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Existence and uniqueness of random solutions of nonlinear stochastic functional integral equations

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1. Introduction. Stochastic or random integral equations arise quite often in the engineering, biological, chemical, and physical sciences (see, e.g., [1], [8] and [6]).

The object of the present paper is to study a nonlinear stochastic functional integral equation of the type

$$(1.1)\qquad x(t,\omega)=F\Big(t,\int_0^{g(t)} f_1(t,s,x(s,\omega),\omega)\,ds,$$
$$\int_0^{g(t)} f_2(t,s,x(s,\omega),\omega)\,dw(s,\omega),\, x(h(t),\omega),\omega\Big)\overset{\mathrm{df}}{=}(Ux)(t,\omega),$$

where

(i) $t\in R_+\overset{\mathrm{df}}{=}[0,+\infty)$, and $\omega\in\Omega$, the supporting set of a complete probability measure space $(\Omega,\mathscr{F},P)$;

(ii) $x\colon R_+\times\Omega\to R$ is the unknown random function;

(iii) $F\colon R_+\times R^3\times\Omega\to R$ and $f_j\colon \Delta\times R\times\Omega\to R$, $j=1,2$, are given random functions, where $\Delta\overset{\mathrm{df}}{=}\{(t,s)\colon 0\leqq s\leqq t<\infty\}$;

(iv) $g,h\colon R_+\to R_+$ are given scalar functions;

(v) $w\colon R_+\times\Omega\to R$ is a Wiener process.

The first integral of the stochastic equation (1.1) is to be understood as an ordinary Lebesgue integral, while the second integral is an Ito stochastic integral. We shall give sufficient conditions which will ensure the existence and uniqueness of a random solution, a second order stochastic process, of the above stochastic functional integral equation. The tool which we utilize to obtain these results is the comparison method. This method is based on the convergence of successive approximations produced by a comparison operator associated with the operator U. The abstract form of the comparison method was introduced by WAŻEWSKI [11] in the case of deterministic equations.

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Almost all authors use the well-known Banach fixed point theorem or the concept of admissibility theory, [1], [8] and [6], in proving the existence and uniqueness of results for cases similar to equation (1.1). Unfortunately these methods involve a strong condition concerning the function F. By the comparison method this condition can be slightly weakened. Consequently in this paper conditions involving some relation between the Lipschitz constants of the function F and the estimations imposed on the functions g and h appear.

Equation (1.1) is a generalization of equations considered by MANOUGIAN, RAO and TSOKOS [6] (if $F(t, u_1, u_2, x, \omega)=h(t, x)+u_1+u_2$, $f_j(t, s, x, \omega)=k_j(t, s, \omega)\cdot\varphi_j(s, x)$ and $g(t)=t$, $h(t)=t$), TURO [10] (if $F(t, u_1, u_2, x, \omega)=F(t, u_1, x, \omega)$), GIHMAN and SKOROHOD [3], and DOOB [2], among others.

2. Preliminaries. We introduce a family $\mathscr{F}_t$, $t\in R_+$, of σ-algebras of subsets of Ω with the following properties:

(i) $\mathscr{F}_{t_1}\subset\mathscr{F}_{t_2}$, for $t_1<t_2$, $\mathscr{F}_t\subset\mathscr{F}$ $t\in R_+$;

(ii) for every t, $w(t, \omega)$ is $\mathscr{F}_t$-measurable;

(iii) for $\lambda\geqq 0$, the increments $w(t+\lambda, \omega)-w(t, \omega)$ are independent (in the probabilistic sense) of $\mathscr{F}_t$.

Definition 2.1. We shall denote by $C(R_+, L_2)$ the space of all continuous maps $x\colon R_+\to L_2(\Omega, \mathscr{F}_t, P)$ with the topology of uniform convergence on compacta.

It may be noted that $C(R_+, L_2)$ is a locally convex space whose topology is defined by the following family of seminorms:

$$\|x\|_n = \sup_{0\leqq t\leqq n} \{E[|x(t, \omega)|^2]\}^{1/2}$$

where E denotes the expected value of the random process.

Definition 2.2. A sequence $\{x_k\}$ of elements of the space $C(R_+, L_2)$ will be called a Cauchy sequence if for every $\varepsilon>0$ and n there exists an N such that for $k>N$ and $l>N$ we have $\|x_k-x_l\|_n<\varepsilon$.

It is clear that the space $C(R_+, L_2)$ is complete, that is, every Cauchy sequence of its elements has a limit in $C(R_+, L_2)$.

Definition 2.3. We shall call x a random solution of the stochastic functional integral equation (1.1) if $x\in C(R_+, L_2)$ and satisfies equation (1.1) P-a.e.

With respect to the functions appearing in equation (1.1) we shall assume the following:

(i) $F(t, u_1, u_2, x, \cdot)$ is $\mathscr{F}_t$-measurable for each $t\in R_+, u_1, u_2, x\in R$, and is continuous in t uniformly in u_1, u_2, x;

(ii) $f_j(t, s, x, \cdot)$, $j=1, 2$, are $\mathcal{F}_s$-measurable for each $(t, s)\in\Delta$, $x\in R$, and are continuous as maps from Δ into $L_2(\Omega, \mathcal{F}, P)$;

(iii) g, h: $R_+ \to R_+$ are continuous and $g(t)\leqq t$, $h(t)\leqq t$, $t\in R_+$.

3. Somme lemmas. Let us define

$$(Ku)(t) \overset{\text{df}}{=} k(t) \int_0^{g(t)} u(s)\, ds, \quad t\in R_+,$$

$$(Lu)(t) \overset{\text{df}}{=} l(t)\, u(h(t)), \quad t\in R_+.$$

Put $Su \overset{\text{df}}{=} \sum_{n=0}^{\infty} L^n u$ with the pointwise convergence of the series in R_+, where $L^n \overset{\text{df}}{=} LL^{n-1}$, $n=1, 2, \ldots$, $L^0=I$ is the identity operator in $C(R_+, R_+)$, the class of all continuous and nonnegative functions defined on R_+.

From the definition of the operator L it follows that

$$(L^n u)(t) = l_n(t)\, u(h_n(t)),$$

where

$$h_0(t) \overset{\text{df}}{=} t, \quad h_{n+1}(t) \overset{\text{df}}{=} h(h_n(t)), \quad n = 0, 1, \ldots, \quad t\in R_+,$$

$$l_0(t) \overset{\text{df}}{=} 1, \quad l_{n+1}(t) \overset{\text{df}}{=} \prod_{k=0}^{n} l(h_k(t)), \quad n = 0, 1, \ldots, \quad t\in R_+.$$

Lemma 3.1. ([9], [5]) *Assume that*

(i) $k, l, g, h, r\in C(R_+, R_+)$ *and* $g(t), h(t)\in[0, t]$, $t\in R_+$;

(ii) $s=Sr<\infty$, $s^*=Sk^*<\infty$, *where* $k^*(t)\overset{\text{df}}{=}k(t)g(t)$;

(iii) $s, s^*\in C(R_+, R_+)$ *and* $\sup_{R_+} \frac{s^*(t)}{t}<\infty$.

Then

(a) *there exists* $u_0\in C(R_+, R_+)$ *which is a unique solution of equation*

$$u = SKu + Sr \tag{3.1}$$

in the class L_{loc} *of all non-negative and locally integrable functions on* R_+;

(b) *the function* u_0 *is the unique solution of the equation*

$$u = Ku + Lu + r \tag{3.2}$$

in the class $L_{\text{loc}}(u_0)\overset{\text{df}}{=}\{u\colon u\in L_{\text{loc}},\ \|u\|_0<\infty\}$, *where* $\|u\|_0\overset{\text{df}}{=}\inf\{c\colon u\leqq cu_0,\ c\in R_+\}$;

(c) *the function* $u=0$ *is the unique solution of the inequality*

$$u \leqq Ku + Lu \tag{3.3}$$

in the class $L_{\text{loc}}(u_0)$.

Proof. First we prove (a). We note that if $u \in L_{\text{loc}}$ and is the solution of equation (3.1), then $u \in C(R_+, R_+)$. Thus we shall prove that equation (3.1) has a unique solution in $C(R_+, R_+)$. We shall obtain a solution first on an arbitrary closed, bounded interval $[0, n]$. Let $C([0, n], R)$ be the space of all continuous functions on $[0, n]$, where we introduce a norm $\|\cdot\|_*$ in the following way:

$$\|u\|_* \overset{\text{df}}{=} \sup_{t \in [0, n]} e^{-\lambda t}|u(t)|, \quad \text{where} \quad \lambda > \bar{\lambda} \overset{\text{df}}{=} \sup \frac{s^*(t)}{t}.$$

Now we can prove that the operator SK is a contraction in $C([0, n], R)$, i.e., $\|SK\| < 1$. Indeed, from the inequality $e^{\alpha t} - 1 \leqq \alpha e^t$ for $\alpha \in [0, 1]$, $t \in R_+$, we have

$$\|SKu\|_* \leqq \sup_{[0,n]} e^{-\lambda t} \sum_{n=0}^{\infty} l_n(t) k(h_n(t)) \int_0^{g(h_n(t))} e^{\lambda s} \sup_{s \in [0,n]} e^{-\lambda s}|u(s)|\, ds \leqq$$

$$\leqq \frac{1}{\lambda} \frac{s^*(t)}{t} \|u\|_* \leqq \frac{\bar{\lambda}}{\lambda} \|u\|_*.$$

Hence it follows that $\|SK\| < 1$. Now from the Banach fixed point theorem it follows that equation (3.1) has a unique solution $u_0 \in C([0, n], R_+)$. Since n is arbitrary, u_0 is a unique solution of equation (3.1) on R_+.

Now we prove (b). It is obvious that the function u_0 satisfies equation (3.2). Next we prove that in the class $L_{\text{loc}}(u_0)$ the function u_0 is the unique solution of equation (3.2).

Indeed, if $u \in L_{\text{loc}}(u_0)$ is a solution of (3.2) then by induction we get

$$\bar{u} = \sum_{i=0}^{n-1} L^i h + \sum_{i=0}^{n-1} L^i K \bar{u} + L^n \bar{u}.$$

Because $\bar{u} \in L_{\text{loc}}(u_0)$, there exists $c \in R_+$ such that $\bar{u} \leqq c u_0$, hence $L^n \bar{u} \leqq c L^n u_0$. We easily find that $L^n u_0 \to 0$ since u_0 is the solution of equation (3.1). As a consequence of this $L^n \bar{u} \to 0$, and we infer that $\bar{u}$ satisfies (3.1). In view of the uniqueness proved for this equation we conclude $\bar{u} = u_0$.

Finally we prove (c). If $u \in L_{\text{loc}}(u_0)$ is the solution of inequality (3.3) then we have $L^n u \to 0$, $n \to \infty$, and by induction we get

$$u \leqq \sum_{i=0}^{n-1} L^i K u + L^n u, \quad n = 0, 1, \dots,$$

Letting $n\to\infty$ we get $u \leqq SKu$, hence we conclude that $u=0$, and the lemma is proved.

Remark 3.1. Now we give some effective conditions under which assumption (ii) of Lemma 3.1 is fulfilled.

a) If we assume that

$$(3.4)\quad k(t) \leqq \bar{k} = \text{const}, \quad l(t) \leqq \bar{l} = \text{const}, \quad g(t) \leqq \bar{g}t, \quad h(t) \leqq \bar{h}t, \quad \bar{g}, \bar{h} \in [0, 1],$$

and $r(t) \leqq \bar{r}t$, $t \in R_+$, for some $\bar{r} \in R_+$, then assumption (ii) of Lemma 3.1 is satisfied provided $\bar{l}\bar{h} < 1$.

b) if $k(t) \leqq \bar{k}$, $l(t) \leqq \bar{l}t$, $g(t) \leqq \bar{g}t$, $h(t) \leqq \bar{h}t$, $r(t) \leqq \bar{r}t$, $\bar{k}, \bar{l}, \bar{r} \in R_+$, $\bar{g} \in [0, 1]$ and $\bar{h} \in [0, 1)$, $t \in R_+$, then assumption (ii) of Lemma 3.1 is satisfied.

c) Finally, if we suppose (3.4) and $r(t) \leqq \bar{r}t^p$, $t \in R_+$, for some $\bar{r}, p \in R_+$, then (ii) of Lemma 3.1 is satisfied provided $\bar{l}\bar{h}^p < 1$.

We construct a sequence as follows:

$$(3.5)\qquad u_{n+1} = Ku_n + Lu_n, \quad n = 0, 1, \ldots,$$

where u_0 is defined in Lemma 3.1.

Lemma 3.2. [4] *If the assumptions of Lemma* 3.1 *are satisfied, then*

$$(3.6)\qquad 0 \leqq u_{n+1} \leqq u_n, \quad n = 0, 1, \ldots,$$

and $u_n \rightrightarrows 0$ *for* $n\to\infty$, *where the sign* $\rightrightarrows$ *denotes uniform convergence in any compact subset of* R_+.

Proof. Relation (3.6) we get by induction. The convergence of the sequence $\{u_n\}$ is implied by (3.6). The limit of this sequence satisfies the inequality (3.3), and by Lemma 3.1 it must be equal to zero identically. The uniform convergence of $\{u_n\}$ follows from Dini's theorem.

4. Main results. In order to prove the existence of a solution of equation (1.1), we define the sequence $\{x_n\}$ of random functions by the relations:

$$(4.1)\qquad x_{n+1} = Ux_n, \quad n = 0, 1, \ldots,$$

where U is defined by (1.1) and x_0 is an arbitrarily fixed element of $C(R_+, L_2)$.

We introduce the following

Assumption H. We assume that

(1) there exist functions $\bar{k}_j, \tilde{k}_j, l \in C(R_+, R_+)$, $j=1,2$, such that

$$|F(t, u_1, u_2, x, \omega) - F(t, \bar{u}_1, \bar{u}_2, \bar{x}, \omega)| \leqq \sum_{j=1}^{2} \bar{k}_j(t)|u_j - \bar{u}_j| + l(t)|x - \bar{x}|,$$

$$|f_j(t, s, x, \omega) - f_j(t, s, \bar{x}, \omega)| \leqq \tilde{k}_j(t)|x - \bar{x}|,$$

for $t \in R_+$, $s \leqq t$, $u_j, \bar{u}_j, x, \bar{x} \in R$, $j = 1, 2$;

(2) $F(t, 0, 0, 0, \cdot) \in L_2(\Omega, \mathscr{F}_t, P)$ for each $t \in R_+$, and $f_j(t, s, 0, \cdot) \in L_2(\Omega, \mathscr{F}_s, P)$ for each $(t, s) \in \Delta$.

Remark 4.1. We note that from condition (1) of Assumption H we obtain the following estimates:

$$|F(t, u_1, u_2, x, \omega)|^2 \leqq 4\bar{k}_1^2(t)|u_1|^2 + 4\bar{k}_2^2(t)|u_2|^2 + 4l^2(t)|x|^2 + 4|F(t, 0, 0, 0, \omega)|^2$$

and

$$|f_j(t, s, x, \omega)|^2 \leqq 2\tilde{k}_j^2(t)|x|^2 + 2|f_j(t, s, 0, \omega)|^2$$

for $t \in R_+$, $s \leqq t$, $u_j, x \in R$, $j = 1, 2$, $\omega \in \Omega$.

Put

$$(4.2) \qquad \begin{aligned} k(t) &= 6[\bar{k}_1^2(t)\tilde{k}_1^2(t)g(t) + \bar{k}_2^2(t)\tilde{k}_2^2(t)], \\ l(t) = 6l^2(t), \quad r(t) &= 2E[|(Ux_0)(t, \omega) - x_0(t, \omega)|^2]. \end{aligned}$$

Theorem 4.1. *If Assumption* H *and assumptions* (ii) *and* (iii) *of Lemma* 3.1 *are satisfied with k, l and r defined by* (4.2), *then there exists a random solution $\bar{x} \in C(R_+, L_2)$ of equation* (1.1) *such that*

$$(4.3) \qquad E[|\bar{x}(t, \omega) - x_n(t, \omega)|^2] \leqq u_n(t), \quad n = 0, 1, \ldots, \quad t \in R_+.$$

The solution $\bar{x}$ is unique in the class $L_{\mathrm{loc}}^*(u_0) \overset{\mathrm{df}}{=} \{x\colon x \in L_{\mathrm{loc}}^*,\ E[|x(t, \omega) - x_0(t, \omega)|^2] \in \in L_{\mathrm{loc}}(u_0)\}$, where L_{loc}^* is the class of all locally integrable random functions defined on R_+ with range in $L_2(\Omega, \mathscr{F}_t, P)$, and $L_{\mathrm{loc}}(u_0)$ is defined in Lemma 3.1.

Proof. From the assumptions of the theorem, Cauchy's inequality, and the properties of the stochastic integral ([3], [7]) it follows that the integrals in equation (4.1) exist for each n (see Remark 4.1) and $x_n \in C(R_+, L_2)$, $n=0, 1, \ldots$.

To prove the existence of a solution of equation (1.1) we first prove the following estimates

$$(4.4) \qquad E[|x_n(t, \omega) - x_0(t, \omega)|^2] \leqq u_0(t), \quad n = 0, 1, \ldots, \quad t \in R_+,$$

$$(4.5) \qquad E[|x_{n+m}(t, \omega) - x_n(t, \omega)|^2] \leqq u_n(t), \quad n, m = 0, 1, \ldots, \quad t \in R_+.$$

It is clear that (4.4) holds for $n=0$. If we suppose that (4.4) holds for some $n>0$, then from $(x+y)^2 \leqq 2(x^2+y^2)$ and $(x+y+z)^2 \leqq 3(x^2+y^2+z^2)$, an application of Cauchy's inequality and the properties of the stochastic integral we have

$$E[|x_{n+1}(t,\omega)-x_0(t,\omega)|^2] \leqq$$
$$\leqq 2E[|(Ux_n)(t,\omega)-(Ux_0)(t,\omega)|^2]+2E[|(Ux_0)(t,\omega)-x_0(t,\omega)|^2] \leqq$$
$$\leqq 6\bar{k}_1^2(t)E\Big[\Big|\int_0^{g(t)} (f_1(t,s,x_n(s,\omega),\omega)-f_1(t,s,x_0(s,\omega),\omega))\,ds\Big|^2\Big]+$$
$$+6\bar{k}_2^2(t)E\Big[\Big|\int_0^{g(t)} (f_2(t,s,x_n(s,\omega),\omega)-f_2(t,s,x_0(s,\omega),\omega))\,dw(s,\omega)\Big|^2\Big]+$$
$$+6l^2(t)E[|x_n(h(t),\omega)-x_0(h(t),\omega)|^2]+2E[|(Ux_0)(t,\omega)-x_0(t,\omega)|^2] \leqq$$
$$\leqq (6\bar{k}_1^2(t)\tilde{k}_1^2(t)g(t)+6\bar{k}_2^2(t)\tilde{k}_2^2(t))\int_0^{g(t)} E[|x_n(s,\omega)-x_0(s,\omega)|^2]\,ds+$$
$$+6l^2(t)E[|x_n(h(t),\omega)-x_0(h(t),\omega)|^2]+2E[|U(x_0)(t,\omega)-x_0(t,\omega)|^2] \leqq$$
$$\leqq k(t)\int_0^{g(t)} u_0(s)\,ds+l(t)u_0(h(t))+r(t) = u_0(t).$$

Now (4.4) follows by induction.

It follows from (4.4) that (4.5) holds for $n=0$, $m=0, 1, \ldots$. Now the inequality (4.5) follows from

$$E[|x_{n+m+1}(t,\omega)-x_{n+1}(t,\omega)|^2] \leqq (Ku_n)(t)+(Lu_n)(t) = u_{n+1}(t), \quad t\in R_+,$$

and by induction.

Since $u_n \rightrightarrows 0$ for $n\to\infty$ (see Lemma 3.2) and from (4.5) it follows that $\{x_n\}$ is a Cauchy sequence (see Definition 2.2) in $C(R_+, L_2)$. Now, since $C(R_+, L_2)$ is a complete space, there exists an $\bar{x}\in C(R_+, L_2)$ such that $x_n\to\bar{x}$. If $m\to\infty$, then (4.5) yields estimation (4.3). By the estimation

$$E[|\bar{x}(t,\omega)-(U\bar{x})(t,\omega)|^2] \leqq$$
$$\leqq 2E[|\bar{x}(t,\omega)-x_n(t,\omega)|^2]+2E[|(Ux_{n-1})(t,\omega)-(U\bar{x})(t,\omega)|^2] \leqq$$
$$\leqq 4u_n(t), \quad n = 0, 1, \ldots, \quad t\in R_+,$$

it follows that the random function $\bar{x}$ satisfies equation (1.1).

The uniqueness part of the theorem follows immediately from assertion (c) of Lemma 3.1. Indeed, if we suppose that there exists another solution $\tilde{x}$ of equation (1.1) belonging to $L^*_{\rm loc}(u_0)$ then we easily infer that $\tilde{u}(t)=E[|\bar{x}(t,\omega)-\tilde{x}(t,\omega)|^2]\in L_{\rm loc}(u_0)$, and $\tilde{u}\leqq K\tilde{u}+L\tilde{u}$. Hence and from (c) of Lemma 3.1 it follows that $E[|\bar{x}(t,\omega)-\tilde{x}(t,\omega)|^2]=0$. This completes the proof of the theorem.

Remark 4.2. By using the Banach fixed point theorem or the concept of admissibility theory it is easy to prove that there exists a unique random solution of stochastic equation (1.1) if Assumption H is fullfiled and

$$k(t)g(t)+l(t) < 1, \quad t\in R_+, \tag{4.6}$$

where k and l are defined by (4.2).

The following theorem, which follows from part c) of Remark 3.1 and Theorem 4.1, shows that condition (4.6) is more restrictive than the assumptions of Theorem 4.1.

Theorem 4.2. *If Assumption* H, *assumption* (iii) *of Lemma* 3.1 *and condition* (3.4) *are satisfied and if* $r(t)\leqq\bar{r}t^p$, $t\in R_+$, *for some* $p, \bar{r}\in R_+$, *then the assertion of Theorem* 4.1 *holds provided* $\bar{l}\bar{h}^p<1$.

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