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The stability of d'Alembert-type functional equations

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In this paper we deal with the following problem: if f, g, h, k are complex valued functions on the Abelian group G with the property, that the function $(x, y) \to f(x+y)+g(x-y)-h(x)k(y)$ is bounded, what can be said about the functions f, g, h, k? Obviously, this problem is a generalization of the well-known functional equations

$$f(x+y)+f(x-y) = 2f(x)g(y), \tag{0}$$

$$f(x+y)+g(x-y) = h(x)k(y). \tag{1}$$

Special cases of this problem has been treated by many authors. The special case $k=1$ is of "additive type" and can be reduced to the problem: if $(x, y) \to f(x+y)-f(x)-f(y)$ is bounded, what can be said about f? The problem in this form is treated in [2], [4], [5], [6], [8]. The special case $g=0$ and $h=k=f$ is treated in [3], and the case $g=0$ and $h=f$ is treated in [9]. Further, the special case where $f=g=h$ and $k=2f$ is treated in [3], and the case where $f=g=h$ is treated in [10]. In this paper we completely solve the above problem.

First we make a simple observation: evidently, if f, g, h, k is a solution of the functional equation (1) and a, b are arbitrary bounded complex valued functions on G, then the functions $f+a, g+b, h, k$ solve our problem. Our main result is the following: if f, g, h, k are unbounded functions, then essentially this is the only solution of our problem.

In the sequel we shall use the following notation and terminology: C denotes the set of complex numbers. If G is a group and $M: G \to C$ is a function for which $M(x+y)=M(x)M(y)$ holds for all x, y in G, then we call M an *exponential.* The function $A: G \to C$ is called *additive*, if $A(x+y)=A(x)+A(y)$ holds whenever x, y is in G. If $F: G \to C$ is a function, then F_e and F_o denotes the even and the odd

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part of F respectively, that is,

$$F_e(x) = -\frac{1}{2}(F(x)+F(-x)), \quad F_o(x) = \frac{1}{2}(F(x)-F(-x))$$

for all x in G.

In what follows we suppose, that G is a fixed Abelian group in which the mapping $x \to 2x$ is an automorphism.

We shall use the following theorem:

Theorem 1. *If* $f, g\colon G \to C$ *satisfy* (0), *then there are an exponential* $M\colon G \to C$, *an additive function* $A\colon G \to C$ *and* α, β *constants such that we have the following possibilities:*

(i) $f = 0$, g *is arbitrary,*

(ii) $f = A+\alpha$, $g = 1$,

(iii) $f = \alpha M_e + \beta M_o$, $g = M_e$.

The proof of this theorem can be obtained by the method of [1], using the results of [7].

Lemma 2. *Let* $f, g, h\colon G \to C$ *be functions for which the function* $(x, y) \to f(x+y) - g(x)h(y)$ *is bounded. Then there are an exponential* $M\colon G \to C$, *a bounded function* $a\colon G \to C$ *and* α, β *constants such that we have the following possibilities:*

(i) f *is bounded,* h *is arbitrary,* $g=0$,

(ii) f *is bounded,* $h=0$, g *is arbitrary,*

(iii) f, g, h *are bounded,*

(iv) $f = \alpha\beta M + a$, $g = \alpha M$, $h = \beta M$.

Proof. The first three cases are trivial, hence we may suppose that f, g, h are unbounded. Let $\alpha = g(0)$, $\beta = h(0)$ and $a = f - \beta g$. Obviously, a is bounded, and the identity

$$f(x+y) - g(x)h(y) - a(x+y) = \beta g(x+y) - g(x)h(y)$$

implies that $\beta \neq 0$, and the function $(x, y) \to g(x+y) - g(x)\beta^{-1}h(y)$ is bounded. By [9], it follows (iv).

Lemma 3. *Let* $f, g\colon G \to C$ *be functions for which the function* $(x, y) \to f(x+y) + f(x-y) - 2f(x)g(y)$ *is bounded. Then there are an exponential* $M\colon G \to C$, *an additive function* $A\colon G \to C$, *a bounded function* $a\colon G \to C$ *and* α, β *constants such that we have the following possibilities:*

(i) $f = 0$, g *is arbitrary,*

(ii) f, g *are bounded,*

(iii) $f = A + a$, $g = 1$,

(iv) $f = \alpha M_e + \alpha M_o$, $g = M_e$.

Proof. The first two cases are trivial. We may suppose that f is unbounded. This implies that $g\neq 0$. If $g=1$, then by [8], (iii) follows. Suppose that $g\neq 1$. Let $F(x, y)=f(x+y)+f(x-y)-2f(x)g(y)$ for all x, y in G. By [10] and Theorem 1, there is an exponential $M: G\to C$ for which $g=M_e$, in particular g is even. Now consider the identity

$$2g(z)F(x, y) = F(x, y+z)+F(x, y-z)-F(x+y, z)-F(x-y, z),$$

which shows that either g is bounded, or $F=0$. Suppose, that g is bounded, and observe that the following identities hold:

$$f_e(y)g(x)-f_e(x)g(y) = \frac{1}{4}\left(F(x, y)-F(y, x)+F(-x, -y)-F(-y, -x)\right), \tag{2}$$

$$f_o(x+y)-f_o(x)g(y)-f_o(y)g(x) = \frac{1}{4}\left(F(x, -y)-F(-y, x)-F(-x, y)+F(y, -x)\right). \tag{3}$$

By (2) we obtain that f_e is bounded, and by (3) we see that the function $x\to f_o(x+y)-f_o(x)g(y)$ is bounded for all fixed y in G. Since f_o cannot be bounded, by [9] it follows that g is an exponential. As $g\neq 0$, we have $g(0)=1$, and for all x in G,

$$1 = g(0) = g\left(\frac{x}{2}\right)g\left(-\frac{x}{2}\right) = g\left(\frac{x}{2}\right)g\left(\frac{x}{2}\right) = g(x),$$

a contradiction. Hence g is unbounded and $F=0$, that is, (iv) follows by Theorem 1.

Theorem 4. *Let f, g, h, k: $G\to C$ be functions for which the function $(x, y)\to f(x+y)+g(x-y)-h(x)k(y)$ is bounded. Then there are an exponential M: $G\to C$, an additive function $A: G\to C$, bounded functions a, b, c: $G\to C$, and constants $\alpha, \beta, \gamma, \delta$ such that we have the following possibilities:*

(i) *f, g, h, k are bounded,*

(ii) *f, g are bounded, $h=0$, k is arbitrary,*

(iii) *f, g are bounded, h is arbitrary, $k=0$,*

(iv) *f is bounded, $g = \alpha\beta M+b$, $h = \alpha M$, $k = \beta M^{-1}$,*

(v) *$f = \alpha\beta M+a$, g is bounded, $h = \alpha M$, $k = \beta M$,*

(vi) *$f = \frac{1}{2}\alpha A+a$, $g = -\frac{1}{2}\alpha A+b$, $h = \alpha$, $k = A+c$,*

(vii) *$f = \frac{1}{2}\beta A+a$, $g = \frac{1}{2}\beta A+b$, $h = A+c$, $k = \beta$,*

(viii) *$f = \frac{1}{4}\alpha\beta A^2+\frac{1}{2}(\alpha\delta+\beta\gamma)A+a$, $g = -\frac{1}{4}\alpha\beta A^2+\frac{1}{2}(\alpha\delta-\beta\gamma)A+b$, $h = \alpha A+\gamma$, $k = \beta A+\delta$,*

$$\text{(ix) } f = \frac{1}{2}(\alpha\gamma+\beta\delta)M_e+\frac{1}{2}(\alpha\delta+\beta\gamma)M_o+a, \quad h = \alpha M_e+\beta M_o,$$
$$g = \frac{1}{2}(\alpha\gamma-\beta\delta)M_e-\frac{1}{2}(\alpha\delta-\beta\gamma)M_o+b, \quad k = \gamma M_e+\delta M_o.$$

Proof. The first three cases are trivial, and if f or g is bounded, then by Lemma 2 we have (iv) or (v). Now we may suppose that f, g are unbounded, and $h\neq 0$, $k\neq 0$. Let $h(x_0)\neq 0$, $k(y_0)\neq 0$, and we introduce the new functions:

$$F(x) = h(x_0)^{-1}k(y_0)^{-1}f(x+x_0+y_0), \quad G(x) = h(x_0)^{-1}k(y_0)^{-1}g(x+x_0-y_0),$$
$$H(x) = h(x_0)^{-1}h(x+x_0), \quad K(x) = k(y_0)^{-1}k(x+y_0).$$

We have that F, G are unbounded, $H(0)=K(0)=1$, and the function D defined by

$$D(x,y) = F(x+y)+G(x-y)-H(x)K(y) \tag{4}$$

is bounded. First we present some simple identities concerning F, G, H, K, D, which we shall need in the sequel:

$$H(x+y)+H(x-y)-2H(x)K_e(y) = \tag{5}$$
$$= D(x,y)+D(x,-y)-D(x+y,0)-D(x-y,0),$$

$$H_o(y)K_o(x)-H_o(x)K_o(y) = \frac{1}{4}(D(x,y)-D(y,x)-D(x,-y)+ \tag{6}$$
$$+D(-y,x)+D(-x,-y)-D(-y,-x)-D(-x,y)+D(y,-x),$$

$$H(x+y)K_o(x-y)-H(x)K_o(x)+H(y)K_o(y) = \tag{7}$$
$$= \frac{1}{2}(D(x,x)-D(x,-x)+D(y,-y)-D(y,y)+D(x+y,y-x)-D(x+y,x-y)),$$

$$H_o(x+y)K_o(x-y)-H_o(x)K_o(x)+H_o(y)K_o(y) = \tag{8}$$
$$= \frac{1}{4}(D(x,x)+D(-x,-x)-D(x,-x)-D(-x,x)+D(y,-y)+D(-y,y)-$$
$$-D(y,y)-D(-y,-y)+D(x+y,y-x)+D(-x-y,x-y)-D(x+y,x-y)-$$
$$-D(-x-y,y-x)),$$

and finally, if $H_o=0$, that is, H is even, then

$$K(x+y)+K(x-y)-2K(x)H(y) = 2D(y,x)-D(0,x+y)-D(0,x-y)- \tag{9}$$
$$-D\left(\frac{x+y}{2},\frac{x+y}{2}\right)+D\left(\frac{-x-y}{2},\frac{x+y}{2}\right)-D\left(\frac{y-x}{2},\frac{x-y}{2}\right)+D\left(\frac{x-y}{2},\frac{x-y}{2}\right).$$

These identities can be checked by an easy computation and they show, that the expressions on the left hand sides are bounded. Finally, we shall need the relations

$$F(x) = H\left(\frac{x}{2}\right)K\left(\frac{x}{2}\right)+D\left(\frac{x}{2},\ \frac{x}{2}\right)-G(0), \tag{10}$$

$$G(x) = H\left(\frac{x}{2}\right)K\left(-\frac{x}{2}\right)+D\left(\frac{x}{2},\ -\frac{x}{2}\right)-G(0).$$

Now we assume that H is bounded, and show that (vi) follows. By (5) K_e is bounded, and if H is not even, then by (6) K_o is bounded, too, which is impossible by (10). Hence H is even, and then by (9) and Lemma 3 either $K=A+a$ and $H=1$, or $K=M_e+\beta M_o$, $H=M_e$. In the latter case M_e is bounded, and by the identity $M_e(x+y)-M_e(x-y)=2M_o(x)M_o(y)$ the function M_o is bounded, too, that is, K is also bounded, which is impossible by (10). This means that $H=1$ and $K=A+a$, where $A\colon G\to C$ is additive, and $a\colon G\to C$ is bounded. By (10) and by the definition of F, G, H, K we have (vi).

Hence we may suppose in the sequel, that H is unbounded.

From (5) by Lemma 3 we have two cases. In the first case $K_e=1$, $H=A+c$, where $A\colon G\to C$ is additive and $c\colon G\to C$ is bounded. Here $A\neq 0$ and $H_o\neq 0$, hence by (6) $K_o=\alpha A+d$, where $d\colon G\to C$ is odd and bounded, and α is a constant. If $\alpha=0$, then by (6) either H_o is bounded, which is impossible, or $K_o=0$, that is $K=K_e=1$ and from (10) we obtain (vii) using the definition of F, G, H, K.

Let $\alpha\neq 0$, then we substitute H_o and K_o into (6) and we have that the function

$$(x, y) \to A(x)\left(\alpha c_o(y)-d(y)\right)-A(y)\left(d(x)-\alpha c_o(x)\right)$$

is bounded. If there is a y in G, for which $d(y)\neq\alpha c_o(y)$, then $A=0$, which is impossible. Hence $d=\alpha c_o$, and $H=A+c_o+c_e$, $K=\alpha A+\alpha c_o+1$. Substituting into (8) we have that the function

$$(x, y) \to A(x)\left(c_o(x+y)+c_o(x-y)-2c_o(x)\right)-A(y)\left(c_o(x+y)-c_o(x-y)-2c_o(y)\right)$$

is bounded. Substituting $x+y$ for x and $x-y$ for y, we have that the function

$$(x, y) \to A(x+y)c_o(x+y)-A(x-y)c_o(x-y)-A(y)c_o(2x)-A(x)c_o(2y) \tag{11}$$

is bounded. Let $p(x)=A(x)c_o(x)$ and $P(x, y)=p(x+y)-p(x-y)-A(x)c_o(2y)$, then (11) implies the boundedness of $x\to P(x, y)$ for all fixed y in G. On the other hand, the identity

$$P(x+y, z)+P(x-y, z)-P(x, y+z)+P(x, y-z) =$$

$$= A(x)\left(c_o(2y+2z)-c_o(2y-2z)-2c_o(2z)\right)$$

shows, that for all fixed y, z in G the function $x \to A(x)\big(c_o(2y+2z)-c_o(2y-2z)-$ $-2c_o(2z)\big)$ is bounded, and hence

$$c_o(2y+2z)-c_o(2y-2z) = 2c_o(2z)$$

holds for all y, z in G. Interchanging y and z, we have that c_o is additive and as it is bounded, $c_o=0$, $H=A+c_e$, $K=\alpha A+1$. Substituting into (7) we get that the function

$$(x, y) \to A(x)\big(c_e(x+y)-c_e(x)\big)-A(y)\big(c_e(x+y)-c_e(y)\big)$$

is bounded. Writing $x+y$ for x and $x-y$ for y we obtain that the function

$$(12) \qquad (x, y) \to A(x+y)c_e(x+y)-A(x-y)c_e(x-y)-2A(y)c_e(2x)$$

is bounded. Let $p(x)=A(x)c_e(x)$ and $P(x, y)=p(x+y)-p(x-y)-2A(y)c_e(2x)$, then (12) implies that P is bounded. On the other hand, the identity

$$P(x+y, z)+P(x-y, z)-P(r, y+z)+P(x, y-z) =$$
$$= -2A(z)\big(c_e(2x+2y)+c_e(2x-2y)-2c_e(2x)\big)$$

shows that the functional equation

$$c_e(2x+2y)+c_e(2x-2y) = 2c_e(2x)$$

holds. Interchanging x and y we get that c_e is constant. Since $H(0)=1$, therefore $c_e=1$ and $H=A+1$, $K=\alpha A+1$. Using (10) and the definition of F, G, H, K we obtain case (viii).

Finally, we have to return to the second case at (5), where by Lemma 3, $H=M_e+\alpha M_o$, $K_e=M_e$. Here $M: G\to C$ is an exponential, and α is a constant. Of course $M_o=0$ is impossible, and so (6) implies $K_o=\beta M_o+a$, where $a: G\to C$ is bounded and β is a constant. Hence by (10) we have for all x in G that

$$F(x) = \frac{1+\alpha\beta}{2} M_e(x)+\frac{\alpha+\beta}{2} M_o(x)+\left[M_e\left(\frac{x}{2}\right)+\alpha M_o\left(\frac{x}{2}\right)\right] a\left(\frac{x}{2}\right)+d(x),$$

$$G(x) = \frac{1-\alpha\beta}{2} M_e(x)-\frac{\alpha-\beta}{2} M_o(x)-\left[M_e\left(\frac{x}{2}\right)+\alpha M_o\left(\frac{x}{2}\right)\right] a\left(\frac{x}{2}\right)+e(x),$$

where $d, e: G\to C$ are bounded functions (we have used that a is obviously odd). Substituting into (4) and using that D is bounded, we have that the function

$$(13) \qquad (x, y) \to H\left(\frac{x+y}{2}\right)a\left(\frac{x+y}{2}\right)-H\left(\frac{x-y}{2}\right)a\left(\frac{x-y}{2}\right)-H(x)a(y)$$

is bounded. Let $p(x)=H\left(\frac{x}{2}\right)a\left(\frac{x}{2}\right)$ and $P(x, y)=p(x+y)-p(x-y)-H(x)a(y)$.

Then (13) implies that P is bounded. On the other hand, using that H is unbounded, we infer from the identity

$$P(x+y,z)+P(x-y,z)+P(x,y-z)-P(x,y+z) =$$
$$= H(x)\big(a(y+z)-a(y-z)-2M_e(y)a(z)\big)$$

that the functional equation

$$a(y+z)-a(y-z) = 2M_e(y)a(z)$$

holds. If $a\neq 0$, then M_e, and consequently H is bounded, which is impossible. Hence $a=0$, and we obtain case (ix). The theorem is proved.

Remark. Theorem 4 shows that for unbounded functions $f, g, h, k\colon G\to C$ the only possibility for $(x,y)\to f(x+y)+g(x-y)-h(x)k(y)$ to be bounded is that $f+a, g+b, h, k$ be a solution of (1) with some bounded functions $a, b\colon G\to C$.

Remark. The proofs of the above theorems and lemmata show that the main result can be generalized for other functional analytic function properties instead of "boundedness". More precisely, let W be a complex linear space of complex valued functions on $G\times G$ with the properties:

(i) if F belongs to W, then $(x,y)\to F(x+u, y+v)$ belongs to W,

(ii) constant functions belong to W,

(iii) if F belongs to W, then all the functions

$$(x,y)\to F(y,x),\quad (x,y)\to F(x,-y),$$

$$(x,y)\to F(x+y,x-y),\quad (x,y)\to F(x+y,0),\quad (x,y)\to F(x-y,0)$$

$$(x,y)\to F\left(\frac{x+y}{2},\frac{x+y}{2}\right),\quad (x,y)\to F\left(\frac{x+y}{2},-\frac{x+y}{2}\right),$$

$$(x,y)\to F(x,x),\quad (x,y)\to F(2x,0),$$

and for all z in G, $(x,y)\to F(x,z)$ belong to W,

(iv) if for a function $f\colon G\to C$ the function $(x,y)\to f(x+y)+f(x-y)-2f(x)$ belongs to W, then there is a function $A\colon G\to C$ such that $A(x+y)+A(x-y)= =2A(x)$ holds for all x, y in G, and $(x,y)\to f(x)-A(x)$ belongs to W.

Then Theorem 4 holds, if we set everywhere "belongs to W" instead of "bounded". For instance, if $W=(0)$, then we obtain from Theorem 4 the general solution of (1). As less trivial examples, "boundedness" can be replaced by "almost periodicity", or in the cases $G=R$ (the real line) or G compact Abelian, by "continuity", provided the mapping $x\to 2x$ is a homeomorphism.

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