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Idempotent distributive semirings. I

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0. Introduction

A *semiring* $(S, +, \cdot)$ is an algebra of type $\langle 2,2 \rangle$ where

(0.1) $$(S, +) \text{ is a semigroup}$$

(0.2) $$(S, \cdot) \text{ is a semigroup}$$

(0.3) $$(a+b)c = ac+bc \quad \text{and} \quad a(b+c) = ab+ac \quad \text{for all} \quad a, b, c \in S.$$

We shall always write ab instead of $a \cdot b$, and we suppose that $\cdot$ links stronger than $+$; with this assumption we can omit brackets.

The semiring $(S, +, \cdot)$ is *idempotent* if

(0.4) $$a+a = a = aa \quad \text{for all} \quad a \in S,$$

i.e. the reducts $(S, +)$ and $(S, \cdot)$ are bands. We say that the semiring $(S, +, \cdot)$ is *distributive* if

(0.5) $$ab+c = (a+c)(b+c) \quad \text{and} \quad a+bc = (a+b)(a+c) \quad \text{for all} \quad a, b, c \in S.$$

The purpose of this paper is to clarify the structure of idempotent distributive semirings which henceforth will be called ID-semirings.

We refer the reader to [6], [7] for a construction and a classification of bands. We shall assume that the reader is acquainted with the definition of the Płonka sum of a semilattice ordered system of algebras [8].

We shall now list diverse examples which supply the motivation for our investigations.

Result 1 ([1], Corollary 8). *Let $(I, \cdot)$ and $(\Lambda, \cdot)$ be any semigroups. On $I \times \Lambda = S$ we define an addition and a multiplication in the following way. For all $(i, \lambda), (j, \mu) \in S$, let*

(0.6) $$(i, \lambda)+(j, \mu) = (i, \mu), \quad \textit{and} \quad (i, \lambda)(j, \mu) = (ij, \lambda\mu).$$

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Then $(S, +, \cdot)$ is a semiring for which the additive reduct is a rectangular band; conversely, every semiring for which the additive reduct is a rectangular band is isomorphic to a semiring constructed in this fashion.

We give a direct proof for the sake of completeness.

Proof. Let S be a semiring for which the additive reduct is a rectangular band, and let us introduce the relations $\mathscr{R}$ and $\mathscr{L}$ on S in the following way:

$$a\mathscr{R}b \text{ if and only if } a+b=b \text{ and } b+a=a,$$

$$a\mathscr{L}b \text{ if and only if } a+b=a \text{ and } b+a=b.$$

It should be evident that $\mathscr{R}$ and $\mathscr{L}$ are equivalence relations and that $\mathscr{L}\cap\mathscr{R}$ is the equality. Let $S/\mathscr{L}=I$ and $S/\mathscr{R}=\Lambda$, and for any $a\in S$, let $i_a\in S/\mathscr{L}=I$ denote the $\mathscr{L}$-class of a, and $\lambda_a\in S/\mathscr{R}=\Lambda$ the $\mathscr{R}$-class containing a. By the foregoing the mapping $\varphi: S\to I\times\Lambda$, $a\to(i_a, \lambda_a)$ is an injective mapping. Let (i, λ) be any element of $I\times\Lambda$. Then $i=i_a$ and $\lambda=\lambda_b$ for some $a, b\in S$; since $(S, +)$ is a rectangular band it follows that $i_a=i_{b+a}$ and $\lambda_b=\lambda_{b+a}$. We conclude that

$$(i, \lambda) = (i_a, \lambda_b) = (i_{b+a}, \lambda_{b+a}) = (b+a)\varphi.$$

Thus the injection φ is in fact a bijection onto $I\times\Lambda$.

From (0.3) it follows that $\mathscr{L}$ and $\mathscr{R}$ are congruence relations. From the above we then have $S\cong S/\mathscr{L}\times S/\mathscr{R}=I\times\Lambda$ by φ.

We consider the semiring $I=S/\mathscr{L}$. Let i_a, i_b be any elements of I. Then

$$i_a+i_b = i_{a+b} = i_b$$

since $(a+b)+b=a+b$ and $b+(a+b)=b$. Hence the additive reduct of I is a right-zero semigroup. Analogously, the additive reduct of Λ is a left-zero semigroup. We conclude that every semiring for which the additive reduct is a rectangular band may be constructed as stated above.

The direct part is obvious.

Corollary 1. *Let $(S, +, \cdot)$ be a semiring for which the additive reduct is a rectangular band. Then $(S, +, \cdot)$ is distributive if and only if $(S, +, \cdot)$ is idempotent.*

Obviously the semiring $(S, +, \cdot)=(I\times\Lambda, +, \cdot)$ of Result 1 is an ID-semiring if and only if the semigroups $(I, \cdot)$ and $(\Lambda, \cdot)$ are bands.

We shall say that an ID-semiring is a *rectangular* [*normal*, *left-zero*, ...] *semiring* if and only if both the reducts are rectangular [normal, left-zero, ...] bands. A *nest* $(S, +, \cdot)$ is an algebra of type $\langle 2,2\rangle$ which satisfies

$$a+b = b \tag{0.7}$$

and

$$ab = a \tag{0.8}$$

for all $a, b \in S$ [2] [18]. It is easy to see that a nest is necessarily an ID-semiring where the additive reduct is a right-zero band and the multiplicative reduct a left-zero band. Using the notations of Result 1 we can state that any nest is of the form $(I \times \Lambda, +, \cdot)$ where $|I|=1$ and where $(\Lambda, \cdot)$ is a left-zero band. A *dual nest* $(S, +, \cdot)$ is an ID-semiring where

$$a+b = a \tag{0.9}$$

and

$$ab = b \tag{0.10}$$

holds for all $a, b \in S$. With the notations of Result 1 we have that a dual nest is of the form $(I \times \Lambda, +, \cdot)$ where $|\Lambda|=1$ and where $(I, \cdot)$ is a right-zero band. A left-zero semiring is of the form $(I \times \Lambda, +, \cdot)$ where $|\Lambda|=1$ and where $(I, \cdot)$ is a left-zero band, whereas a right-zero semiring is of the form $(I \times \Lambda, +, \cdot)$ where $|I|=1$ and where $(\Lambda, \cdot)$ is a right-zero band.

Corollary 2. *A semiring is rectangular if and only if it is the direct product of a left-zero semiring, a right-zero semiring, a nest and a dual nest.*

Proof. Let $(S, +, \cdot)$ be a rectangular semiring. Since the additive reduct is a rectangular band, $(S, +, \cdot)=(I \times \Lambda, +, \cdot)$ can be constructed as in Result 1, where $(I, \cdot)$ and $(\Lambda, \cdot)$ are rectangular bands. It follows that $(I, \cdot)$ is of the form $(I_1 \times I_2, \cdot)$, where

$$(i_1, i_2)(j_1, j_2) = (i_1, j_2)$$

for all $(i_1, i_2), (j_1, j_2) \in I_1 \times I_2$, whereas $(\Lambda, \cdot)$ is of the form $(\Lambda_1 \times \Lambda_2, \cdot)$, where

$$(\lambda_1, \lambda_2)(\mu_1, \mu_2) = (\lambda_1, \mu_2).$$

Therefore the rectangular semiring $(S, +, \cdot)$ must be isomorphic to $(I_1 \times I_2 \times \Lambda_1 \times \times \Lambda_2, +, \cdot)$, where

$$(i_1, i_2, \lambda_1, \lambda_2)+(j_1, j_2, \mu_1, \mu_2) = (i_1, i_2, \mu_1, \mu_2),$$

$$(i_1, i_2, \lambda_1, \lambda_2)(j_1, j_2, \mu_1, \mu_2) = (i_1, j_2, \lambda_1, \mu_2)$$

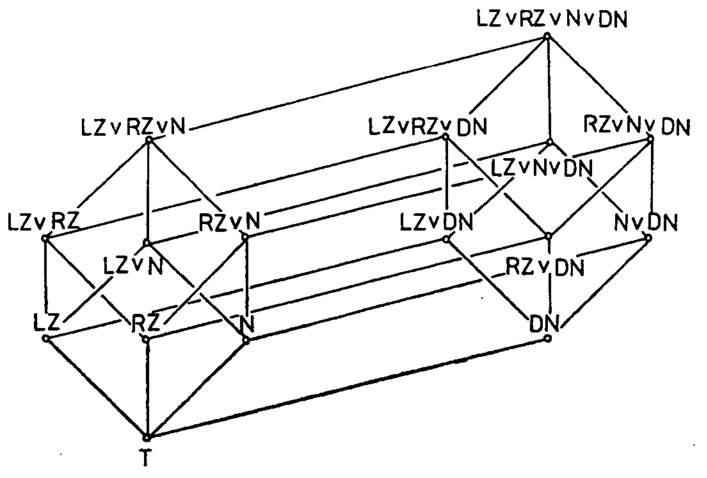
for all $i_1, j_1 \in I_1$, $i_2, j_2 \in I_2$, $\lambda_1, \mu_1 \in \Lambda_1$, and $\lambda_2, \mu_2 \in \Lambda_2$. Hence $(S, +, \cdot)$ is isomorphic to the direct product of the left-zero semiring I_1, the right-zero semiring Λ_2, the nest Λ_1 and the dual nest I_2. Conversely, any such direct product must yield a rectangulan semiring.

Let us consider the variety N of nests, DN of dual nests, LZ of left-zero semirings and RZ of right-zero semirings. Since all the equivalence relations of algebras in these classes are congruence relations, all subdirectly irreducible algebras contain exactly two elements. Hence the varieties *LZ, RZ, N* and *DN* are atoms in the lattice of varieties of rectangular semirings. It is easy to see that an arbitrary algebra in a join of some of the varieties *N, DN, LZ* and *RZ* is a direct product of algebras in the component varieties. This follows from [11] Theorem 1 (the suitable polynomials are just xy or $x+y$). Hence any subdirectly irreducible algebra in a join of some of the varieties *N, DN LZ* and *RZ* belongs to one of the component varieties. These remarks imply the following description of the lattice of varieties of rectangular semirings.

Corollary 3. *There are exactly* 16 *varieties of rectangular semirings. They form a Boolean lattice.*

The variety $LZ\vee RZ\vee N\vee DN$ *is equal to the variety of rectangular semirings. Each one of the "join" subvarieties of the variety of rectangular semirings is defined by the identities for rectangular semirings and by one additional identity:*

$LZ\vee RZ$	$x+y = xy$
$LZ\vee N$	$xy = x$
$LZ\vee DN$	$x+y = x$
$RZ\vee N$	$x+y = y$
$RZ\vee DN$	$xy = y$
$N\vee DN$	$x+y = yx$
$LZ\vee RZ\vee N$	$xy+x = x$
$LZ\vee RZ\vee DN$	$x+yx = x$
$LZ\vee N\vee DN$	$x+xy = x$
$RZ\vee N\vee DN$	$yx+x = x$



Proof. The first part of the corollary is clear from the previous remarks. The second part can be easily proved using standard methods. Let us show for example that the variety $LZ \vee RZ$ is defined by the identities for rectangular semirings and the identity $x+y=xy$. If the ID-semiring $(S, +, \cdot)$ belongs to the variety $LZ \vee RZ$, it is a product of a left-zero semiring and a right-zero semiring. So it is easy to see that the identity $x+y=xy$ is satified. Conversely, if a rectangular semiring $(S, +, \cdot)$ satisfies this identity, then by Corollary 2, we conclude that it is in fact isomorphic to the direct product of a left-zero semiring and a right-zero semiring.

Let us again consider the semiring $(S, +, \cdot)=(I\times\Lambda, +, \cdot)$ of Result 1. Then $(S, +)$ is a left-zero [right-zero] band if and only if $|\Lambda|=1$ $[|I|=1]$, and if this is the case then $(S, +, \cdot)$ may be identified with $(I, +, \cdot)$ $[(\Lambda, +, \cdot)]$. It follows that any semigroup may be represented as the multiplicative reduct of a semiring. In particular, any band may be represented as the multiplicative reduct of an ID-semiring. This fact has been observed by many authors (see e.g. [20] Example 2.3.4).

A semiring $(S, +, \cdot)$ is called a *mono-semiring* if

$$a+b = ab \tag{0.11}$$

holds for all $a, b \in S$, that is, if the two operations $+$ and $\cdot$ coincide. If $(S, +, \cdot)$ is an idempotent semiring which is also a mono-semiring, then $(S, +)$ and $(S, \cdot)$ must be normal bands, and $(S, +, \cdot)$ needs to be distributive. Conversely, any normal band is the additive (or multiplicative) reduct of an ID-mono-semiring ([20], Theorem 4.4.2). Left-zero semirings and right-zero semirings need to be mono-semirings. A rectangular mono-semiring is the direct product of a left-zero semiring and a right-zero semiring. An ID-mono-semiring is the Płonka sum (in the sense of [8]) of a semilattice ordered system of rectangular mono-semirings [19].

An idempotent semiring $(S, +, \cdot)$ where the reducts $(S, +)$ and $(S, \cdot)$ are semilattices has been called a $\cdot$*-distributive bisemilattice* in [4][12][13][14]. If moreover $(S, +, \cdot)$ is distributive, then $(S, +, \cdot)$ is called a *distributive bisemilattice* [15] (*distributive quasilattices* in [9]). Here the distributive lattices form the main example. Another particular case consists of the mono-semirings where the two reducts are semilattices: they will be called *mono-bisemilattices*. We remark that mono-bisemilattices could as well be identified with semilattices. A distributive bisemilattice is the Płonka sum of a semilattice ordered system of distributive lattices [9].

1. Normal semirings

In this section we investigate the structure of normal semirings, i.e. semirings satisfying the identities

(1.1) $$xyzw = xzyw,$$

(1.2) $$x+y+z+w = x+z+y+w.$$

We shall see later (Theorem 2.3) that any ID-semiring can "in principle" be constructed from normal semirings.

First, let $(S, +, \cdot)$ be any ID-semiring. On S we introduce the equivalence relations $\overset{+}{\mathscr{L}}$, $\overset{+}{\mathscr{R}}$, $\dot{\mathscr{L}}$, $\dot{\mathscr{R}}$ which are given by

(1.3) $a\overset{+}{\mathscr{L}}b$ if and only if $a+b=a$ and $b+a=b$

(1.4) $a\overset{+}{\mathscr{R}}b$ if and only if $a+b=b$ and $b+a=a$

(1.5) $a\dot{\mathscr{L}}b$ if and only if $ab=a$ and $ba=b$

(1.6) $a\dot{\mathscr{R}}b$ if and only if $ab=b$ and $ba=a.$

Obviously $\overset{+}{\mathscr{L}}, \overset{+}{\mathscr{R}}, \overset{+}{\mathscr{L}}\circ\overset{+}{\mathscr{R}}=\overset{+}{\mathscr{R}}\circ\overset{+}{\mathscr{L}}=\overset{+}{\mathscr{D}}$ and $\dot{\mathscr{L}}, \dot{\mathscr{R}}, \dot{\mathscr{L}}\circ\dot{\mathscr{R}}=\dot{\mathscr{R}}\circ\dot{\mathscr{L}}=\dot{\mathscr{D}}$ are the usual relations of Green for the bands $(S, +)$ and $(S, \cdot)$ respectively [7]. It is easy to see that $\dot{\mathscr{L}}$ and $\dot{\mathscr{R}}$ are congruence relations on the reduct $(S, +)$ [1]. Hence $\dot{\mathscr{D}}$ is a congruence relation on the reduct $(S, +)$. On the other hand $\dot{\mathscr{D}}$ is the least congruence on $(S, \cdot)$ for which $(S/\dot{\mathscr{D}}, \cdot)$ is a semilattice. Thus, $\dot{\mathscr{D}}$ is a congruence relation on the ID-semiring $(S, +, \cdot)$ and it is in fact the least congruence on the semiring $(S, +, \cdot)$ for which the quotient has a multiplicative reduct which is a semilattice. By interchanging the role of $+$ and $\cdot$ we can state a similar result for $\overset{+}{\mathscr{D}}$. Consequently $\dot{\mathscr{D}}\vee\overset{+}{\mathscr{D}}$ is the least congruence on the ID-semiring $(S, +, \cdot)$ for which the quotient is a distributive bisemilattice.

Theorem 1. *Every normal semiring* $(S, +, \cdot)$ *is a subdirect product of* (i) *a normal semiring* S_1 *which has a left normal additive reduct and a right normal multiplicative reduct*, (ii) *a left normal semiring* S_2, (iii) *a normal semiring* S_3 *which has a right normal additive reduct and a left normal multiplicative reduct and* (iv) *a right normal semiring* S_4, *where* S, S_1, S_2, S_3 *and* S_4 *have the same greatest bisemilattice homomorphic image.*

Proof. Since $(S, +)$ is a normal band, $\overset{+}{\mathscr{R}}$ and $\overset{+}{\mathscr{L}}$ are congruences on $(S, +)$ [6][19]. Therefore $\overset{+}{\mathscr{R}}$ and $\overset{+}{\mathscr{L}}$ are congruences on S, and since $\overset{+}{\mathscr{R}} \cap \overset{+}{\mathscr{L}}$ is the equality, we have that S is a subdirect product of the normal semirings $V = S/\overset{+}{\mathscr{R}}$ and $W = S/\overset{+}{\mathscr{L}}$. Since $\overset{+}{\mathscr{R}} \subseteq \overset{+}{\mathscr{D}} \vee \dot{\mathscr{D}}$ and $\overset{+}{\mathscr{L}} \subseteq \overset{+}{\mathscr{D}} \vee \dot{\mathscr{D}}$, it follows that S, V and W have the same greatest bisemilattice homomorphic image. Let $\overset{+}{\mathscr{L}}_V, \overset{+}{\mathscr{R}}_V, \overset{+}{\mathscr{D}}_V$, $\dot{\mathscr{L}}_V$, $\dot{\mathscr{R}}_V$, $\dot{\mathscr{D}}_V$ and $\overset{+}{\mathscr{L}}_W, \overset{+}{\mathscr{R}}_W, \overset{+}{\mathscr{D}}_W, \dot{\mathscr{L}}_W, \dot{\mathscr{R}}_W, \dot{\mathscr{D}}_W$ be the corresponding relations of Green for the normal semirings V and W respectively. Since $\overset{+}{\mathscr{L}}_V = \overset{+}{\mathscr{D}}_V$, and $\overset{+}{\mathscr{R}}_V$ is trivial, we know that the additive reduct of V is a left normal band. Dually, the additive reduct of W is a right normal band. Again, $\dot{\mathscr{L}}_V$ and $\dot{\mathscr{R}}_V$ are congruences on V. Putting $S_1 = V/\dot{\mathscr{L}}_V$ and $S_2 = V/\dot{\mathscr{R}}_V$, we have that V is a subdirect product of S_1 and S_2, where V, S_1 and S_2 have the same greatest bisemilattice homomorphic image, and where S_1 and S_2 are as stated in the theorem. On the other hand, W is a subdirect product of $S_3 = W/\dot{\mathscr{R}}_W$ and $S_4 = W/\dot{\mathscr{L}}_W$, where W, S_3 and S_4 have the same greatest bisemilattice homomorphic image, and where S_3 and S_4 are as stated in the theorem.

Theorem 2. *An ID-semiring $(S, +, \cdot)$ is left [right] normal if and only if S divides the direct product of* (i) *a left-zero [right-zero] semiring T_1,* (ii) *the greatest bisemilattice homomorphic image T_2 of S,* (iii) *a normal semiring T_3 for which the additive reduct is a left-zero [right-zero] band and the multiplicative reduct a semilattice, and* (iv) *a normal semiring T_4 for which the additive reduct is a semilattice and the multiplicative reduct a left-zero [right-zero] band.*

Proof. Let us suppose that S is a left normal semiring.

1. Let $(V, +, \cdot)$ be the semiring where $V = S$, such that the additive reduct $(V, +)$ coincides with the additive reduct $(S, +)$, and such that for all $a, b \in S = V$, $ab = a$ in $(V, \cdot)$. Clearly V is a normal semiring for which the additive reduct is a left normal band and the multiplicative reduct a left-zero band.

$\dot{\mathscr{L}} = \dot{\mathscr{D}}$ is the least congruence relation on S for which the multiplicative reduct of the quotient is a semilattice. Hence $W = S/\dot{\mathscr{L}}$ is a normal semiring for which the additive reduct is a left normal band and the multiplicative reduct a semilattice, and the greatest bisemilattice homomorphic image of W is exactly the same as the greatest bisemilattice homomorphic image of S. For every $x \in S$, let $\dot{L}_x$ denote the $\dot{\mathscr{L}}$-class containing x. From [6][19] it follows that for any $a, x \in S$, we have $\dot{L}_x \leqq \dot{L}_a$ in $(W, \cdot)$, that is $\dot{L}_x = \dot{L}_x \dot{L}_a = \dot{L}_a \dot{L}_x$, if and only if $x = xa$ in $(S, \cdot)$ or, if and only if $x \dot{\mathscr{L}} ax$. In this case ax is the only element of $\dot{L}_x$ which commutes with a.

Let us consider the semiring $V \times W$, and the subset R of $V \times W$ which is given by $R = \{(a, \dot{L}_x) | \dot{L}_x \leqq \dot{L}_a \text{ in } S/\dot{\mathscr{L}}\}$. Let $(a, \dot{L}_x)$ and $(b, \dot{L}_y)$ be any elements of R.

Since $\dot{L}_x\dot{L}_y\leqq\dot{L}_x\leqq\dot{L}_a$, we have $(a,\dot{L}_x)(b,\dot{L}_y)=(a,\dot{L}_x\dot{L}_y)=(a,\dot{L}_{xy})\in R$. From $xa=x$ and $yb=y$, we have $(x+y)(a+y)=xa+y=x+y$ and $(a+y)(a+b)=a+yb=a+y$, thus $\dot{L}_{x+y}\leqq\dot{L}_{a+y}\leqq\dot{L}_{a+b}$, and so $(a,\dot{L}_x)+(b,\dot{L}_y)=(a+b,\dot{L}_x+\dot{L}_y)=(a+b,\dot{L}_{x+y})\in R$. We conclude that R is a subsemiring of $V\times W$.

We now introduce the mapping $\varphi\colon R\to S$, $(a,\dot{L}_x)\to ax$. This mapping is well-defined, since ax is the unique element of $\dot{L}_x$ which commutes with a. Further, φ is surjective, since $(x,\dot{L}_x)\in R$ for all $x\in S$, and $(x,\dot{L}_x)\varphi=x$. Again, let $(a,\dot{L}_x)$ and $(b,\dot{L}_y)$ be any elements of R. Since $ax\in\dot{L}_x$ and $by\in\dot{L}_y$, we have $ax+by\in\in\dot{L}_x+\dot{L}_y=\dot{L}_{x+y}$. Further, $ax+by=a(ax)+by=(a+by)(ax+by)=(a+b(by))\cdot\cdot(ax+by)=(a+b)(a+by)(ax+by)$ from which it follows that $ax+by$ is the unique element of $\dot{L}_{x+y}$ which commutes with $a+b$. Thus $(a+b,\dot{L}_{x+y})\varphi=ax+by$, and so $(a,\dot{L}_x)\varphi+(b,\dot{L}_y)\varphi=ax+by=(a+b,\dot{L}_{x+y})\varphi=((a,\dot{L}_x)+(b,\dot{L}_y))\varphi$. Since $yb=y$, we have by the left normality of $(S,\cdot)$ that $(a,\dot{L}_x)\varphi(b,\dot{L}_y)\varphi=axby=axyb=axy==(a,\dot{L}_{xy})\varphi=((a,\dot{L}_x)(b,\dot{L}_y))\varphi$. We conclude that φ is a homomorphism of R onto S. Hence, S divides the direct product of V and W, where the greatest bisemilattice homomorphic image of V is trivial, and where S and W have the same greatest bisemilattice homomorphic image.

2. By interchanging the role of $+$ and $\cdot$, and using the same method as in 1, we can now show that V divides the direct product of a left-zero semiring T_1 and a normal semiring T_4 for which the additive reduct is a semilattice and the multiplicative reduct a left-zero band. Here T_1 is the left-zero semiring with carrier $S=V$, and T_4 is the normal semiring for which the additive reduct is the semilattice $(V/\overset{+}{\mathscr{L}}_V,+)=(S/\overset{+}{\mathscr{L}},+)$ and the multiplicative reduct the left-zero band on the set $V/\overset{+}{\mathscr{L}}_V=S/\overset{+}{\mathscr{L}}$. Similarly, W divides the direct product of a normal semiring T_3 for which the additive reduct is a left-zero band and the multiplicative reduct a semilattice, and a normal semiring T_2 which is actually a distributive bisemilattice. Here the additive reduct of T_3 is the left-zero band on the set $S/\dot{\mathscr{L}}$, whereas the multiplicative reduct of T_3 coincides with the semilattice $(S/\dot{\mathscr{L}},\cdot)$; on the other hand $T_2=(W/\overset{+}{\mathscr{L}}_W)\cong\cong S/(\dot{\mathscr{L}}\vee\overset{+}{\mathscr{L}})$, where $S/(\dot{\mathscr{L}}\vee\overset{+}{\mathscr{L}})$ is the greatest bisemilattice homomorphic image of S. We conclude that S divides the direct product of the semirings T_1, T_2, T_3 and T_4.

3. The semirings T_1, T_2, T_3, T_4 mentioned in the theorem are all left normal semirings. Therefore every semiring which divides their direct product must also be left normal.

In a similar fashion we can prove the following theorem.

Theorem 3. *An ID-semiring* $(S,+,\cdot)$ *has a right* [*left*] *normal additive reduct and a left* [*right*] *normal multiplicative reduct if and only if S divides the direct product of* (i) *a nest* [*dual nest*] T_1, (ii) *the greatest bisemilattice homomorphic image* T_2 *of*

S, (iii) *a normal semiring* T_3 *for which the additive reduct is a right-zero [left-zero] band and the multiplicative reduct a semilattice, and* (iv) *a normal semiring* T_4 *for which the additive reduct is a semilattice and the multiplicative reduct a left-zero [right-zero] band.*

By Result 0.1, Corollary 0.2, Theorems 1.1, 1.2 and 1.3, we can now conclude to the following division theorem for normal semirings in general.

Theorem 4. *An ID-semiring* $(S, +, \cdot)$ *is normal if and only if* S *divides the direct product of* (i) *a rectangular semiring* T_1, (ii) *the greatest bisemilattice homomorphic image* T_2 *of* S, (iii) *a normal semiring* T_3 *for which the additive reduct is a rectangular band and the multiplicative reduct a semilattice, and* (iv) *a normal semiring* T_4 *for which the additive reduct is a semilattice and the multiplicative reduct a rectangular band.*

It should be remarked that the components T_1, T_2, T_3 and T_4 can be constructed in terms of sets, semilattices and distributive lattices (Result 0.1, Corollary 0.2, [9]).

Lemma 5. *The greatest bisemilattice homomorphic image of a normal semiring* $(S, +, \cdot)$ *is a distributive lattice if and only if* S *satisfies the identity*

$$x(x+y+x)x = x. \tag{1.7}$$

Proof. A distributive bisemilattice which satisfies (1.7) must be a distributive lattice [9]. Therefore the greatest bisemilattice homomorphic image of a normal semiring satisfying (1.7) must be a distributive lattice. Let us conversely suppose that the greatest bisemilattice homomorphic image of the normal semiring $(S, +, \cdot)$ is a distributive lattice. By the foregoing theorem we know that S divides the direct product of T_1, T_2, T_3 and T_4, where T_1, T_3 and T_4 are as stated in Theorem 4, and where T_2 is a distributive lattice. One easily checks that T_1, T_2, T_3 and T_4 satisfy (1.7). Thus S satisfies (1.7).

Theorem 6. *An ID-semiring* $(S, +, \cdot)$ *is a normal semiring if and only if* S *is the Płonka sum of a semilattice ordered system of normal semirings that satisfy the generalized absorption law* (1.7).

Proof. Let E be the set of all equations which hold for all normal semirings for which the greatest bisemilattice homomorphic image is a distributive lattice. Let $R(E)$ denote the set of equations which are consequences of E and which are regular. Let K_E and $K_{R(E)}$ denote the equational classes defined by E and $R(E)$, respectively. Since (1.7) belongs to E we have from Theorem 1 of [10] that $K_{R(E)}$ consists of those algebras which are the Płonka sum of a semilattice ordered system of algebras which belong to K_E. Evidently the algebras which belong to K_E are exactly the normal semirings satisfying (1.7), and since the equations (0.1), (0.2), (0.3), (0.4), (0.5), (1.1) and (1.2) are all regular, it is obvious that $K_{R(E)}$ consists of normal semirings. The normal semirings T_1, T_3 and T_4 which appear in Theorem 4 all belong to K_E since their greatest bisemilattice homomorphic image is trivial. It follows from [9] that distributive bisemilattices belong to $K_{R(E)}$. Thus the normal semirings T_1, T_2, T_3 and T_4 of Theorem 4 all belong to $K_{R(E)}$. Since $K_{R(E)}$ is an equational class it follows from Theorem 4 that every normal semiring belongs to $K_{R(E)}$.

Remark. If we restrict ourselves to mono-semirings, then Theorem 6 is equivalent with the result of [19] which states that every normal band is the Płonka sum of a semilattice ordered system of rectangular bands. Our Theorem 6 also generalizes Płonka's decomposition of distributive bisemilattices [9]; this latter generalization goes in another direction than Padmanabhan's generalization of Płonka's result: we keep distributivity but abandon commutativity, Padmanabhan keeps commutativity but abandons distributivity [5].

It will be manifest for the reader that we actually used [19] and [9] in our proofs towards Theorem 6. We could have given a direct proof by showing that for every normal semiring $(S, +, \cdot)$

$$f\colon S^2 \to S, \quad (x, y) \to x(x+y+x)x$$

is a partition function of $(S, +, \cdot)$ (in the sense of [8]). Our procedure via Result 0.1, Corollary 0.2, Theorems 1 to 4, has the advantage that it gives insight in part of the lattice of subvarieties of the variety of normal semirings.

In the next section we generalize Theorem 6 for arbitrary ID-semirings.

2. A decomposition of ID-semirings

Let $(S, +, \cdot)$ be any ID-semiring. Then $S/\overset{+}{\mathscr{D}}$ and $S/\dot{\mathscr{D}}$ are ID-semirings where one of the reducts is a semilattice. Our first result here states that $S/\overset{+}{\mathscr{D}}$ and $S/\dot{\mathscr{D}}$ must be normal semirings.

Theorem 1. *Let $(S, +, \cdot)$ be an ID-semiring where $(S, +)$ is a semilattice. Then $(S, \cdot)$ is a normal band.*

Proof. Clearly $\dot{\mathscr{D}}$ is the least congruence on $(S, +, \cdot)$ for which the quotient is a distributive bisemilattice. For any $a \in S$ we denote the $\dot{\mathscr{D}}$-class containing a by $\bar{a}$. The distributive bisemilattice $T = S/\dot{\mathscr{D}}$ is the Płonka sum of a semilattice ordered system of distributive lattices

$$\langle Y, \langle T_\alpha \rangle_{\alpha \in Y}, \langle \varphi_{\beta,\alpha} \rangle_{\alpha \leqq \beta;\, \alpha, \beta \in Y} \rangle$$

[9].

Let a and b any elements of S, such that $ab=ba=b$. Clearly $\bar{a}\bar{b}=\bar{b}\bar{a}=\bar{b}$ in the distributive bisemilattice T, and so $\bar{a}\in T_\alpha, \bar{b}\in T_\beta$ for some $\alpha, \beta\in Y$, with $\beta\leqq\alpha$. We distinguish two cases: 1. $\alpha=\beta$, and 2. $\alpha>\beta$.

1. If $\alpha=\beta$, then $\bar{a}$ and $\bar{b}$ belong to the same distributive lattice T_α. Let x be any element of $\bar{b}$ which satisfies $xa=ax=x$. Then $x+a=a+x\in\bar{a}+\bar{b}=\bar{a}$. One readily checks that also $bx+b\in\bar{b}$, and $(bx+b)a=a(bx+b)=bx+b$, and so $bx+b+a=a$. Consequently $b=ba=b(bx+b+a)=bx+b$ and dually $b=xb+b$. If we interchange the role of x and b we also have $x=xb+x=bx+x$. Further, $xb+bx=(xb+bx)^2=(xb)^2+(xb)(bx)+(bx)(xb)+(bx)^2=xb+x+b+bx=(b+x)^2=$ $=b+x$ and so $b=b+xb+bx=xb+bx=xb+bx+x=x$. We conclude that b is the only element of its $\dot{\mathscr{D}}$-class which is a solution of $xa=ax=x$.

2. Let us now suppose that $\alpha>\beta$. From $ab=b$ it follows that $(\bar{a}\varphi_{\alpha,\beta})\bar{b}=\bar{a}\bar{b}=\bar{b}$, where $\bar{a}\varphi_{\alpha,\beta}, \bar{b}\in T_\beta$. Since T_β is a distributive lattice, we also have $\bar{a}+\bar{b}=\bar{a}\varphi_{\alpha,\beta}+\bar{b}=$ $=\bar{a}\varphi_{\alpha,\beta}$, and consequently $a+b\in\bar{a}\varphi_{\alpha,\beta}$. One can see that $(a+b)a=a(a+b)=a+b$. Let y be any element of $\bar{a}\varphi_{\alpha,\beta}$ which satisfies $ya=ay=y$. Clearly $y+a=a+y\in\bar{a}\varphi_{\alpha,\beta}$, and since $\bar{a}\varphi_{\alpha,\beta}$ forms a multiplicative rectangular band, we have $a+y=y+a=$ $=(a+y)y(a+y)=(a+y)(ya+y^2)=(a+y)y=ay+y^2=y$. Similarly $y(a+b)$, $(a+b)y\in$ $\in\bar{a}\varphi_{\alpha,\beta}$ and $a+y(a+b)=y(a+b)$, $a+(a+b)y=(a+b)y$.

Therefore,

$$\begin{aligned} a+b &= (a+b)y(a+b) = (a+b)y\big((a+b)+a\big) = (a+b)y(a+b)+(a+b)y = \\ &= (a+b)+(a+b)y = (a+b)(a+y) = (a+b)y = \big(a+(a+b)\big)y = \\ &= y+(a+b)y = y(a+b)y+(a+b)y = (y+a)(a+b)y = y(a+b)y = y, \end{aligned}$$

and we conclude that $a+b$ is the only element of the $\dot{\mathscr{D}}$-class $\bar{a}\varphi_{\alpha,\beta}$ which is a solution of $ya=ay=y$. Let x be any element of $\bar{b}$ which satisfies $xa=ax=x$. Again $a+x\in\bar{a}\varphi_{\alpha,\beta}$ and $a(a+x)=(a+x)a=a+x$, and so by the foregoing we have $a+x=$ $=a+b$. Furthermore $b=(a+b)b=b(a+b)$, $x=(a+x)x=(a+b)x=x(a+x)=$ $=x(a+b)$, where $x, b\in\bar{b}$, and where $\bar{b}$ and $\overline{a+b}$ belong to the same distributive lattice T_β. By 1. we conclude that $b=x$. Therefore, b is the only element of its $\dot{\mathscr{D}}$-class which is a solution of $ax=xa=x$.

From 1. and 2. we conclude that $\bar{a}\bar{b}=\bar{b}$ in $S/\dot{\mathscr{D}}$ implies that with every $a\in\bar{a}$ there corresponds a unique $b\in\bar{b}$ which is a solution of $ax=xa=x$. Therefore $(S, \cdot)$ is a normal band [6].

Theorem 2. *For an ID-semiring $(S, +, \cdot)$ the following are equivalent:*

(i) $\dot{\mathscr{D}}\cap\overset{+}{\mathscr{D}}$ *is the equality,*

(ii) $(S, +, \cdot)$ *satisfies*

$$xy+yx=yx+xy, \tag{2.1}$$

(iii) $(S, +, \cdot)$ *is a subdirect product of ID-semirings for which one of the reducts is a semilattice,*

(iv) $(S, +, \cdot)$ *divides the direct product of* (1) *a distributive bisemilattice,* (2) *a normal semiring for which the additive reduct is a rectangular band and the multiplicative reduct a semilattice, and* (3) *a normal semiring for which the additive reduct is a semilattice and the multiplicative reduct a rectangular band.*

Proof. Let $(S, +, \cdot)$ be an ID-semiring which satisfies (i). Then $(S, +, \cdot)$ is a subdirect product of $S/\dot{\mathscr{D}}$ and $S/\overset{+}{\mathscr{D}}$, where the multiplicative reduct of $S/\dot{\mathscr{D}}$ is a semilattice, and the additive reduct of $S/\overset{+}{\mathscr{D}}$ is a semilattice. Thus (i) implies (iii). It follows from Theorems 1.1 to 1.4 that (iii) implies (iv). Let us now suppose that $(S, +, \cdot)$ satisfies (iv). Clearly the normal semirings listed in (iv) all satisfy (2.1), and so $(S, +, \cdot)$ satisfies (2.1): consequently (iv) implies (ii). Let us suppose that $(S, +, \cdot)$ satisfies (2.1). We know that the $\dot{\mathscr{D}}\cap\overset{+}{\mathscr{D}}$-classes are rectangular. On the other hand, the $\dot{\mathscr{D}}\cap\overset{+}{\mathscr{D}}$-classes form subalgebras which satisfy (2.1). It is an easy matter to check that a left-zero semiring, a right-zero semiring, a nest or a dual nest can only satisfy (2.1) if they are trivial. From Corollary 0.2 we conclude that the $\dot{\mathscr{D}}\cap\overset{+}{\mathscr{D}}$-classes of $(S, +, \cdot)$ must be trivial. Thus (ii) implies (i).

Let K be a fixed class of algebras of finite type and K_1, K_2 subclasses of K. The *product* $K_1 \circ K_2$ is the class of all algebras A from K on each of which one can find a congruence θ such that $A/\theta \in K_2$ and such that all θ-classes form subalgebras which belong to K_1 [3].

Theorem 3. *The variety of ID-semirings is the product of the variety of rectangular semirings and the variety of ID-semirings which satisfy the equivalent conditions of Theorem* 2.

Proof. The canonical homomorphism $\theta\colon S \to S/\dot{\mathscr{D}} \times S/\overset{+}{\mathscr{D}}$ induces the congruence relation $\dot{\mathscr{D}}\cap\overset{+}{\mathscr{D}}$ on S. Clearly $S/\dot{\mathscr{D}}\cap\overset{+}{\mathscr{D}} \cong S\theta$ satisfies the equivalent conditions (i) to (iv) of Theorem 2, and the $(\dot{\mathscr{D}}\cap\overset{+}{\mathscr{D}})$-classes form rectangular semirings.

Remark that we can construct rectangular semirings and the semirings which appear in (iv) of Theorem 2 in terms of sets, semilattices and distributive lattices by our results of Section 0. We now give a decomposition for arbitrary ID-semirings; we use Theorem 1.6, and we shall generalize it.

Lemma 4. *The greatest bisemilattice homomorphic image of an ID-semiring* $(S, +, \cdot)$ *is a distributive lattice if and only if S satisfies the generalized absorption law* (1.7).

Proof. It should be clear that the greatest bisemilattice homomorphic image of an ID-semiring which satisfies (1.7) must be a distributive lattice. Let us now suppose that the greatest bisemilattice homomorphic image of the ID-semiring $(S, +, \cdot)$ is a distributive lattice $S/\dot{\mathcal{D}}\vee\overset{+}{\mathcal{D}}$. Since $\dot{\mathcal{D}}\cap\overset{+}{\mathcal{D}}\subseteq\dot{\mathcal{D}}\vee\overset{+}{\mathcal{D}}$, we have that $\dot{\mathcal{D}}\vee\overset{+}{\mathcal{D}}/\dot{\mathcal{D}}\cap\overset{+}{\mathcal{D}}$ is the least congruence on $S/\dot{\mathcal{D}}\cap\overset{+}{\mathcal{D}}$ for which the quotient is a bisemilattice, and $(S/\dot{\mathcal{D}}\cap\overset{+}{\mathcal{D}})/(\dot{\mathcal{D}}\vee\overset{+}{\mathcal{D}}/\dot{\mathcal{D}}\cap\overset{+}{\mathcal{D}})\cong S/\dot{\mathcal{D}}\vee\overset{+}{\mathcal{D}}$ is a distributive lattice. By Theorem 1 and Theorem 3 we know that $S/\dot{\mathcal{D}}\cap\overset{+}{\mathcal{D}}$ is a normal semiring, and so by Lemma 1.5 we may conclude that the normal semiring $S/\dot{\mathcal{D}}\cap\overset{+}{\mathcal{D}}$ satisfies (1.7). Let a and b be any elements of S. We denote the $\dot{\mathcal{D}}\cap\overset{+}{\mathcal{D}}$-class containing a and b by $\bar{a}$ and $\bar{b}$ respectively. Then $a(a+b+a)a\in\bar{a}(\bar{a}+\bar{b}+\bar{a})\bar{a}=\bar{a}$ since $S/\dot{\mathcal{D}}\cap\overset{+}{\mathcal{D}}$ satisfies (1.7). Then a, $a(a+b+a)a\in\bar{a}$, and since the multiplicative reduct of $\bar{a}$ is rectangular we have $a(a+b+a)a=a$. Thus $(S, +, \cdot)$ satisfies (1.7).

Theorem 5. *Every ID-semiring is the Płonka sum of a semilattice ordered system of ID-semirings which satisfy the generalized absorption law* (1.7).

Proof. Let $(S, +, \cdot)$ be any ID-semiring, and let us consider the normal semiring $S/\dot{\mathcal{D}}\cap\overset{+}{\mathcal{D}}=\bar{S}$. For any $a\in S$, $\bar{a}$ will denote the $\dot{\mathcal{D}}\cap\overset{+}{\mathcal{D}}$-class containing a. It follows from Theorem 1.6 that $\bar{S}$ is the Płonka sum of a semilattice ordered system of normal semirings satisfying (1.7)

$$\langle Y, \langle \bar{S}_\alpha\rangle_{\alpha\in Y}, \langle\varphi_{\alpha,\beta}\rangle_{\beta\leqq\alpha;\,\alpha,\beta\in Y}\rangle. \tag{2.2}$$

For any $\alpha\in Y$, let $S_\alpha=\{a|\bar{a}\in\bar{S}_\alpha\}$.

Let $a\in S_\alpha$, and let $\beta\leqq\alpha$ in Y. Let x be any element of $\bar{a}\varphi_{\alpha,\beta}$, and consider $b=axa$. Then $axa\in\bar{a}\bar{x}\bar{a}=\bar{a}\varphi_{\alpha,\beta}$, and so b is an element of $\bar{a}\varphi_{\alpha,\beta}$ which satisfies $ab=ba=b$. Then $a+b\in\bar{a}+\bar{b}=\bar{a}\varphi_{\alpha,\beta}$, and so $a+b=(a+b)b(a+b)=(a+b)(ba+b^2)=$ $=(a+b)b=ab+b^2=b$, and similarly $b+a=b$. By symmetry we may conclude that an element b of $\bar{a}\varphi_{\alpha,\beta}$ is a solution of $ay=ya=y$ if and only if b is a solution of $a+y=y+a=y$. Let b and b' be any elements of $\bar{a}\varphi_{\alpha,\beta}$ such that $ab=ba=b$ and $ab'=b'a=b'$. Since $\bar{a}\varphi_{\alpha,\beta}$ forms a rectangular semiring, we have from the foregoing

$$b=(b+b')+b=(b+b')+aba=(b+b'+a)(b+b'+b)(b+b'+a)=$$
$$=(b+b')b(b+b')=b+b'=(b+b')b'(b+b')=$$
$$=(a+b+b')(b'+b+b')(a+b+b')=ab'a+(b+b')=b'+b+b'=b'.$$

Hence $a+y=y+a=y$ has a unique solution in $\bar{a}\varphi_{\alpha,\beta}$, and this solution coincides with the unique solution of $ay=ya=y$ in $\bar{a}\varphi_{\alpha,\beta}$. We define the mapping $\psi_{\alpha,\beta}$: $S_\alpha\to S_\beta$ by the requirement that for every $a\in S_\alpha$, $a\psi_{\alpha,\beta}$ is the unique solution in

$\bar{a}\varphi_{\alpha,\beta}$ of the above considered equations. It follows from our considerations that $\psi_{\alpha,\beta}$ is well-defined.

Let us consider the system

$$(2.3) \qquad \langle Y, \langle S_\alpha \rangle_{\alpha \in Y}, \langle \psi_{\beta,\alpha} \rangle_{\alpha \leqq \beta; \alpha, \beta \in Y} \rangle.$$

It should be obvious that for every $\alpha \in Y$, S_α and $\bar{S}_\alpha$ have the same greatest bisemilattice homomorphic image which is a distributive lattice. Therefore S_α is an ID-semiring which satisfies (1.7) for all $\alpha \in Y$. Evidently $S_\alpha \cap S_\beta = \emptyset$ if $\alpha \neq \beta$ since $\bar{S}_\alpha \cap \bar{S}_\beta = \emptyset$ in that case.

Let $\alpha, \beta \in Y$, with $\beta \leqq \alpha$, and let us suppose that $a, a' \in S_\alpha$. Then $b = a\psi_{\alpha,\beta}$, $b' = a'\psi_{\alpha,\beta} \in S_\beta$, and $bb' \in \bar{b}\bar{b}' = \bar{a}\varphi_{\alpha,\beta}\bar{a}'\varphi_{\alpha,\beta} = \bar{a}\bar{a}'\varphi_{\alpha,\beta}$. Further,

$$bb' = (b+a)(b'+a') = bb'+ba'+ab'+aa'$$

implies that $bb'+aa' = bb'$, and similarly $aa'+bb' = bb'$. Hence $bb' = (aa')\psi_{\alpha,\beta}$, and so $(aa')\psi_{\alpha,\beta} = (a\psi_{\alpha,\beta})(a'\psi_{\alpha,\beta})$. Analogously $(a+a')\psi_{\alpha,\beta} = a\psi_{\alpha,\beta} + a'\psi_{\alpha,\beta}$. We conclude that $\langle \psi_{\alpha,\beta} \rangle_{\beta \leqq \alpha; \alpha, \beta \in Y}$ is a family of homomorphisms. It should be obvious that for every $\alpha \in Y$, $\psi_{\alpha,\alpha}$ is the identity mapping on S_α. Further, let $\gamma \leqq \beta \leqq \alpha$ in Y, and let $a \in S_\alpha$. Let us put $a\psi_{\alpha,\beta} = b$ and $b\psi_{\beta,\gamma} = c$. Then $b \in \bar{a}\varphi_{\alpha,\beta}$, and $c \in \bar{b}\varphi_{\beta,\gamma} = \bar{a}\varphi_{\alpha,\beta}\varphi_{\beta,\gamma} = \bar{a}\varphi_{\alpha,\gamma}$. Further $ab = ba = b$ and $bc = cb = c$ imply $ac = ca = c$, and so $c = a\psi_{\alpha,\gamma}$. Thus $\psi_{\alpha,\beta}\psi_{\beta,\gamma} = \psi_{\alpha,\gamma}$. We conclude that (2.3) is a semilattice ordered system of ID-semirings which satisfy the generalized absorption law (1.7).

Finally, let $a \in S_\alpha$ and $b \in S_\beta$, and put $a\psi_{\alpha,\alpha\beta} = a'$ and $b\psi_{\beta,\alpha\beta} = b'$. Then $a'b' \in \bar{a}'\bar{b}' = \bar{a}\varphi_{\alpha,\alpha\beta}\bar{b}\varphi_{\beta,\alpha\beta} = \bar{a}\bar{b} = \overline{ab}$, hence $a'b'$ and ab belong to the same rectangular semiring $\overline{a'b'} = \overline{ab}$. Further, $a'b' = (a'+a)(b'+b) = a'b'+a'b+ab'+ab$ and so $a'b'+ab = a'b'$. Similary $ab+a'b' = a'b'$, and thus $ab = ab+a'b'+ab = a'b'$ We conclude that $ab = (a\psi_{\alpha,\alpha\beta})(b\psi_{\beta,\alpha\beta})$ and by symmetry, $a+b = (a\psi_{\alpha,\alpha\beta})+(b\psi_{\beta,\alpha\beta})$. Consequently, $(S, +, \cdot)$ is the Płonka sum of the system (2.3).

The reader may check that for any $\alpha, \beta \in Y$, with $\beta \leqq \alpha$, and $a \in S_\alpha$, we have $a\psi_{\alpha,\beta} = b$ if and only if $a(a+b+a)a = b$. Thus we can conclude to the following.

Corollary 6. *For any ID-semiring* $(S, +, \cdot)$,

$$f\colon S^2 \to S, \quad (x, y) \to x(x+y+x)x$$

is a partition function.

Proof. Immediate from the proof of the foregoing and from [8].

Problem. Let $(S, +, \cdot)$ be an ID-semiring where $(S, +)$ is a semilattice. By Theorem 2.1 $(S, \cdot)$ must be a normal band. Characterize the normal bands which can be realized in this way. Remark that [16] and [17] give information about the normal bands $(S, \cdot)$ where $(S, +, \cdot)$ is an ID-semiring for which $(S, +)$ is a chain.

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