense that for all $f, g\in \mathbf{C}[G]$

$$\text{(10)}\qquad \mathrm{trace}\,(\lambda_*(f)\cdot\lambda_*(g)) = \#(G)\langle f, g\rangle.$$

For $\langle f, g\rangle=\langle fg^*, e\rangle=(fg^*)(e)$, and since $\lambda_*(f)x=fx=\Sigma f(s)sx=\Sigma f(tx^{-1})t$, whence $\langle\lambda_*(f)x, t\rangle=f(tx^{-1})$ for $x, t\in G$, we have also $\mathrm{trace}\,(\lambda_*(f))=\#(G)f(e)$. As evidently $\lambda_*(g)^*=\lambda_*(g^*)$ we have finally $\mathrm{trace}\,(\lambda_*(f)\cdot\lambda_*(g))=\mathrm{trace}\,(\lambda_*(fg^*))= \#(G)(fg^*)(e)=\#(G)\langle f, g\rangle$, which is (10). By the lemma we therefore conclude that $\mathbf{S}(\mathbf{C}[G]u_j)={}'\mathbf{S}(\mathbf{C}[G]u_j)$ for all j, and tracing this back through (9) and (8) we reach the assertion of the theorem, q.e.d.

We turn now to the question whether one can define "positive" over finite groups in such a way as to substantiate the Fejér—Riesz relation. By (4) the elements of ${}'\mathbf{S}$ go over to positive operators in the regular representation. If a definition of $\mathbf{P}$ consistent with this fact can be formulated, then automatically one will have ${}'\mathbf{S}=\mathbf{P}$, and also automatically, by Theorem 3, the Fejér—Riesz relation $\mathbf{P}=\mathbf{S}$. The following considerations lead to such a formulation.

Definition 1. By the *unitary dual object* $\hat{G}_u$ of a finite group G we mean the set of all equivalence classes of irreducible unitary complex representations of G.[8])

Let $[X]$ denote the similarity class of the operator X or the equivalence class of the representation X, as context requires. For any representation ϱ we write ϱ_* for the extension of ϱ to the discrete group algebra $\mathbf{C}[G]$.

Definition 2. The *Fourier transform* $\hat{f}$ of $f\in\mathbf{C}[G]$, G finite, is the map of $\hat{G}_u$ to similarity classes of operators given by

$$\text{(11)}\qquad \hat{f}([\varrho]) = [\varrho_*(f)]$$

for $[\varrho]\in\hat{G}_u$.

[8]) This is a variant of a procedure discussed in [6] without attribution.

Operator positivity is of course a unitary invariant, and it is known that equivalent irreducible unitary representations are in fact unitarily equivalent.[9]) It follows that, for any $[\varrho]\in\hat{G}_u$ and $f\in\mathbf{C}[G]$, $\alpha_*(f)\geqq 0$ for a single $\alpha\in[\varrho]$ if and only if $\alpha_*(f)\geqq 0$ for all $\alpha\in[\varrho]$. This makes possible the following definition.

Definition 3. For $f\in\mathbf{C}[G]$ and $\xi\in\hat{G}_u$ we say $\hat{f}$ is *positive at* ξ, and write $\hat{f}(\xi)\geqq 0$, if $\alpha_*(f)\geqq 0$ for any, hence all, $\alpha\in\xi$; we say $\hat{f}$ is *positive*, and write $\hat{f}\geqq 0$, if $\hat{f}(\xi)\geqq 0$ for all $\xi\in\hat{G}_u$.

Having formulated this concept of positive transform we now say, as in the previous case, that $f\in\mathbf{C}[G]$ for G finite is *positive* if $\hat{f}\geqq 0$, and we denote by $\mathbf{P}(\mathbf{C}[G])$, as before, the set $\{f\colon \hat{f}\geqq 0\}$ of such positive elements. The consistency with (4) of this definition of $\mathbf{P}$ follows from the reducibility of the left regular representation of G, as we now show.

Theorem 4. *For any finite group G we have* $\mathbf{P}(\mathbf{C}[G])={}'\mathbf{S}(\mathbf{C}[G])$.

Proof. Let λ denote the left regular representation of G. It is well known that every irreducible unitary representation of G is (equivalent to) a direct summand of λ with multiplicity equal to its degree.[10]) Let $\lambda^{(j)}$ be the irreducible subrepresentations of λ, and d_j their degrees. Then $\lambda\cong\oplus d_j\lambda^{(j)}$. For the extension λ_* of λ to $\mathbf{C}[G]$, which is of course nothing but the left regular representation of $\mathbf{C}[G]$[11]), we then have $\lambda_*(f)\cong\oplus d_j\lambda_*^{(j)}(f)$ for all $f\in\mathbf{C}[G]$. Now $f\in\mathbf{P}(\mathbf{C}[G])$ if and only if $\lambda_*^{(j)}(f)\geqq 0$ for all j, hence if and only if $\lambda_*(f)\geqq 0$, which is to say if and only if $f\in\mathbf{A}(\mathbf{C}[G])$. The proof is now completed by an appeal to (4).

Corollary. *For any finite group G we have the Fejér—Riesz relation* $\mathbf{P}(\mathbf{C}[G])=\mathbf{S}(\mathbf{C}[G])$.

5. The group $\mathbf{Z}\oplus\mathbf{Z}$

For finite groups we have $\mathbf{S}={}'\mathbf{S}$ as a matter of pure algebra, for discrete abelian groups generally we have $\mathbf{S}\subset\mathbf{P}={}'\mathbf{P}={}'\mathbf{S}$, and for $\mathbf{Z}$ in particular we have the self duality $\mathbf{S}={}'\mathbf{S}$, this being an equivalent formulation over discrete abelian groups of the Fejér—Riesz relation $\mathbf{S}=\mathbf{P}$. One naturally inquires into the extent of validity of this self duality, or equivalently, of the validity of the extended Fejér—Riesz theorem. In this inquiry the next case to check after $G=\mathbf{Z}$ is $G=\mathbf{Z}\oplus\mathbf{Z}$. We find that for $\mathbf{Z}\oplus\mathbf{Z}$ the extended theorem fails, $\mathbf{P}\neq\mathbf{S}$. As we shall see in a moment, this will

[9]) See for instance [7], (3.2), page 19.

[10]) [8], page 1—18.

[11]) In agreement with our previous use of the symbol λ_*.

show that the Fejér—Riesz theorem fails for trigonometric polynomials in several variables.

We will demonstrate this by exhibiting a class of counterexamples. For this purpose it proves convenient to employ Laurent polynomials, as follows. Over **Z** we may express the generic $f=\Sigma f(n)n\in\mathbf{C}[\mathbf{Z}]$ as the Laurent polynomial $f(z)=\Sigma f(n)z^n$ in the complex variable z. Evidently the addition and multiplication of Laurent polynomials duplicate the corresponding operations in $\mathbf{C}[\mathbf{Z}]$, or, in algebraic language, the set of Laurent polynomials is **C**-algebra isomorphic to $\mathbf{C}[\mathbf{Z}]$. If we put $f^*(z)= =\Sigma\overline{f(n)}z^{-n}$ and use the obvious scalar product then the isomorphism preserves the Ambrose law (1) as well. In the Laurent version the Fourier transform $\hat{f}$ of $f\in\mathbf{C}[\mathbf{Z}]$ is the restriction to $\mathbf{T}^1$ of the corresponding Laurent polynomial. Over $\mathbf{Z}\oplus\mathbf{Z}$ we proceed analogously. We have elements $f=\Sigma f(n,m)(n\oplus m)\in\mathbf{C}[\mathbf{Z}\oplus\mathbf{Z}]$ and their Fourier transforms $\hat{f}(\chi_t,\chi_s)=\Sigma f(n,m)\exp(-int)\exp(-ins)$, trigonometric polynomials in two variables, which we may view as the restrictions to $\mathbf{T}^2$ of the Laurent polynomials $f(z,w)=\Sigma f(n,m)z^n w^m$ in two complex variables. Our discussion of **P** and **S** over $\mathbf{Z}\oplus\mathbf{Z}$ will thus also be a treatment of the Fejér—Riesz theorem in two variables. We note for reference that the involution in two variables reads $f^*(z,w)= =\Sigma\overline{f(n,m)}z^{-n}w^{-m}$.

If $f\in\mathbf{C}[\mathbf{Z}\oplus\mathbf{Z}]$ is a hermitian square, so that $f(z,w)=h(z,w)h^*(z,w)$ for some Laurent polynomial $h(z,w)$, then we claim that without loss of generality $h(z,w)$ may be assumed to be *analytic* in z and w, and to have non-zero coefficient of z^0 when written as a polynomial in z, which is to say $h(z,w)=\sum_0^n a_k(w)z^k$ with $a_k(w)\in \in\mathbf{C}[w]$[12]) and $a_0(w)\neq 0$. For by definition of the involution we have $h^*(z,w)= =\bar{h}(z^{-1},w^{-1})$, the bar denoting the conjugation of all constants. Therefore the lowest negative powers (negative exponents of greatest absolute value) of z and w occurring in $h(z,w)$ are the negatives of the highest (positive) powers of z and w occurring in $h^*(z,w)$; hence by factoring out of $h(z,w)$ the lowest negative powers of z and w, and out of $h^*(z,w)$ the highest powers, we shall have cancellation. Therefore we can substitute for $h(z,w)$ an analytic polynomial. If $a_s(w)$, $s>0$, is the non-zero coefficient of least index in $h(z,w)$, now assumed analytic, we can factor z^s out of $h(z,w)$ and z^{-s} out of $h^*(z,w)$ and cancel them, arriving thus at new polynomials with non-zero coefficients of z^0 ("constant terms").

Example. The element $f\in\mathbf{C}[\mathbf{Z}\oplus\mathbf{Z}]$ whose Laurent polynomial is $f(z,w)= =(1/4)\{z^2+z^{-2}+w^2+w^{-2}+4\}$ is a hermitian square because $\hat{f}(\chi_t,\chi_s)=$

[12]) We adhere to the standard notations $F[t]$ for the ring of polynomials and $F(t)$ for the field of rational functions over the field F. There is no conflict with the notation $\mathbf{C}[G]$ where G is a group.

$= 1+(1/2)\cos(2t)+(1/2)\cos(2s) = \cos^2(t)+\cos^2(s) = |\cos(t)+i\cdot\cos(s)|^2$. From this factorization we have $f(z, w)=(1/2)\{(z+z^{-1})+i(w+w^{-1})\}\cdot(1/2)\{(z+z^{-1})-i(w+w^{-1})\}$, which is to say $f(z, w)=g(z, w)g^*(z, w)$ with $g(z, w)=(1/2)\{(z+z^{-1})+i(w+w^{-1})\}$. Factoring out the lowest negative powers we have

$$g(z, w) = z^{-1}w^{-1}(1/2)\{(z^2w+w)+i(w^2z+z)\} = z^{-1}w^{-1}\left\{\frac{w}{2}+\frac{i}{2}(w^2+1)z+\frac{w}{2}z^2\right\}.$$

Thus $f=hh^*$ where $h\in\mathbf{C}[\mathbf{Z}\oplus\mathbf{Z}]$ has the Laurent polynomial

$$h(z, w) = \frac{w}{2}+\frac{i}{2}(w^2+1)z+\frac{w}{2}z^2.$$

By means of the foregoing reduction one might hope to be able to characterize the hermitian squares in a purely algebraic way. But the system of non-linear equations one would have to discuss has so far proved intractable, and we are forced to curcumvent this difficulty by the following special arguments, which enable us to proceed a little farther.

Let $p(z, w)\in\mathbf{C}(w)[z]$ be given, and suppose its degree in z is 2. We then have $p(z, w)=p_0(w)z^2+p_1(w)z+p_2(w)$ with $p_j(w)\in\mathbf{C}[w]$. In general the equation $p(z, w)=0$ defines two branches $r_\pm=\{-p_1\pm\sqrt{(p_1^2-4p_0p_2)}\}/2p_0$; $r_\pm$ are algebraic over $\mathbf{C}(w)$, and $\mathbf{C}(w, r_+, r_-)$ is the splitting field of $p(z, w)$. Therefore $p(z, w)$ is reducible in $\mathbf{C}(w)[z]$ if and only if one, hence both, of $r_\pm$ are in $\mathbf{C}(w)$.

Consider now

$$f(z, w) = (z+z^{-1})(w+w^{-1})+c, \quad c\in\mathbf{R}. \tag{12}$$

We rewrite this as $f(z, w)=z^{-1}\{(w+w^{-1})z^2+cz+(w+w^{-1})\}$ and put $p(z, w)=(w+w^{-1})z^2+cz+(w+w^{-1})$. The equation $p(z, w)=0$ determines the branches $r_\pm=\{-c\pm\sqrt{(c^2-4(w+w^{-1})^2)}\}/2(w+w^{-1})$. To ascertain the character of these functions we examine the radical $\sqrt{(c^2-4(w+w^{-1})^2)}=\sqrt{(c-2(w+w^{-1}))}\cdot\sqrt{(c+2(w+w^{-1}))}$. We have $c\pm2(w+w^{-1})=0$ if and only if $w^2\pm(c/2)w+1=0$, which is to say $w=\{\pm c\mp\sqrt{(c^2-16)}\}/4$. Thus except for $c=\pm4$ the functions $r_\pm$ both have branch points at these values of w, so that $r_\pm(w)\notin\mathbf{C}(w)$ and $p(z, w)$ is irreducible in $\mathbf{C}(w)[z]$ for $c\neq\pm4$.

If f were a hermitian square, so that $f(z, w)=h(z, w)h^*(z, w)$ with $h(z, w)$ analytic, $h(z, w)=\sum_0^n a_k(w)z^k$, $a_k(w)\in\mathbf{C}[w]$, $a_0(w)\neq0$, then precisely because the constant term is not zero the highest power of z occurring in $h(z, w)h^*(z, w)$ is the highest power of z occurring in $h(z, w)$. But since $f=hh^*$ this is the highest power of z occurring in $f(z, w)$, namely 1. Therefore $h(z, w)$ must have the form $\alpha(w)(z-\beta(w))$, with $\alpha(w)\in\mathbf{C}[w]$ and $\alpha(w)\beta(w)\in\mathbf{C}[w]$. Since $\beta(w)$ is at worst rational we have

$\beta(w)\in\mathbf{C}(w)$. By definition of the involution we have

$$h^*(z,w)=\bar{\alpha}(w^{-1})\big(z^{-1}-\bar{\beta}(w^{-1})\big)=\bar{\alpha}(w^{-1})z^{-1}\big(1-z\bar{\beta}(w^{-1})\big)=$$

$$=(-1)z^{-1}\bar{\alpha}(w^{-1})\bar{\beta}(w^{-1})\big(z-1/\bar{\beta}(w^{-1})\big).$$

Substituting this into our relation $f(z,w)=h(z,w)h^*(z,w)$ and recalling that $zf(z,w)=p(z,w)$ we get $p(z,w)=(-1)\alpha(w)\bar{\alpha}(w^{-1})\bar{\beta}(w^{-1})\big(z-\alpha(w)\big)\big(z-1/\bar{\beta}(w^{-1})\big)$, a factorization of $p(z,w)$ in $\mathbf{C}(w)[z]$. But we have just observed, in the previous paragraph, that $p(z,w)$ is irreducible in $\mathbf{C}(w)[z]$ if $c\neq\pm 4$. Hence there can be no factorization of the form $f=hh^*$ if $c\neq\pm 4$. Since $\hat{f}(\chi_t,\chi_s)=2\cos(t)\cdot 2\cos(s)+c$ we have $f\in\mathbf{P}(\mathbf{C}[\mathbf{Z}\oplus\mathbf{Z}])$ for $c>4$. We have established.

Theorem 5. *For each real* $c>4$ *the Laurent polynomial* $f(z,w)=(z+z^{-1})\cdot(w+w^{-1})+c$ *defines an element of* $\mathbf{C}[\mathbf{Z}\oplus\mathbf{Z}]$ *which is positive but is not a hermitian square.*

By the same methods we have the following further result.

Theorem 6. *The set* $\mathbf{S}(\mathbf{C}[\mathbf{Z}\oplus\mathbf{Z}])$ *of hermitian squares over* $\mathbf{Z}\oplus\mathbf{Z}$ *is not a cone.*

Proof. With $h(z,w)=z+w$ put $g=hh^*$, $f(z,w)=g(z,w)+c$ with $c\in\mathbf{R}$, and $p(z,w)=z^{-1}f(z,w)$. One checks that there exists $0<c_0\in\mathbf{R}$ such that $p(z,w)$ is irreducible in $\mathbf{C}(w)[z]$. Hence $f_0(z,w)=g(z,w)+c_0$ does not correspond to a hermitian square in $\mathbf{C}[\mathbf{Z}\oplus\mathbf{Z}]$ even though f is the sum $f=hh^*+(\sqrt{c_0})(\sqrt{c_0})^*$ of hermitian squares.

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