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Definable principal congruence relations: Kith and kin

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This paper has two aims. Firstly, it seeks to illumine the way in which principal congruence relations are constructed. To this end, a hierarchy of "definability" of congruences is presented. Notions both weaker and stronger than (first order) *definable principal congruences* (dpc) are considered. Secondly, it attacks the problem posed by BURRIS and LAWRENCE [13], "If **K** is a class of algebras and if the quasivariety generated by **K**, $Q(\mathbf{K})$, has definable principal congruences must the variety generated by **K**, $V(\mathbf{K})$, also have dpc"? These two aims are linked by several results of the following form, "If $V(\mathbf{K})$ has (some weak notion of) definable principal congruences and $Q(\mathbf{K})$ has definable principal congruences, then $V(\mathbf{K})$ has definable principal congruences."

The discussion of the hierarchy mentioned above includes a survey of the literature on such notions and it attempts to connect these properties with others of a quite different nature. For example, several levels of the hierarchy are linked with n-permutability of (principal) congruence relations.

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1. Definitions and notation

In general, we denote algebras by A, B, and C while **K** denotes a class of algebras of some fixed similarity type. The variety generated by **K** is $V(\mathbf{K})$; and $Q(\mathbf{K})$ denotes the quasivariety generated by the class **K**. A class of algebras is said to be *locally finite* if every finitely generated algebra in the class is finite; the class is *uniformly locally finite* if there exists a function f such that for all n every n-generated algebra in the class has cardinality at most $f(n)$.

If A is an algebra, and if x and y are elements of A, then the principal congruence relation generated by x and y is the smallest congruence relation on A for which x

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and y are congruent. It is denoted by $\Theta(x, y)$. For a given algebra A of some similarity type τ, Malcev has provided a description of the principal congruence generated by the elements a_0 and a_1 solely in terms of the polynomials of the algebra A, e.g. [23] or [16, p. 54]. Namely,

$$b_0 \equiv b_1 \quad \Theta(a_0, a_1) \leftrightarrow \exists n \exists p_1, \dots, p_n \exists s \exists \mathbf{z}_1, \dots, \mathbf{z}_n \quad \text{such that}$$

$$\text{(M)} \quad b_0 = p_1(a_{s(1)}, \mathbf{z}_1), \quad p_i(a_{1-s(i)}, \mathbf{z}_i) = p_{i+1}(a_{s(i+1)}, \mathbf{z}_{i+1}) \quad \text{for} \quad 1 \leqq i < n, \quad \text{and}$$

$$b_1 = p_n(a_{1-s(n)}, \mathbf{z}_n)$$

where n is a positive integer, the p_i are k_i-ary polynomials of type τ, s is a switching function, i.e. $s\colon \{1, 2, \dots, n\} \to \{0, 1\}$, and $\mathbf{z}_i$ are $k_i - 1$ tuples from A, for $1 \leqq i \leqq n$. Each fixed instance of the polynomials p_i and the switching function s is called a *Malcev formula.*

We write $\psi(a_0, a_1, b_0, b_1, p_1, \dots, p_n, s, \mathbf{z}_1, \dots, \mathbf{z}_n)$ for the last 2 lines of (M). Note that the integer n is implicit in this formula. Also observe that $\exists \mathbf{z}_1, \dots, \mathbf{z}_n \, \psi$ is a positive existential formula. Such a formula was called a congruence formula in [3].

There is an easy abstract form of this result which is implicit in [25]. (Also see [26] for a nice application.) Let $\mathrm{Diag}^+(A) = \{R(\mathbf{z}) |\ R$ is atomic, $A \models R(\mathbf{z})\}$. Note that $\mathrm{Diag}^+(A)$ is in a language $L(A)$ which has names for all members of A.

Lemma 1.1. (Folklore) *If a, b, c, d are in A and $c \equiv d \ \Theta(a, b)$, then there is a positive existential L-formula $S(x, y, u, v)$ such that*

i) $\models S(x, y, u, u)$ *implies* $x = y$,

ii) $A \models S(c, d, a, b)$.

Proof. Note $\mathrm{Diag}^+(A) \cup (a = b) \cup (c \neq d)$ is consistent if and only if there is a homomorphic image of A which identifies a with b but does not identify c with d. Thus if $c \equiv d \ \Theta(a, b)$ then $\models \bigwedge \{R_i(c, d, a, b, \mathbf{e}) | 1 \leqq i \leqq m\} \,\&\, (a = b)$ implies $(c = d)$ for some finite set $R_1, \dots, R_m$ of atomic formulas. But then

$$\models \forall x, y, u, v (\exists \mathbf{w} (\bigwedge \{R_i(x, y, u, v, \mathbf{w}) | 1 \leqq i \leqq m\}) \,\&\, (u = v))$$

implies $x = y$ and thus $A \models \exists \mathbf{w} \bigwedge \{R_i(c, d, a, b) | 1 \leqq i \leqq m\}$.

Either of these characterizations allow us to "define" principal congruences in the language $L(\omega_1, \omega)$ i.e. the language which extends first order logic by allowing infinite disjunctions. Namely $c \equiv d \ \Theta(a, b) \leftrightarrow \bigvee \{S(c, d, a, b) | S \in P\}$ where P is the collection of positive existential formulas satisfying $S(x, y, u, u) \to x = y$ (or the collection of Malcev formulas). Although some information can be obtained from this weak definability of principal congruences (cf. [2]), in this paper we want to discuss various stronger notions of definability. The first formalization of this kind occurred as follows in [3]. A class **K** is said to have *definable principal conguences* if there is a 4-ary first order formula φ in the language of **K** such that

$$\forall A \in \mathbf{K}; \forall a_0, a_1, b_0, b_1 \in A, \ (b_0 \equiv b_1 \, \Theta(a_0, a_1) \leftrightarrow \varphi(a_0, a_1, b_0, b_1)).$$

For further details on definable principal congruences see [3] and sections 2 and 5 below.

Our earlier work on definable principal congruences focused on varieties. However, [12] shows the advantages of dealing with more general classes. Thus in the following we define these notions for arbitrary classes of algebras. Occasionally it will be necessary to assume that such a class satisfies the compactness theorem. Of course any elementary class satisfies this condition. For this reason many of our results have a dual character, we first describe the effect of a property on a specified algebra and then note the effect on algebras in a variety satisfying this condition. We call the former a local result, and the latter a global one.

Since a disjunction of all possible ψ describes the principal congruences in any class **K** of algebras, it follows from the compactness theorem that if a class **K** which satisfies the compactness theorem has definable principal congruences, then this defining formula is equivalent to some finite disjunction of the ψ, i.e.

$$\varphi(a_0, a_1, b_0, b_1) \leftrightarrow \bigvee_i \psi(a_0, a_1, b_0, b_1, p_1^i, \ldots, p_n^i, s_i, \mathbf{z}_1^i, \ldots, \mathbf{z}_n^i),$$

where i ranges over some finite index set. Note there is a uniform subscript n in this formula. This is possible since the diagonal elements are in any principal congruence relation, i.e. $w \equiv w \ \Theta(x, y)$ allowing the "padding out" of formulas ψ having different lengths to one uniform length.

2. DPC and its relatives

In this section we discuss first some weakenings and then some strengthenings of the notion of a definable principal congruence relation. These arise in a natural way via a reshuffling of the quantifiers. That is, we require that certain of the existentially quantified variables in (M) do not depend on the particular a_0, a_1, b_0, and b_1.

We start with Malcev's characterization of principal congruences. For any algebra A in the class **K**, and for all a_0, a_1, b_0, b_1 in A,

$$b_0 \equiv b_1 \ \Theta(a_0, a_1) \leftrightarrow \exists n \exists p_1, \ldots, p_n \exists s \exists \mathbf{z}_1, \ldots, \mathbf{z}_n$$
$$\psi(a_0, a_1, b_0, b_1, p_1, \ldots, p_n, s, \mathbf{z}_1, \ldots, \mathbf{z}_n).$$

We proceed to pull out existential quantifiers from the right side of this expression.

A) There is a bound n on the number of steps in determining principal congruences for all algebras in $\mathbf{K}$. In this case we say $\mathbf{K}$ has *n-step principal congruences.* By previous remarks we can, without loss of generality, assume all principal congruences can be described using a Malcev formula with the same fixed n. Formally, this gives

$$\exists n \quad \text{such that} \quad \forall A \in \mathbf{K} \quad \forall a_0, a_1, b_0, b_1 \in A$$

$$(b_0 \equiv b_1\, \Theta(a_0, a_1) \leftrightarrow \exists p_1, \ldots, p_n \exists s \exists \mathbf{z}_1, \ldots, \mathbf{z}_n$$

$$\psi(a_0, a_1, b_0, b_1, p_1, \ldots, p_n, s, \mathbf{z}_1, \ldots, \mathbf{z}_n)).$$

Note that this is not as strong as definable principal congruences since the polynomials which are to be used are not specified (and the language does not allow variables having polynomials for values). But, by restricting n in this way there are only finitely many choices for the switching functions, i.e. the 2^n functions from $\{1, \ldots, n\}$ to $\{0, 1\}$. Algebras with n-step principal congruences are discussed in section 3.

B) The class $\mathbf{K}$ has n-step principal congruences and there is a *specified list of switching functions* for determining all principal congruences of algebras in $\mathbf{K}$. This becomes in the notational pattern we have adopted:

$$\exists n, \exists s_1, \ldots, s_k \quad \text{such that} \quad \forall A \in \mathbf{K} \quad \forall a_0, a_1, b_0, b_1 \in A$$

$$(b_0 \equiv b_1\, \Theta(a_0, a_1) \leftrightarrow \exists p_1, \ldots, p_n \exists i \exists \mathbf{z}_1, \ldots, \mathbf{z}_n$$

$$\psi(a_0, a_1, b_0, b_1, p_1, \ldots, p_n, s_i, \mathbf{z}_1, \ldots, \mathbf{z}_n)).$$

We will be mainly interested in the case that $k=1$. Results concerning this situation will be given in section 4.

C) The class $\mathbf{K}$ has n-step principal congruences with a specified list of switching functions, and there is a *specified list of polynomials* to be used for determining principal congruences of all the algebras in $\mathbf{K}$. This is of course, *definable principal congruences.* Thus it formally becomes:

$$\exists n, \exists s_1, \ldots, s_k \exists p_1^i, \ldots, p_n^i \quad (1 \leq i \leq k) \quad \text{such that} \quad \forall A \in \mathbf{K} \quad \forall a_0, a_1, b_0, b_1 \in A$$

$$(b_0 \equiv b_1 \quad \Theta(a_0, a_1) \leftrightarrow \exists i \quad \exists \mathbf{z}_1, \ldots, \mathbf{z}_n \quad \psi(a_0, a_1, b_0, b_1, p_1^i, \ldots, p_n^i, s_i, \mathbf{z}_1, \ldots, \mathbf{z}_n)).$$

A discussion of definable principal congruences in this context is found in section 5.

D) The class $\mathbf{K}$ has n-step principal congruences with a *single switching function* s and a specified list of polynomials $p_1, \ldots, p_n$ to be used for determining principal congruences in **K**. Formally this becomes:

$$\exists n, \exists s, \exists p_1, \ldots, p_n \quad \text{such that} \quad \forall A \in \mathbf{K} \quad \forall a_0, a_1, b_0, b_1 \in A$$

$$(b_0 \equiv b_1 \, \Theta(a_0, a_1) \leftrightarrow \exists z_1, \ldots, z_n \quad \psi(a_0, a_1, b_0, b_1, p_1, \ldots, p_n, s, z_1, \ldots, z_n)).$$

If φ defines principal congruences for a class **K** then φ is a finite disjunction of Mal-cev formulas. In D) there is only one disjunct (this is exactly the distinction implicit in [3] between the congruences being defined by a weak congruence formula or a congruence formula). This notion is now called "**K** has a uniform congruence scheme" and has been investigated in [15], [7], [8], [21]. In particular, [15] shows uniform congruence scheme is equivalent to their notion of equationally definable principal congruences.

We thus have a hierarchy of properties A), B), C) and D). It is a natural question to investigate how this hierarchy behaves in the presence of other properties the class **K** may possess. These questions will not figure in the remainder of this paper, so we briefly mention them at this point. Fried and Kiss [33] have also considered related questions.

One possibility is to restrict the variables z_i to take on values from $\{a_0, a_1, b_0, b_1\}$. By a result of Day [14] this is equivalent to the congruence extension property. (For details on the congruence extension property see [17] or [16].) If this restriction on the z_i is added to condition A) and if **K** also is uniformly locally finite (or at least 4-generated algebras in **K** have bounded cardinality), then **K** has definable principal congruences, and the defining formula can be made quantifier free as discussed in [3]. Moreover, if in D) all $z_i \in \{a_0, a_1, b_0, b_1\}$, then this gives the restricted uniform congruence scheme discussed in [15], [9], [10], [21] and [22].

Another possible restriction is to place a bound m on the possible arities of the polynomials which appear. In this case, if **K** is also uniformly locally finite, then conditions A) and B) both collapse to C). This is similar to the property CEP_n discussed in [21]. This also has some bearing on the notion of P_0-principal congruence relations which are defined in the next section. Baker [31] considers restrictions on the polynomials for varieties of lattices.

If **K** is a variety with distributive congruence lattices, then it is shown in [15] that conditions C) and D) are the same. We do not know how this affects conditions A) and B).

3. Bounded number of steps

In this section we investigate condition A) for a class **K** of algebras. We provide some curious examples, pose some questions, and deal with some work of Burris and Lawrence on definable principal congruences.

The simplest example of a class of algebras having a finite bound on the number of steps required in principal congruence relations is provided by sets, i.e. algebras with no fundamental operations. For sets, the only polynomials are the projection operations. If A is a set, then for $x, y \in A$, the principal congruence relation $\Theta(x, y)$ consists of the ordered pairs (x, y), (y, x), and (w, w) for all $w \in A$. It is easily seen that these three cases can be handled with polynomial projection functions; but both possible switching functions are required, and two different projection functions are needed. This simple example is instructive since it shows that condition A) is not preserved under extension of varieties, and hence is not a Malcev condition. (For details on Malcev conditions see [16], [27], and [3].) This is to be contrasted with some of the results in section 4.

An instance of the significance of n-step principal congruences may be found in the work of BURRIS and LAWRENCE [12], [13] on definable principal congruences in groups and rings. In the second of these two papers, they define the notion of P_0-projective principal congruences for a class **K** of algebras. Essentially this is condition A) with $n=1$ and with the added stipulation that the polynomials which are to be used are drawn from some class P_0 of polynomials. They posed the following problem:

Problem 1. (Burris and Lawrence) Let **K** be a class of algebras such that $Q(\mathbf{K})$ has definable principal congruences. Does $V(\mathbf{K})$ also have definable principal congruences?

They prove the following in [13]:

Theorem 3.1. (Burris & Lawrence) *Let* **K** *be a class of algebras such that* $Q(\mathbf{K})$ *has definable principal congruences and such that* $V(\mathbf{K})$ *has* P_0*-projective principal congruence relations for some set* P_0. *Then* $V(\mathbf{K})$ *also has definable principal congruences.*

For an arbitrary class **K** of algebras, the class $Q(\mathbf{K})$ and the class $ISP(\mathbf{K})$ of all algebras isomorphic to subalgebras of products of members of **K** need not be the same. For example [4] and [5] contain a discussion of this. However, for any class **K**, $HQ(\mathbf{K})=HSP(\mathbf{K})=V(\mathbf{K})$ and the following variant on Problem 1 is possible.

Problem 1a. Let **K** be a class of algebras such that $SP(\mathbf{K})$ has definable principal congruences. Does $V(\mathbf{K})$ also have definable principal congruences?

We now present a generalization of Theorem 3.1 which answers Problem 1 and 1a for certain classes **K** of algebras.

Theorem 3.2. *Let* **K** *be a class of algebras such that* **K** *has definable principal congruences and such that* $H(\mathbf{K})$ *has* n*-step principal congruences for some integer* n. *Then* $H(\mathbf{K})$ *also has definable principal congruences.*

Proof. Let φ be any positive 4-ary formula defining principal congruences for the class **K**, and suppose $H(\mathbf{K})$ has n-step principal congruences. We claim $H(\mathbf{K})$ has definable principal congruences given by

$$(*)\qquad b_0 \equiv b_1 \quad \Theta(a_0, a_1) \ \text{ iff } \ \exists w_0, \dots, w_n$$
$$(w_0 = b_0,\ w_n = b_1, \ \text{ and } \ \varphi(a_0, a_1, w_{i-1}, w_i) \ \text{ for } \ 1 \leqq i \leqq n).$$

To verify this claim consider $B \in H(\mathbf{K})$, $A \in \mathbf{K}$, and a homomorphism h from A onto B. Let $b_0 \equiv b_1 \quad \Theta(a_0, a_1)$ in B. So B models the formula

$$\psi(a_0, a_1, b_0, b_1, p_1, \dots, p_n, s, \mathbf{z}_1, \dots, \mathbf{z}_n)$$

for some choice of p_j, s, and $\mathbf{z}_j$, where the p_j are k_j-ary polynomials. Arbitrarily choose $a_i' \in A$ and k_j-1 tuples $\mathbf{z}_j'$ with $h(a_i')=a_i$ and $h(\mathbf{z}_j')=\mathbf{z}_j$. Note that $h(p_j(a_i', \mathbf{z}_j'))=p_j(a_i, \mathbf{z}_j)$. Also, in the algebra A it is the case that $p_j(a_0', \mathbf{z}_j') \equiv$ $\equiv p_j(a_1', \mathbf{z}_j') \ \Theta(a_0', a_1')$ for $1 \leqq j \leqq n$. Hence $A \models \varphi(a_0', a_1', p_j(a_0', \mathbf{z}_j'), p_j(a_1', \mathbf{z}_j'))$ and since φ is positive and h is a homomorphism $B \models \varphi(a_0, a_1, p_j(a_0, \mathbf{z}_j), p_j(a_1, \mathbf{z}_j))$. Moreover, in B, $p_j(a_{1-s(j)}, \mathbf{z}_j)=p_{j+1}(a_{s(j+1)}, \mathbf{z}_{j+1})$. So choosing $w_0=p_1(a_{s(j)}, \mathbf{z}_1)$, $w_n=p_n(a_{1-s(n)}, \mathbf{z}_n)$, and $w_i=p_{i+1}(a_{s(i+1)}, \mathbf{z}_{i+1})$ with $1 \leqq i < n$ shows $(*)$ holds in B.

Corollary 3.3. *Let* **K** *be a class of algebras such that the class* $Q(\mathbf{K})$ *has definable principal congruences and the class* $V(\mathbf{K})$ *has n-step principal congruences for some integer n. Then* $V(\mathbf{K})$ *also has definable principal congruences. Moreover, this result holds if Q is replaced by SP.*

Because of Theorem 3.2 and because the formula ψ is positive, it would be tempting to conjecture that n-step principal congruences is preserved under homomorphism. The following shows this is not the case.

Example 3.4. Let A have universe consisting of the positive integers and suppose for each positive integer i there is a unary operation g_i such that

$$g_i(1) = 2i+1, \quad g_i(2) = 2i+2, \quad \text{and} \quad g_i(k) = k \quad \text{for all} \quad k > 2.$$

Then A satisfies condition A) with $n=2$, but A has a homomorphic image B which does not satisfy condition A) for any n. In order to verify this, first observe that there are only four types of principal congruences on A (we list the nontrivial blocks):

$\Theta(1, 2) = 1, 2/3, 4/5, 6/\dots,$
$\Theta(1, k) = /k, \text{ odds}/, \ (k > 2)$
$\Theta(2, k) = /k, \text{ evens}/, \ (k > 2)$
$\Theta(k, m) = /k, m/ \ (k, m > 2).$

It is easily seen that the first and last of these congruences can be done in one step, while for the others, two steps will suffice. Define a homomorphism h of A so that h has kernel consisting of $\bigvee_i \Theta(2i, 2i+1)$ as i ranges over all $i > 1$. Then in the algebra

$h(A)$, the principal congruence relation $\Theta(h(1), h(2))$ will require an arbitrarily large number of steps.

Problem 2. Let A be an algebra that satisfies condition A) for its principal congruences with $n=1$. Does every homomorphic image of A also have this property?

We now present another approach to Problem 1. If A is an algebra, and if h is a homomorphism of A and if Θ is some congruence relation on A, then the image of Θ under the homomorphism h, denoted $h(\Theta)$, will consist of $\{(h(x), y(h)) | (x, y) \in \Theta\}$. Note that in general, because of transitivity, $h(\Theta)$ need not be a congruence in $h(A)$. We say that the homomorphism h *preserves* the congruence Θ if $h(\Theta)$ is a congruence relation of $h(A)$. Consider the following property for an algebra A:

(∗∗) For any homomorphism h of A any principal congruence relation $\Theta(h(x), h(y))$ of the algebra $h(A)$ is the image of some principal congruence relation of A.

Note that if h preserves $\Theta(x, y)$, then $\Theta(h(x), h(y)) = h(\Theta(x, y))$. Thus if every homomorphism of A preserves every principal congruence relation of A, then A has (∗∗). In section 4 we investigate the condition that a given homomorphism preserve a given congruence relation. Our interest in (∗∗) stems from the following.

Theorem 3.5. *Let* $\mathbf{K}$ *be a class of algebras with definable principal congruences, and suppose each algebra in* $\mathbf{K}$ *has property* (∗∗). *Then* $H(\mathbf{K})$ *also has definable principal congruences.*

Proof. Let $\mathbf{K}$ have definable principal congruences given by some positive formula φ. Consider $A \in \mathbf{K}$ and some homomorphism h of A onto an algebra B. If $u \equiv v \;\; \Theta(x, y)$ in B, then by property (∗∗), there are u', v', x', y' in A such that $h(\Theta(x', y')) = \Theta(x, y)$ and $h(u') = u$, $h(v') = v$, and $u' \equiv v' \;\; \Theta(x', y')$. Hence, $A \models \varphi(x', y', u', v')$, and since φ is positive, it follows that $B \models \varphi(x, y, u, v)$. Thus φ serves to define principal congruences in B as well.

We can relate property (∗∗) to our hierarchy by the following observation. If $\mathbf{K}$ is a class of algebras for which the class $H(\mathbf{K})$ satisfies condition A) with $n=1$, then every algebra A in $\mathbf{K}$ has property (∗∗). The proof of this is immediate since transitivity will not be violated. This observation gives Theorem 3.1 as a corollary of Theorem 3.5.

We note that condition (∗∗) is not trivial.

Example 3.6. There exist finite algebras A and B and a homomorphism h from A onto B such that the algebra B has a principal congruence relation that is not the image under h of any principal congruence relation on A. For example, let $A=\{1,2,3,4,5,6\}$ and let A have two unary operations f and g given by:

	1	2	3	4	5	6
f	3	4	3	4	5	6
g	5	6	3	4	5	6.

Let h be the homomorphism of A which identifies only 4 and 5. The principal congruence $\Theta\big(h(1), h(2)\big)$ in the algebra $h(A)$ is not the image under h of any principal congruence of A; for if it were, a simple argument shows it would be the image of the congruence $\Theta(1, 2)$, but the image of this relation under the homomorphism h is not transitive.

We conclude this section by discussing the relation between n-step principal congruences and n-permutability of principal congruences. This is motivated, in part, by work of MAGARI [22].

If R and S are binary relations on the same set, then the composition of them, denoted $R\circ S$, consists of all pairs (x, y) for which there is some z such that $(x, z)\in R$ and $(z, y)\in S$. Let $R\circ R\circ \dots \circ R$ with n factors be denoted by R^n. Also, $R^{-1}=\{(y, x)|(x, y)\in R\}$. The equivalence relations R and S are said to be n-permutable if

$$R\circ S\circ R\dots = S\circ R\circ S\dots, \text{ each side having } n \text{ factors.}$$

Thus 2-permutable is the usual permutable: $R\circ S=S\circ R$. If R and S are n-permutable then

$$R\vee S = R\circ S\circ R\dots \quad (n \text{ factors}).$$

Theorem 3.7. *Let A be an algebra such that A and every homomorphic image of A has n-step principal congruences. Then the principal congruences of A are $2n+1$ permutable.*

Proof. We will prove a slightly stronger result: If Θ is any principal congruence relation, $\Theta=\Theta(a_0, a_1)$, and if Σ is any congruence relation, then

$$\Theta\vee\Sigma = \Sigma\circ\Theta\circ\Sigma\circ\dots\circ\Sigma \quad (2n+1 \text{ factors}).$$

To this end, let $b_0\equiv b_1$ $(\Theta\vee\Sigma)$. So there exists a sequence of elements $t_0, t_1, \dots, t_k$ such that $t_0=b_0$, $t_k=b_1$, $t_{2i}\equiv t_{2i+1}$ Θ, and $t_{2i+1}\equiv t_{2i+2}$ Σ. Let h be a homomorphism with kernel Σ. So $h(t_{2i+1})=h(t_{2i+2})$ and $h(t_{2i})\equiv h(t_{2i+1})$ $\Theta(h(a_0), h(a_1))$. Therefore, $h(t_0)\equiv h(t_k)$ $\Theta(h(a_0), h(a_1))$. By hypothesis $h(A)$ has n-step principal congruences, so there exist polynomials p_i and elements z_i, $1\leqq i\leqq n$, and a switching function s to establish that $h(t_0)\equiv h(t_k)$ $\Theta\big(h(a_0), h(a_1)\big)$ in $h(A)$. But then in

A, $p_i(a_{1-s(i)}, z_i) \equiv p_{i+1}(a_{s(i+1)}, z_{i+1})$ Σ and $p_i(a_0, z_i) \equiv p_i(a_1, z_i)$ Θ. Therefore, the chain $b_0 = t_0$, $p_i(a_{s(1)}, z_1)$, $p_1(a_{1-s(1)}, z_1)$, $p_2(a_{s(2)}, z_2)$, ..., $t_n = b_1$, establishes the claim.

Note that the algebra $A = \langle\{0, 1, 2, ...\}, f\rangle$ where $f(i) = i-1$, and $f(0) = 0$ has permutable congruences as do all its subalgebras and homomorphic images; but principal congruences in A require an arbitrary number of steps. Hence Theorem 3.7 has no natural converse in this local form. However, a reasonable converse for the global version might be:

Problem 3. If a variety **K** has the property that there is some m for which all principal congruences are m-permutable, does **K** have n-step principal congruences for some n.

4. Bounded steps and specified switching

We next investigate condition B) for a class **K** of algebras. This general problem has not received much attention in the literature. However, if **K** is a variety and if only one switching function is allowed, and if this function is a constant function, then such **K** have been studied in some detail, although from a different point of view. For the remainder of this section we confine ourselves to classes **K** satisfying condition B) and having only one switching function s.

As in section 2, the easiest example is furnished by sets. For if **K** is the variety of sets, then all principal congruences in **K** can be obtained using only 2 steps and with a switching function $s(1) = 0$ and $s(2) = 1$. Note of course that several different polynomials will be required in the different cases. One consequence of this example is that condition B), even with only one switching function, is not a Malcev condition. This has also been observed by P. Köhler. However, compare this with Theorem 4.2 below.

In [22] Magari considers the notion of a good n-family for a class **K** of algebras. In the case of $n=1$ this reduces to **K** having m-step principal congruences for some integer m and some fixed switching function s, with all of the $z_i \in \{a_0, a_1, b_0, b_1\}$. He shows in the proof on pp. 695—696 that if an algebra has this property then if Θ is any principal congruence relation and Σ is any congruence relation, then

$$\Theta \circ \Sigma \circ \Theta \circ \ldots \circ \Theta \subseteqq \Sigma \circ \Theta \circ \Sigma \circ \ldots \circ \Sigma \quad (2m+1 \text{ factors in both}).$$

It follows then that in **K** principal congruence relations are $2m+1$ permutable.

We have shown in Theorem 3.7 that this result of Magari does not depend on fixing a particular switching function. In fact, by specifying the switching function (to be a constant) and working in a global setting, an even stronger result can be obtained.

We now consider varieties with n-permutable congruences. The following is due to HAGEMANN and MITSCHKE [19] and HAGEMANN [18].

Theorem 4.1. *For a variety* **K** *of algebras, the following are equivalent:*
(i) *The congruence relations of every algebra in* **K** *are n-permutable.*
(ii) *There exist ternary algebraic operations* $q_1, \ldots, q_n$ *on* **K** *such that*

$$q_1(x, y, y) = x, \ q_{i-1}(x, x, y) = q_i(x, y, y) \ \text{ and } \ q_{n-1}(x, x, y) = y.$$

(iii) *For any* A *in* **K** *and any reflexive subalgebra* R *of* A^2, $R^{-1} \subseteqq R^{n-1}$.
(iv) *For any* A *in* **K** *and for any reflexive subalgebra* R *of* A^2, $R^n = R^{n-1}$.

The following is implicit in HAGEMANN and MITSCHKE [19] and is an unpublished result of H. Lakser. Also see CHAJDA and RACHUNEK [32].

Theorem 4.2. *For a variety* **K** *of algebras, the following are equivalent:*
(i) **K** *has* $n+1$ *permutable congruence relations.*
(ii) *There is a constant function* $s\colon \{1, \ldots, n\} \to \{0, 1\}$ *such that all principal congruences of algebras in* **K** *can be done in n steps using s as the switching function.*

Proof. To show (i)→(ii) let $A \in \mathbf{K}$ and suppose $c \equiv d \ \Theta(a, b)$ in A. Define a relation $R = \{(p(a, \mathbf{z}), p(b, \mathbf{z})) \mid p$ is any k-ary polynomial and $\mathbf{z}$ is any $k-1$ tuple of elements of A, $k = 1, 2, \ldots\}$. We wish to show $(c, d) \in R^n$. Note that the relation R is reflexive and is a subalgebra of A^2. Also, by Malcev's lemma, there exists an m such that $(c, d) \in R_1 \circ R_2 \circ \ldots \circ R_m$, where each R_i is either R or R^{-1}. By Theorem 4.1 $R^{-1} \subseteqq R^n$, so $(c, d) \in R^t$ for some $t \geqq m$. But again by Theorem 4.1, $R^t \subseteqq R^n$, and hence $(c, d) \in R^n$ as desired.

For the opposite direction, assume, without loss, that $s(i) = 0$ for all i. Let F be the free **K** algebra on the three free generators a, b, and c. Note $b \equiv a \ \Theta(a, b)$. So there exist polynomials $p_1, \ldots, p_n$ such that

$$b = p_1(a, \mathbf{z}_1); \quad p_i(b, \mathbf{z}_i) = p_{i+1}(a, \mathbf{z}_{i+1}), \quad a = p_n(b, \mathbf{z}_n).$$

Each $\mathbf{z}_i$ is a sequence of elements of F, and each element of F is itself a polynomial in the variables a, b, and c. Denote this sequence of polynomials by $\mathbf{z}_i(a, b, c)$. Finally, defining $q_i(x, y, w) = p_i(y, \mathbf{z}_i(w, x, w))$ gives the desired polynomial identities of Theorem 4.1.

Combining Theorem 4.2 with Theorem 3.2 we have:

Corollary 4.3. *If the variety* **V** *has n-permutable congruences for some n,* $\mathbf{K} \subseteqq \mathbf{V}$, *and* $Q(\mathbf{K})$ *has definable principal congruences, then* $V(\mathbf{K})$ *has definable principal congruences.*

Corollary 4.4. (Burris and Lawrence) *If* **K** *is a class of groups or rings and*

$Q(\mathbf{K})$ has definable principal congruences, then $V(\mathbf{K})$ has definable principal congruences.

Problem 4. Can the condition that $\mathbf{K}$ is a variety in Theorems 4.1 or 4.2 be relaxed in some way?

Problem 5. Are there any results, analogous to Theorem 4.2, for switching functions s that are not constant? Observe that the argument using the free algebra can still be used to produce polynomial identities.

With regard to Problem 5, Peter Köhler has observed that the variety of distributive lattices has a restricted uniform congruence scheme with $n=4$ and $s(1)=s(3)=0$ and $s(2)=s(4)=1$, but by a result of Wille [29, p. 79], distributive lattices are not m-permutable for any integer m.

We conclude this section by exhibiting a few curiosities concerning 3-permutability. It is easily seen that if an algebra A has principal congruences that are 2-permutable (i.e. permutable), then all congruence relations of A are permutable. This is not the case for 3-permutability; witness e.g., sets have all principal congruences 3-permutable, but congruences in general for sets are not. Indeed, congruences on sets are not n-permutable for any n, since a constant switching function will not suffice for generating them.

If a class $\mathbf{K}$ has the property that $H(\mathbf{K})$ has 2-permutable congruences, then by Theorem 4.2 and by the remarks following the proof of Theorem 3.5, it follows that every algebra in $\mathbf{K}$ has property ($**$). A similar result is possible for 3-permutability as well. In [29, Satz 6.19] Wille proved that a variety $\mathbf{K}$ has the property that arbitrary homomorphisms preserve congruence relations iff $\mathbf{K}$ has 3-permutable congruences. We now present a local version of his result, via a similar proof, and thereby give another sufficient condition for ($**$).

Theorem 4.5. *Let A be an arbitrary algebra. A congruence relation Θ of A is preserved by a homomorphism h iff $\Theta\circ\ker(h)\circ\Theta\subseteq\ker(h)\circ\Theta\circ\ker(h)$.*

Proof. To show h preserves Θ, it suffices to show $h(\Theta)$ is transitive. So let $(h(w), h(x))$ and $(h(y), h(z))$ be in $h(\Theta)$, with $h(x)=h(y)$. Then there exist w', x', y', and z' in A such that $h(w')=h(w)$, $h(x')=h(x)$, etc. with (w', x') and (y', z') in Θ. Note $h(x')=h(y')$. By hypothesis, $\exists u, v\in A$ such that $h(w')=h(u), h(z')=h(v)$, and $(u, v)\in\Theta$. Hence $h(w)=h(u), h(z)=h(v)$, and $(h(u), h(v))\in h(\Theta)$. So $(h(w), h(z))\in h(\Theta)$ as desired.

Conversely, let $(w, x), (y, z)\in\Theta$ with $h(x)=h(y)$. Apply h to give $(h(w), h(x))\in h(\Theta)$, $(h(y), h(z))\in h(\Theta)$. So there exist $w', z'\in A$ such that $h(w')=h(w)$, $h(z')=h(z)$ and $(w', z')\in\Theta$. This gives $(w, z)\in\ker(h)\circ\Theta\circ\ker(h)$ as desired.

Note that the variety **K** of 1-unary algebras in which $f(x)=f(y)$ for all x and y has the property that any homomorphism preserves principal congruences, but not arbitrary congruence relations.

A specialized version of Theorem 4.5 gives the next result. The proof is an easy modification of the proof of 4.5.

Theorem 4.6. *Let* **K** *be a class of algebras. The following are equivalent;*

(i) *every homomorphism of an algebra in* **K** *preserves principal congruences;*

(ii) *principal congruences are 3-permutable for algebras in* **K**.

5. Definable principal congruences

The notion of definable principal congruences was introduced in [3] in an attempt to describe the behavior of subdirectly irreducible algebras in a variety. Interest in the concept has continued, not only for its own sake, but also as a crucial hypothesis for other theorems in universal algebra and equational logic. One direction of research has been to classify varieties by whether or not they have definable principal congruences. This was done in [3] and has continued with the previously cited work of Burris and Lawrence [12], [13] for groups and rings, and by Baker [1] for groups. Negative results have also been obtained by Burris [11] who exhibited a 4-element algebra that generates a variety without definable principal congruences (but which does have distributive congruence lattices); by McKenzie [24] who showed that every nondistributive variety of lattices fails to have definable principal congruences; and by Taylor [28] who showed that the variety of commutative semigroups satisfying the law $xy=uv$ (which is generated by a 3-element semigroup), does not have definable principal congruences. In [6] it is shown that every 2-element algebra generates a variety with definable principal congruences. A useful theorem of McKenzie [24] states that if a variety **V** of finite similarity type has definable principal congruences and if there is a finite bound on the cardinality of subdirectly irreducible members of **V**, then **V** has a finite basis for its polynomial identities. (Also Jónsson [20] has a similar result.) This was used, for example, in [30] to show that a certain variety of upper bound algebras is finitely axiomatizable. Several strengthened versions of definable principal congruence relations have been given in the literature; some of these were discussed in section 2. Recently Tulipani [34] has shown that if a variety has definable principal congruences, then for any n there is a first order formula for describing the join of n principal congruence relations.

One possible way to obtain a positive solution to Problem 1 would be to show that whenever an algebra A has definable principal congruences with defining formula φ, then every homomorphic image of this algebra also has its principal congruences defined by φ. We now present another example to show that this is not the case. Note that we cannot use Example 3.4 since the algebra in that example did not have definable principal congruences.

Example 5.1. There exists a groupoid A and a homomorphic image B of A such that A has definable principal congruences and B does not.

Let the groupoid A have universe $\{0, 1, 2, ...\}$ and define $\oplus$ on A by

$$\begin{aligned} 1\oplus x &= x\oplus 1 = x \quad \forall x, \\ 0\oplus 0 &= 0, \\ x\oplus y &= x \quad \forall x, y > 1, \\ 0\oplus x &= x\oplus 0 = x \quad \forall x = 3i\pm 1, \quad i > 0, \\ 0\oplus 3i &= 3i-1 \quad \forall i > 0, \\ 3i\oplus 0 &= 3i+1 \quad \forall i > 0. \end{aligned}$$

The principal congruence relations of A are listed below, where only nontrivial blocks are given and $x, y > 1$.

$$\begin{aligned} \Theta(0,1) &= /0, 1/2, 3, 4/5, 6, 7/... \\ \Theta(0, x) &= /0, 2, 3, 4, .../ \\ \Theta(1, x) &= /0; 1, 2, 3, .../ \\ \Theta(x, y) &= /x, y/ \quad x, y \not\equiv 0 \,(\mathrm{mod}\ 3) \\ \Theta(x, y) &= /x-1, x, x+1, y/ \quad x \equiv 0 \,(\mathrm{mod}\ 3), \quad y \not\equiv 0 \,(\mathrm{mod}\ 3) \\ \Theta(x, y) &= /x-1, y-1/x, y/x+1, y+1/ \quad x, y \equiv 0 \,(\mathrm{mod}\ 3). \end{aligned}$$

All of these principal congruences can be achieved in at most six steps using unary algebraic polynomials of the form $q(x) = z_4\oplus(z_2\oplus(x\oplus z_1)\oplus z_3)$ where the z_i are in A. Thus, A has definable principal congruences. Let Θ denote the congruence relation

$$\Theta = \bigvee_i \Theta(3i+1, 3i+2), \quad i = 1, 2, \dots.$$

Let h be any homomorphism with kernel Θ and let $B = h(A)$. Consider the principal congruence relation $\Theta(h(0), h(1))$ in B. $x \equiv y \ \Theta(0, 1)$ in the algebra A implies $h(x) \equiv h(y) \ \Theta(h(0), h(1))$. So the block of $\Theta(h(0), h(1))$ includes $\{h(3i-1), h(3i), h(3i+1)\}$, for all $i > 0$. Thus $\Theta(h(0), h(1)) = /h(0), h(1)/$ the rest of $B/$. But for any polynomial p,

$$\{p(h(0), h(\mathbf{z})), p(h(1), h(\mathbf{z}))\} \subseteqq h(\{p(0, \mathbf{z}'), p(1, \mathbf{z}')|\ \mathbf{z}' \equiv \mathbf{z}\ \Theta\}) \subseteqq$$

$$\subseteqq h(\{3(i-1)-1,\ 3(i-1), 3(i-1)+1, 3i-1, 3i, 3i+1)\})$$

for some i, and so an arbitrarily large number of steps are required.

With regard to Example 5.1, note that $SP(A)$ does not have definable principal congruences. For let C be the subalgebra of A^n generated by $\{\mathbf{3}, \mathbf{4}, \mathbf{6}$, and t^i $(1 \leqq i \leqq n)\}$, wehre $\mathbf{k}$ is the n-tuple of all k's, and t^i consists of all 8's, except for $t_i^i = 0$. We leave it to the reader to verify that $\mathbf{4} \equiv \mathbf{6}$ $\Theta(\mathbf{3}, \mathbf{6})$ in the algebra C, and that polynomials of arity $n+1$ are required.

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