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Moment theorems for operators on Hilbert space

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Introduction

The present note raises and solves moment like problems on the existence of a contraction, a subnormal operator and of a continuous semigroup of contractions, respectively, on a (complex) Hilbert space:

(A) Given a sequence $\{h_n\}_{n\geqq 0}$ of elements of the Hilbert space H, under what condition does there exist a contraction or a subnormal operator T on H such that

$$h_n = T^n h_0 \quad \text{holds for} \quad n = 1, 2, \dots . \tag{1}$$

(B) Given a continuous family $\{h_t\}_{t\geqq 0}$ of elements of the Hilbert space H, under what condition does there exist a continuous semigroup $\{T_t\}_{t\geqq 0}$ of contractions on H such that

$$h_t = T_t h_0 \quad \text{holds for} \quad t \geqq 0. \tag{2}$$

The key to the solution (and of the source of these questions) is the theory of unitary and normal dilations.

The author is indepted to Professor B. Sz.-Nagy for his valuable advices, for his personal stimulation.

For normal extension of subnormal operators we refer to BRAM [1], HALMOS [2] and SZ.-NAGY [3].

Results

Theorem A. *Let $\{h_n\}_{n\geqq 0}$ be a sequence of elements of the Hilbert space H. There exists a contraction T on H satisfying* (1) *if and only if*

$$\text{(i)} \quad \Big\|\sum_{n,n'} c_{n,n'} h_{n+n'}\Big\|^2 \leqq \sum_{\substack{m\geqq n\\ m',n'}} c_{m,m'}\bar{c}_{n,n'}(h_{m-n+m'}, h_{n'}) + \sum_{\substack{m<n\\ m',n'}} c_{m,m'}\bar{c}_{n,n'}(h_{m'}, h_{n-m+n'})$$

holds for any finite double sequence $\{c_{n,n'}\}_{n\geqq 0, n'\geqq 0}$ of complex numbers.

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Theorem B. *Let* $\{h_t\}_{t\geqq 0}$ *be a continuous family of elements of a Hilbert space* H. *There exists a continuous semigroup* $\{T_t\}_{t\geqq 0}$ *of contractions in* H *satisfying* (2) *if and only if*

(ii) $$\Big\|\sum_{t,t'} c_{t,t'} h_{t+t'}\Big\|^2 \leqq \sum_{\substack{s\geqq t\\ s',t'}} c_{s,s'}\bar{c}_{t,t'}(h_{s-t+s'}, h_{t'}) + \sum_{\substack{s<t\\ s',t'}} c_{s,s'}\bar{c}_{t,t'}(h_{s'}, h_{t-s+t'})$$

holds for any finite double sequence $\{c_{t,t'}\}_{t\geqq 0,\, t'\geqq 0}$ *of complex numbers.*

Theorem C. *Let* $\{h_n\}_{n\geqq 0}$ *be a sequence of elements of the Hilbert space* H *such that*

(iii) $\{h_n\}$ *spans the space* H,

(iv) $\|h_n\|\leqq \mathscr{K}^n$ $(n=0, 1, 2, \ldots)$ *for some constant* $\mathscr{K}$.

There exists a subnormal operator T *on* H *satisfying* (1) *if and only if there exists a double sequence* $\{h_n^{n'}\}_{n,\, n'\geqq 0}$ *of elements of* H *such that*

(v) $h_n^0=h_n$ *for* $n=0, 1, 2, \ldots$,

(vi) $(h_n^{n'}, h_m)=(h_n, h_{m+n'})$ *for* $m, n, n'\geqq 0$, *and that*

(vii) $$\Big\|\sum_{n,n'} c_{n,n'} h_n^{n'}\Big\|^2 \leqq \sum_{m,m',n,n'} c_{m,m'}\bar{c}_{n,n'}(h_{m+n'}, h_{m'+n})$$

holds for all finite double sequence $\{c_{n,n'}\}_{n,n'\geqq 0}$ *of complex numbers.*

Necessity

(A) Let U be a unitary dilation of the contraction T on the Hilbert space K containing H such that

(3) $$PU^n h = T^n h \quad (h\in H;\ n = 1, 2, \ldots)$$

holds with the orthogonal projection P of K onto H. Let further $\{c_{n,n'}\}_{n\geqq 0,\, n'\geqq 0}$ be a finite double sequence of complex numbers. We have then by (1) and (3)

$$\Big\|\sum_{n,n'} c_{n,n'} h_{n+n'}\Big\|^2 = \Big\|\sum_{n,n'} c_{n,n'} T^n h_{n'}\Big\|^2 = \Big\|\sum_{n,n'} c_{n,n'} PU^n h_{n'}\Big\|^2 \leqq$$

$$\leqq \Big\|\sum_{n,n'} c_{n,n'} U^n h_{n'}\Big\|^2 = \sum_{m,m',n,n'} c_{m,m'}\bar{c}_{n,n'}(U^m h_{m'}, U^n h_{n'}) =$$

$$= \sum_{\substack{m\geqq n\\ m',n'}} c_{m,m'}\bar{c}_{n,n'}(U^{m-n}h_{m'}, h_{n'}) + \sum_{\substack{m<n\\ m',n'}} c_{m,m'}\bar{c}_{n,n'}(h_{m'}, U^{n-m}h_{n'}) =$$

$$= \sum_{\substack{m\geqq n\\ m',n'}} c_{m,m'}\bar{c}_{n,n'}(T^{m-n}h_{m'}, h_{n'}) + \sum_{\substack{m<n\\ m',n'}} c_{m,m'}\bar{c}_{n,n'}(h_{m'}, T^{n-m}h_{n'}) =$$

$$= \sum_{\substack{m\geqq n\\ m',n'}} c_{m,m'}\bar{c}_{n,n'}(h_{m-n+m'}, h_{n'}) + \sum_{\substack{m<n\\ m',n'}} c_{m,m'}\bar{c}_{n,n'}(h_{m'}, h_{n-m+n'}).$$

(B) The unitary dilation of a continuous semigroup $\{T_t\}_{t\geqq 0}$ of contractions is a continuous semigroup $\{U_t\}_{t\geqq 0}$ of unitaries on the dilations space K, such that

$$PU_t h = T_t h \quad (h \in H,\, t \geqq 0) \tag{4}$$

holds, where P is the orthogonal projection of K onto H. Assume further $\{c_{t,t'}\}_{t\geqq 0, t'\geqq 0}$ is a finite double sequence of complex numbers indexed by nonnegative real numbers. (2) and (4) imply (ii) exactly in the same manner as before.

(C) Suppose N is a normal extension of T acting on a Hilbert space K containing H, and such that

$$PN^{*n'}N^n h = T^{*n'}T^n h \quad (h \in H;\, n, n' \geqq 0) \tag{5}$$

holds with the orthogonal projection P of K onto H. Let further

$$h_n^{n'} = T^{*n'}T^n h_0 \quad (n, n' = 0, 1, 2, \ldots). \tag{6}$$

Assuming (1) we have then $h_n^0 = T^n h_0 = h_n$ for $n = 0, 1, 2, \ldots$; and we have by (6) also that

$$(h_n^{n'}, h_m) = (T^{*n'}T^n h_0, T^m h_0) = (T^n h_0, T^{m+n'} h_0) =$$

$$= (h_n, h_{m+n'}) \quad (m, n, n' = 0, 1, 2, \ldots)$$

and, finally, that

$$\Big\|\sum_{n,n'} c_{n,n'} h_n^{n'}\Big\|^2 = \Big\|\sum_{n,n'} c_{n,n'} T^{*n'}T^n h_0\Big\|^2 = \Big\|P\sum_{n,n'} c_{n,n'} N^{*n'}N^n h_0\Big\|^2 \leqq$$

$$\leqq \Big\|\sum_{n,n'} c_{n,n'} N^{*n'}N^n h_0\Big\|^2 = \sum_{m,m',n,n'} c_{m,m'}\bar c_{n,n'} (N^{m+n'}h_0, N^{m'+n}h_0) =$$

$$= \sum_{m,m',n,n'} c_{m,m'}\bar c_{n,n'} (T^{m+n'}h_0, T^{m'+n}h_0) = \sum_{m,m',n,n'} c_{m,m'}\bar c_{n,n'} (h_{m+n'}, h_{m'+n})$$

holds for any finite double sequence $\{c_{n,n'}\}_{n,n'\geqq 0}$ of complex numbers.

Sufficiency

(A) Let F_0 be the (complex) linear space of all finite double sequences $\{c_{n,n'}\}_{n\geqq 0, n'\geqq 0}$ of complex numbers with the shift operation

$$U_0\{c_{n,n'}\} = \{c'_{n,n'}\}, \quad \text{where} \quad c'_{n,n'} = c_{n-1,n'}\,(n \geqq 1) \quad \text{and} \quad c'_{0,n'} = 0.$$

Let us introduce a semi-inner product in F_0 (in view of (i)) by

$$\langle\{c_{m,m'}\}, \{d_{n,n'}\}\rangle = \sum_{\substack{m\geqq n\\ m',n'}} c_{m,m'}\bar d_{n,n'} (h_{m-n+m'}, h_{n'}) + \sum_{\substack{m<n\\ m',n'}} c_{m,m'}\bar d_{n,n'} (h_{m'}, h_{n-m+n'}). \tag{7}$$

U_0 is an isometry with respect to this semi-inner product. Defining

$$V_0\{c_{n,n'}\} = \sum_{n,n'} c_{n,n'} h_{n+n'} \quad \text{for} \quad \{c_{n,n'}\} \in F_0$$

we obtain a contraction V_0 from F_0 into H.

Let F be the Hilbert space resulting from F_0 by factoring with respect to the null space of $\langle\cdot,\cdot\rangle$ and by completing. At the same time U_0 induces an isometry U on F and V_0 induces a contraction V from F into H. In what follows the equivalence class represented by $\{c_{n,n'}\}$ is also denoted shortly by $\{c_{n,n'}\}$. We show that

$$(8) \qquad V^* h_k = \{d_{n,n'}\}, \quad \text{where} \quad d_{n,n'} = \begin{cases} 1 & \text{if} \quad n = 0, \quad \text{and} \quad n' = k \quad (k = 0, 1, \ldots), \\ 0 & \text{otherwise.} \end{cases}$$

To show this let $k \geqq 0$, $\{c_{m,m'}\} \in F$ so that (7) gives

$$\langle \{c_{m,m'}\}, V^* h_k \rangle = \langle V\{c_{m,m'}\}, h_k \rangle = \sum_{m,m'} c_{m,m'} (h_{m+m'}, h_k) = \langle \{c_{m,m'}\}, \{d_{n,n'}\} \rangle$$

as desired. Because of (8) we get

$$UV^* h_k = \{d_{n,n'}\}, \quad \text{where} \quad d_{n,n'} = \begin{cases} 1 & \text{if} \quad n = 1 \quad \text{and} \quad n' = k \quad (k = 0, 1, \ldots), \\ 0 & \text{otherwise.} \end{cases}$$

Defining

$$T = VUV^*$$

we have $Th_k = VUV^* h_k = h_{k+1}$ for all $k = 0, 1, \ldots$, but this is actually identical with (1).

(B) Let F_0 be, similarly as before, the linear space of all double sequences $\{c_{s,s'}\}_{s \geqq 0, s' \geqq 0}$ of complex numbers indexed by nonnegative real numbers. Define, for all $t \geqq 0$, by

$$U_t\{c_{s,s'}\} = \{c_{s-t,s'}\} \quad \text{for} \quad \{c_{s,s'}\} \in F_0$$

a shift operation and a semi-inner product (in view of (i)) by

$$\langle \{c_{r,r'}\}, \{d_{s,s'}\} \rangle = \sum_{\substack{r \geqq s \\ r,s'}} c_{r,r'} \bar{d}_{s,s'} (h_{r-s+r'}, h_{s'}) + \sum_{\substack{r < s \\ r,s'}} c_{r,r'} \bar{d}_{s,s'} (h_{r'}, h_{s-r+s'});$$

$\{U_t\}_{t \geqq 0}$ is then a continuous semigroup of isometries of the Hilbert space F derived from F_0 as before. By defining

$$V\{c_{s,s'}\} = \sum_{s,s'} c_{s,s'} h_{s+s'} \quad \text{for} \quad \{c_{s,s'}\} \in F_0$$

we get a contraction operator from F into H. The proof that $T_t = VU_tV^*$ $(t \geqq 0)$ is a continuous semigroup of contractions satisfying (2) only needs a slight modification of the argument used above, so we omit it.

(C) Let $\{h_n^{n'}\}_{n,n'\geqq 0}$ be in H such that conditions (iii—iv) are satisfied. Take the (complex) linear space F_0 of all finite double sequences $\{c_{n,n'}\}_{n,n'\geqq 0}$ of complex numbers with a shift operation

$$N_0\{c_{n,n'}\} = \{c'_{n,n'}\}, \quad \text{where} \quad c'_{n,n'} = c_{n-1,n'}\,(n \geqq 1), \quad \text{and} \quad c'_{0,n'} = 0;$$

and (in view of (vii)) with a semi-inner product in F_0 defined by

$$\langle\{c_{m,m'}\}, \{d_{n,n'}\}\rangle = \sum_{m,m',n,n'} c_{m,m'}\,\bar{d}_{n,n'}\,(h_{m+n'}, h_{m'+n}). \tag{9}$$

We are going to prove that

$$\|N_0\| \leqq \mathscr{K} \tag{$*$}$$

with the same $\mathscr{K}$ as that in (iv). First of all, for any $\{c_{n,n'}\}\in F_0$ and $i,j=0, 1, 2, \ldots$ we define

$$c_{n,n'}^{(i,j)} = \begin{cases} c_{n-i,n'-j}, & \text{if} \quad n \geqq i,\ n' \geqq j, \\ 0 & \text{otherwise.} \end{cases}$$

Now, by (9) we have

$$\|\{c_{n,n'}^{(i,j)}\}\|^2 = \sum_{m,m',n,n'} c_{m,m'}\,\bar{c}_{n,n'}\,(h_{m+n'+i+j}, h_{m'+n+i+j}) =$$

$$= \langle\{c_{n,n'}^{(i+j,i+j)}\}, \{c_{n,n'}\}\rangle \leqq \|\{c_{n,n'}^{(i+j,i+j)}\}\| \cdot \|\{c_{n,n'}\}\|.$$

So by induction we can derive

$$\|\{c_{n,n'}^{(1,0)}\}\|^{2^{k+1}} \leqq \|\{c_{n,n'}^{(2^k,2^k)}\}\| \cdot \|\{c_{n,n'}\}\|^{1+2+\ldots+2^k} \quad \text{for} \quad k = 0, 1, \ldots .$$

The definition of N_0 shows that $N_0\{c_{n,n'}\}=\{c_{n,n'}^{(1,0)}\}$ and so the above inequality, (9) and (iv) imply that

$$\|N_0\{c_{n,n'}\}\|^{2^{k+1}} \leqq \|\{c_{n,n'}^{(2^k,2^k)}\}\| \cdot \|\{c_{n,n'}\}\|^{1+2+\ldots+2^k} =$$

$$= \Big\{\sum_{m,m',n,n'} c_{m,m'}\,\bar{c}_{n,n'}\,(h_{m+n'+2^{k+1}}, h_{m'+n+2^{k+1}})\Big\}^{1/2} \|\{c_{n,n'}\}\|^{2^{k+1}-1} \leqq$$

$$\leqq \Big\{\sum_{m,m',n,n'} |c_{m,m'}|\,|\bar{c}_{n,n'}|\,\|h_{m+n'+2^{k+1}}\| \cdot \|h_{m'+n+2^{k+1}}\|\Big\}^{1/2} \|\{c_{n,n'}\}\|^{2^{k+1}-1} \leqq$$

$$\leqq \|\{c_{n,n'}\}\|^{2^{k+1}-1} \sum_{n,n'} |c_{n,n'}|\,\mathscr{K}^{n+n'+2^{k+1}}.$$

This gives

$$\|N_0\{c_{n,n'}\}\| \leqq \|\{c_{n,n'}\}\|^{1-2^{-k-1}} \cdot \mathscr{K}\Big\{\sum_{n,n'} |c_{n,n'}|\,\mathscr{K}^{n+n'}\Big\}^{2^{-k-1}}.$$

Let $k\to\infty$, so we obtain $(*)$.

Defining

$$V_0\{c_{n,n'}\} = \sum_{n,n'} c_{n,n'}\,h_n^{n'} \quad \text{for} \quad \{c_{n,n'}\}\in F_0,$$

(vii) shows that V_0 is a contraction from F_0 into H. We obtain a Hilbert space F from F_0 by factoring with respect to the null space of $\langle\cdot,\cdot\rangle$ and then by completing.

At the same time, V_0 induces a contraction V from F into H and N_0 induces a bounded linear operator N on F.

Finally define the operator

$$T = VNV^* \tag{10}$$

on H. We are going to show that this operator is the desired one. First of all, for any $k \geqq 0$

$$V^* h_k = \{d_{n,n'}\}, \quad \text{where} \quad d_{n,n'} = \begin{cases} 1, & \text{if} \quad n = k \quad \text{and} \quad n' = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Indeed,

$$\langle \{c_{m,m'}\}, V^* h_k \rangle = \langle V\{c_{m,m'}\}, h_k \rangle = \sum_{m,m'} c_{m,m'} (h_m^{m'}, h_k) =$$

$$= \sum_{m,m'} c_{m,m'} (h_m, h_{m'+k}) = \langle \{c_{m,m'}\}, \{d_{n,n'}\} \rangle.$$

Thus

$$T h_k = VNV^* h_k = V\{d_{n-1,n'}\} = \sum_{n,n'} d_{n,n'} h_{n+1}^{n'} = h_{k+1}^0 = h_{k+1}$$

holds for all $k = 0, 1, 2, \ldots$. We have (1) also as was desired. We have only to show that T in (10) is subnormal, that is,

$$\sum_{m,n} (T^m g_n, T^n g_m) \geqq 0 \tag{11}$$

holds for all finite sequence $\{g_n\}_{n \geqq 0}$ in H. We have (11) for elements of the form $g_n = \sum_{n'} \bar{c}_{n,n'} h_{n'}$ (where $\{c_{n,n'}\} \in F$) as a consequence of (vii). Indeed,

$$\sum_{m,n} (T^m g_n, T^n g_m) = \sum_{m,n} \Big(\sum_{n'} \bar{c}_{n,n'} T^m h_{n'}, \sum_{m'} \bar{c}_{m,m'} T^n h_{m'} \Big) =$$

$$= \sum_{m,m',n,n'} c_{m,m'} \bar{c}_{n,n'} (T^m h_{n'}, T^n h_{m'}) = \sum_{m,m',n,n'} c_{m,m'} \bar{c}_{n,n'} (h_{m+n'}, h_{m'+n}) \geqq$$

$$\geqq \Big\| \sum_{n,n'} c_{n,n'} h_n^{n'} \Big\|^2 \geqq 0,$$

which implies (11) in general by (iii). The theorem is proved.

Note that the proof of the theorem yields the following

Proposition. *Let $\{h_n^{n'}\}_{n,n' \geqq 0}$ be a double sequence in H which spans H. There exists a normal operator T on H such that*

$$T^{*n'} T^n h_0^0 = h_n^{n'} \quad (n, n' = 0, 1, 2, \ldots) \tag{12}$$

holds if and only if

$$\|h_n^{n'}\| \leqq \mathscr{K}^{n+n'} \quad \textit{for some constant} \quad \mathscr{K} > 0 \quad (n, n' \geqq 0) \tag{13}$$

and

$$(h_m^{m'}, h_n^{n'}) = (h_{m+n'}^0, h_{m'+n}^0) \quad (m, m', n, n' \geqq 0). \tag{14}$$

Proof. Assume (12), then (13) is trivial and (14) is elementary

$$(h_m^{m'}, h_n^{n'}) = (T^{*m'}T^m h_0^0, T^{*n'}T^n h_0) = (T^{m+n'}h_0^0, T^{m'+n}h_0) = (h_{m+n'}^0, h_{m'+n}^0).$$

Assume now (13) and (14) and denote h_n^0 by h_n (n=0, 1, 2, ...). We have then (v—vii) with equality in (vii), consequently the operator V, appearing in the proof of Theorem C, is a unitary operator from F onto H. Simple calculation shows that

$$N^*\{c_{n,n'}\} = \{c_{n,n'-1}\} \quad \text{for} \quad \{c_{n,n'}\} \in F \tag{15}$$

holds which yields $NN^* = N^*N$, that is, N is a normal operator. Since V is unitary, $T = VNV^*$ is normal, too. We have finally to show (12). T satisfies (1), and, by similar argument as in the proof of Theorem C, (15) implies that

$$VN^{*n'}V^*h_n = h_n^{n'}.$$

So we have

$$T^{*n'}T^n h_0^0 = (VN^*V^*)^{n'}h_n = VN^{*n'}V^*h_n = h_n^{n'} \quad \text{for} \quad n, n' = 0, 1, \ldots.$$

The proof is complete.

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