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Generalized resolvents of contractions

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1. Let T be a contraction (that is a linear operator of norm ≤ 1), defined on a closed subspace $\mathfrak{D}(T)(\neq\mathfrak{H})$ of some Hilbert space $\mathfrak{H}$ and with values in $\mathfrak{H}$. By a contraction extension (c.e.) of T we mean an extension $\tilde{T}$ of T to some Hilbert space $\tilde{\mathfrak{H}}\supset\mathfrak{H}$, which is also a contraction. If $\tilde{\mathfrak{H}}=\mathfrak{H}$, the c.e. $\tilde{T}$ is called canonical.

Let $\tilde{T}$ on $\tilde{\mathfrak{H}}$ be a c.e. of T, and denote by $\tilde{P}$ the orthogonal projector of $\tilde{\mathfrak{H}}$ onto $\mathfrak{H}$. The function

$$z\to R_z:=\tilde{P}(z\tilde{T}-I)^{-1}|_{\mathfrak{H}}\quad(|z|<1)\tag{1}$$

which is defined and holomorphic on the open unit disc $\mathbf{D}:=\{|z|<1\}$ and whose values are bounded linear operators in $\mathfrak{H}$, is called a generalized resolvent of T (generated by $\tilde{T}$). The generalized resolvent R_z is called canonical if $\tilde{T}=T$.

It is the aim of this note to give a description of all generalized resolvents of a nondensely defined contraction T in a Hilbert space $\mathfrak{H}$. This result is an analogue of the formula for the generalized resolvents of an isometric operator, proved in [1] for equal and in [2] (see also [3]) for arbitrary defect numbers.* In their turn these results have their origin in the classical formula of M. G. KREIN on the generalized resolvents of an hermitian operator with equal defect numbers ([4], [5]).

2. Let T be as above. By $\mathring{T}$ we denote the c.e. of T given by

$$\mathring{T}x:=\begin{cases}Tx & x\in\mathfrak{D}(T),\\ 0 & x\in\mathfrak{D}(T)^{\perp},\end{cases}$$

and set

$$D:=(I-\mathring{T}^*\mathring{T})^{1/2},\ D_*:=(I-\mathring{T}\mathring{T}^*)^{1/2},\ \mathscr{D}:=\overline{\mathfrak{R}(D)},\ \mathscr{D}_*:=\overline{\mathfrak{R}(D_*)}.$$

The characteristic function of $\mathring{T}^*$ is denoted by $X(z)$ (see [6, Chap. VI]):

$$X(z):=(-\mathring{T}^*-zD\mathring{R}_zD_*)|_{\mathscr{D}_*},\ \mathring{R}_z:=(z\mathring{T}-I)^{-1},\ z\in\mathbf{D}.$$

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* In these papers the more general case of an isometric operator in a $\pi_\varkappa$-space (Pontrjagin space with index $\varkappa$) has been considered.

It is defined and holomorphic on the open unit disc $\mathbf{D}$ and its values are contractions, mapping $\mathfrak{D}_*$ into $\mathfrak{D}$, see [6, chap. VI]. By $\mathcal{K}$ (or, sometimes, more explicitly by $\mathcal{K}(\mathfrak{D}(T)^{\perp}, \mathfrak{D}_*)$) we denote the set of all functions $G(z)$, defined and holomorphic on $\mathbf{D}$ and whose values are contractions from $\mathfrak{D}(T)^{\perp}$ into $\mathfrak{D}_*$, by $\mathcal{K}_0$ (or $\mathcal{K}_0(\mathfrak{D}(T)^{\perp}, \mathfrak{D}_*)$) the subset of $\mathcal{K}$, consisting of all $G\in\mathcal{K}$ which are independent of z. Finally, Γ is the orthogonal projector of $\mathfrak{H}$ onto $\mathfrak{D}(T)^{\perp}$.

Theorem. *Let T be a contraction in the Hilbert space $\mathfrak{H}$ with a closed domain $\mathfrak{D}(T)\neq\mathfrak{H}$. The formula*

$$R_z = \mathring{R}_z - z\mathring{R}_z D_* G(z)\big(I-\Gamma X(z)G(z)\big)^{-1}\Gamma D\mathring{R}_z \quad (|z|<1) \tag{2}$$

establishes a 1,1-*correspondence between the set of all generalized resolvents R_z of T and all $G\in\mathcal{K}$. The generalized resolvent R_z is canonical if and only if $G\in\mathcal{K}_0$.*

Proof. a) Let $\tilde{T}$ be a canonical c.e. of T. We define an operator F from $\mathfrak{D}(T)^{\perp}$ into $\mathfrak{H}$ by the formula $Fx:=\tilde{T}x$ $(x\in\mathfrak{D}(T)^{\perp})$. Then we have

$$\mathring{T}\mathring{T}^*+FF^* \leqq I \quad \text{or} \quad FF^* \leqq I-\mathring{T}\mathring{T}^* = D_*^2.$$

Therefore the operator $F_1:=F^*D_*^{-1}$ is a contraction, which is densely defined on $\mathfrak{D}_*$ and with values in $\mathfrak{D}(T)^{\perp}$. The adjoint of its closure $G:=(\bar{F}_1)^*$ belongs to $\mathcal{K}_0$. Observing $\mathring{T}\Gamma=0$ we find with $R_z:=(z\tilde{T}-I)^{-1}$:

$$R_z-\mathring{R}_z = z\mathring{R}_z(\mathring{T}-\tilde{T})R_z = z\mathring{R}_z(\mathring{T}-\tilde{T})\Gamma R_z = -z\mathring{R}_z F\Gamma R_z. \tag{3}$$

It follows

$$R_z = (I+z\mathring{R}_z F\Gamma)^{-1}\mathring{R}_z,\; \Gamma R_z = (I+z\Gamma\mathring{R}_z F)^{-1}\Gamma\mathring{R}_z,$$

and (3) can be written as

$$R_z-\mathring{R}_z = -z\mathring{R}_z F(I+z\Gamma\mathring{R}_z F)^{-1}\Gamma\mathring{R}_z = -z\mathring{R}_z D_* G(I+z\Gamma\mathring{R}_z D_* G)^{-1}\Gamma\mathring{R}_z.$$

Furthermore,

$$\Gamma D = \Gamma, \quad \Gamma\mathring{T}^* = 0 \tag{4}$$

and we get

$$\begin{aligned} R_z-\mathring{R}_z &= -z\mathring{R}_z D_* G(I+z\Gamma D\mathring{R}_z D_* G)^{-1}\Gamma\mathring{R}_z = \\ &= -z\mathring{R}_z D_* G\big(I-\Gamma(X(z)-\mathring{T}^*)G\big)^{-1}\Gamma\mathring{R}_z = \\ &= -z\mathring{R}_z D_* G\big(I-\Gamma X(z)G\big)^{-1}\Gamma\mathring{R}_z = -z\mathring{R}_z D_* G\big(I-\Gamma X(z)G\big)^{-1}\Gamma D\mathring{R}_z. \end{aligned} \tag{5}$$

b) Let now $\tilde{T}$ be an arbitrary (not necessarily canonical) c.e. of T in $\tilde{\mathfrak{H}}\supset\mathfrak{H}$, R_z the corresponding generalized resolvent. We shall prove the following statement:

(i) *If z is fixed in* $\mathbf{D}$, *then the operator* R_z^{-1} *exists and*

$$T_z := \frac{1}{z}(R_z^{-1}+I) \tag{6}$$

is a canonical c.e. of T.

Indeed, $R_z x=0$ for some $x\in\mathfrak{H}, x\neq 0$, implies $\big((z\tilde{T}-I)^{-1}x, x\big)=0$ and with $\tilde{u} := (z\tilde{T}-I)^{-1}x$ we get

$$0=\big((z\tilde{T}-I)\tilde{u}, \tilde{u}\big) \quad \text{or} \quad \|\tilde{u}\|^2 = z(\tilde{T}\tilde{u}, \tilde{u}),$$

hence $\tilde{u}=0$ as $|z|<1$ and $\|\tilde{T}\|\leqq 1$, a contradiction. In the same way it follows that the inverse of R_z^* exists, therefore the range of R_z is dense in $\mathfrak{H}$.

In order to see that T_z is a contraction we first show that the operator $S_z := R_z^{-1}+I$ $(|z|<1)$ is a contraction, that is,

$$\|R_z^{-1}x+x\|^2 \leqq \|x\|^2 \quad \text{or} \quad \|R_z^{-1}x\|^2+2\mathrm{Re}\,(R_z^{-1}x, x) \leqq 0 \tag{7}$$

holds for arbitrary $x\in\mathfrak{R}(R_z)$. Putting $R_z^{-1}x=y$, $(z\tilde{T}-I)^{-1}y=\tilde{v}$ we have

$$\begin{aligned}\|R_z^{-1}x\|^2+2\mathrm{Re}\,(R_z^{-1}x, x) &= \|y\|^2+2\mathrm{Re}\big(y, (z\tilde{T}-I)^{-1}y\big) = \\ &= \|(z\tilde{T}-I)\tilde{v}\|^2+2\mathrm{Re}\big((z\tilde{T}-I)\tilde{v}, \tilde{v}\big) = \|z\tilde{T}\tilde{v}\|^2-\|\tilde{v}\|^2 \leqq 0,\end{aligned}$$

and (7) follows. Further, for an arbitrary pair $x, y\in\mathfrak{H}$, $\|x\|=\|y\|=1$, the function

$$f(z) := (S_z x, y) \quad (|z|<1)$$

is a holomorphic function of modulus $\leqq 1$, which vanishes at $z=0$. By Schwarz' lemma, $\frac{1}{z}f(z)$ is of modulus $\leqq 1$ in $\mathbf{D}$, hence also $T_z=\frac{1}{z}S_z$ is a contraction. Finally, if $x\in\mathfrak{D}(T)$ we find

$$(T_z-T)x=\frac{1}{z}R_z^{-1}(I+R_z-zR_zT)x=\frac{1}{z}R_z^{-1}\tilde{P}(z\tilde{T}-I)^{-1}(z\tilde{T}-zT)x=0,$$

therefore T_z is an extension of T. The statement (i) is proved.

Now the results of a) can be applied to the canonical c.e. T_z of T. Observing the relation $(zT_z-I)^{-1}=R_z$, the representation (5) gives

$$R_z-\mathring{R}_z=-z\mathring{R}_z D_* G(z)\big(I-\Gamma X(z)G(z)\big)^{-1}\Gamma D\mathring{R}_z,$$

where $G(z) := \big(\overline{F_1(\bar{z})}\big)^*$, $F_1(z) := F(z)^* D_*^{-1}$ and $F(z) := T_z|_{\mathfrak{D}(T)^\perp}$. As T_z is holomorphic in $\mathbf{D}$, the function $G(z)$ belongs to $\mathscr{K}$. Therefore, an arbitrary generalized resolvent of T admits a representation (2) with some $G\in\mathscr{K}$.

c) Let now, conversely, a function $G\in\mathscr{K}$ be given. According to [6, Chap. V, Prop. 2.1] its domain $\mathfrak{D}(T)^\perp$ and range $\mathscr{D}_*$ decompose as

$$\mathfrak{D}(T)^\perp = \mathfrak{D}'\oplus\mathscr{D}^0, \quad \mathscr{D}_* = \mathscr{D}'_*\oplus\mathscr{D}^0_* \quad \text{resp.},$$

such that $G^0(z):=G(z)|_{\mathfrak{D}^0}$ is a purely contractive holomorphic function (see [6, Chap. V, 2.2], whose values are operators from $\mathfrak{D}^0$ into $\mathscr{D}_*^0$, and $G'(z):=G(z)|_{\mathfrak{D}'}$ is a unitary operator from $\mathfrak{D}'$ onto $\mathscr{D}'_*$, independent of z, $|z|<1$.

The purely contractive holomorphic function $G^0(z)$ is the characteristic function of some contraction S in a Hilbert space $\mathfrak{H}_1$, that is, $\mathfrak{D}^0$ and $\mathscr{D}_*^0$ can be identified with the subspaces $\mathscr{D}_s=\overline{\mathfrak{R}(D_s)}$ and $\mathscr{D}_{s*}=\overline{\mathfrak{R}(D_{s*})}$ resp. of $\mathfrak{H}_1$, and we have

$$G^0(z)=\left(-S-zD_{s*}(zS^*-I)^{-1}D_S\right)|_{\mathscr{D}_s}\quad (|z|<1).$$

Thus, $\mathfrak{D}^0$ and $\mathscr{D}_*^0$ can be considered as subspaces of $\mathfrak{H}$ as well as of $\mathfrak{H}_1$. Besides Γ, projecting $\mathfrak{H}$ orthogonally onto $\mathfrak{D}(T)^\perp$, we introduce the orthogonal projectors $\Gamma^0, \Gamma', \Gamma_*^0$ and Γ'_* in $\mathfrak{H}$ onto $\mathfrak{D}^0, \mathfrak{D}', \mathscr{D}_*^0$ and $\mathscr{D}'_*$ respectively and the orthogonal projectors P and P_* onto $\mathscr{D}_s$ and $\mathscr{D}_{S*}$ in $\mathfrak{H}_1$.

Now an extension $\tilde{T}$ of T, acting in the space $\mathfrak{H}\oplus\mathfrak{H}_1$, will be defined as follows: With respect to the decomposition

$$\mathfrak{H}\oplus\mathfrak{H}_1=\mathfrak{D}(T)\oplus\mathfrak{D}'\oplus\mathfrak{D}^0\oplus\mathfrak{H}_1$$

of the initial space it has the matrix representation

$$\tilde{T}=\begin{pmatrix}\mathring{T}(1-\Gamma) & D_*\Gamma'_*G' & -D_*P_*S & D_*\Gamma_*^0D_{S*}\\ 0 & 0 & D_S & S^*\end{pmatrix}. \tag{8}$$

Clearly, $\tilde{T}$ is an extension of T. In order to see that $\tilde{T}$ is contractive we consider the operator $\tilde{T}\tilde{T}^*=(\tau_{ij})_{i,j=1,2}$ in $\mathfrak{H}\oplus\mathfrak{H}_1$. Observing

$$\tilde{T}^*=\begin{pmatrix}(1-\Gamma)\mathring{T}^* & 0\\ G'^*\Gamma'_*D_* & 0\\ -PS^*\Gamma_*^0D_* & D_S\\ D_{S*}\Gamma_*^0D_* & S\end{pmatrix}$$

and the fact that G'^* maps $\mathscr{D}'_*$ unitarily onto $\mathfrak{D}'$: $G'G'_*=I|_{\mathfrak{D}'_*}$, we find

$$\begin{aligned}\tau_{11}&=\mathring{T}(I-\Gamma)\mathring{T}^*+D_*\Gamma'_*D_*+D_*P_*SPS^*\Gamma_*^0D_*+D_*\Gamma_*^0D_{S*}^2\Gamma_*^0D_*\leqq\\ &\leqq\mathring{T}(I-\Gamma)\mathring{T}^*+D_*\Gamma'_*D_*+D_*P_*SS^*\Gamma_*^0D_*+D_*\Gamma_*^0D_{S*}^2\Gamma_*^0D_*=\\ &=\mathring{T}(I-\Gamma)\mathring{T}^*+D_*\Gamma'_*D_*+D_*\Gamma_*^0D_*\leqq\mathring{T}\mathring{T}^*+D_*^2=I,\\ \tau_{12}&=-D_*P_*SD_S+D_*\Gamma_*^0D_{S*}S=D_*P_*(-SD_S+D_{S*}S)=0,\\ \tau_{22}&=D_S^2+S^*S=I.\end{aligned}$$

Therefore, $\tilde{T}$ is a c.e. of T. Next we have to calculate the generalized resolvent of T, generated by $\tilde{T}$. In order to do this we observe the following proposition, whose simple proof will be left to the reader.

(ii) *If the c.e.* $\tilde{T}$ *of* T, *acting in* $\tilde{\mathfrak{H}}=\mathfrak{H}\oplus\mathfrak{H}_1$ *has the matrix form*

$$\tilde{T}=\begin{pmatrix}\hat{T} & C\\ B & A\end{pmatrix},$$

then we have

$$\tilde{P}(z\tilde{T}-I)^{-1}|_{\mathfrak{H}}=\left(z\hat{T}-I-z^2C(zA-I)^{-1}B\right)^{-1}.$$

We apply this proposition to the operator $\tilde{T}$ in (8). With respect to the decomposition $\mathfrak{H}\oplus\mathfrak{H}_1$ of initial and range space $\tilde{T}$ can be written as

$$\tilde{T}=\begin{pmatrix}\mathring{T}+D_*\Gamma'_*G'\Gamma'-D_*P_*S\Gamma^0 & D_*\Gamma^0_*D_{S^*}\\ D_S\Gamma^0 & S^*\end{pmatrix}$$

and we get for the corresponding generalized resolvent

$$\begin{aligned}(9)\quad R_z&=\left(z\mathring{T}-I+zD_*\Gamma'_*G'\Gamma'-zD_*P_*S\Gamma^0-z^2D_*\Gamma^0_*D_{S^*}(zS^*-I)^{-1}D_S\Gamma^0\right)^{-1}=\\ &=\left(z\mathring{T}-I+zD_*(\Gamma'_*G'\Gamma'-P_*S\Gamma^0-z\Gamma^0_*D_{S^*}(zS^*-I)^{-1}D_{S^*}\Gamma^0\right)^{-1}=\\ &=\left(z\mathring{T}-I+zD_*(\Gamma'_*G'\Gamma'+\Gamma^0_*G^0(z)\Gamma^0)\right)^{-1}=\left(z\mathring{T}-I+zD_*G(z)\Gamma\right)^{-1}=\\ &=\mathring{R}_z\left(I+zD_*G(z)\Gamma\mathring{R}_z\right)^{-1}=\mathring{R}_z-z\mathring{R}_zD_*G(z)\left(I-\Gamma X(z)G(z)\right)^{-1}\Gamma\mathring{R}_z.\end{aligned}$$

The last equality follows easily if one observes the relation (4) and

$$\Gamma X(x)G(z)=-z\Gamma D\mathring{R}_zD_*G(z).$$

As the formula (2) can be written as

$$R_z=\left(z\mathring{T}-I+zD_*G(z)\Gamma\right)^{-1}$$

(see (9)), the correspondence between the generalized resolvents R_z of T and the functions $G\in\mathscr{K}$ is bijective.

d) It remains to prove the last statement of the theorem. If $G_0\in\mathscr{K}_0$ is given we consider the operator $\tilde{T}$:

$$(10)\qquad \tilde{T}x:=\begin{cases}Tx & x\in\mathfrak{D}(T)\\ D_*G_0x & x\in\mathfrak{D}(T)^\perp.\end{cases}$$

Then $\tilde{T}$ is an extension of T and, moreover, we have

$$\tilde{T}\tilde{T}^*=\mathring{T}\Gamma\mathring{T}^*+D_*G_0G_0^*D\leqq\mathring{T}\mathring{T}^*+D_*^2=I,$$

hence $\tilde{T}$ is a c.e. of T. With the notation of part a) of the proof we find

$$F=D_*G_0,\quad F_1=G_0^*\quad\text{and}\quad G=G_0.$$

That is, the generalized resolvent of T, generated by $\tilde{T}$ from (10), is given by (2) with $G=G_0$. The theorem is proved.

Remark 1. In the case of an isometric operator T, the unitary extension $\tilde{T}$, generating a given generalized resolvent of T, is uniquely determined (up to isomorphisms) if some minimality condition is imposed on $\tilde{T}$. This is not true in the situation considered here. E.g., if $G\in\mathscr{K}_0$ and G is not a unitary constant, in the proof of the Theorem (parts c) and d)) two different extensions of T, which generate the same generalized resolvent, have been given.

Remark 2. With the notation in the proof of the theorem, the operator

$$T' : = (\mathring{T}+D_* G'\Gamma')|_{\mathfrak{D}(T)\oplus\mathfrak{D}},$$

is a contraction in $\mathfrak{H}$ which extends T. Evidently, the operator $\tilde{T}$ in (8) is a c.e. of T'. Thus the generalized resolvent R_z of T in (9) is also a generalized resolvent of T'. That is, there exists a function $H\in\mathscr{K}(\mathfrak{D}^0, \mathscr{D}_{1,*})$, such that we have

$$R_z = \mathring{R}_{1,z} - z\mathring{R}_{1,z} D_{1,*} H(z)\big(I-X_1(z)H(z)\big)^{-1}\Gamma^0 \mathring{R}_{1,z}$$

$\big(\mathring{R}_{1,z} := (z\mathring{T}'-I)^{-1}, D_{1,*} := (I-\mathring{T}'(\mathring{T}')^*)^{1/2}, \mathscr{D}_{1,*} : = \overline{\mathscr{R}(D_{1,*})}$ and X_1 is the corresponding characteristic function$\big)$. It is not hard to see that the functions G^0 and H are connected by the relation

$$D_{1,*}H(z) = D_* G^0(z) \quad (|z|<1).$$

We mention that the construction of the operator $\tilde{T}$ in part c) of the proof can also be used if G decomposes as

$$G(z) = G_1 \oplus G_2(z) \quad (|z|<1)$$

where $G_1\in\mathscr{K}_0(\mathfrak{D}_1, \mathscr{D}_{*,1})$, $G_2\in\mathscr{K}(\mathfrak{D}_2, \mathscr{D}_{*,2})$ $(\mathfrak{D}(T)^\perp = \mathfrak{D}_1\oplus\mathfrak{D}_2, \mathscr{D}=\mathscr{D}_{*,1}\oplus\mathscr{D}_{*,2})$ and G_2 is purely contractive. Then the above remark holds also true for the operator

$$T_1 : = (\mathring{T}+D_* G_1\Gamma_1)|_{\mathfrak{D}(T)\oplus\mathfrak{D}_1}$$

(Γ_1 orthogonal projector onto $\mathfrak{D}_1$) instead of T'.

Remark 3. If $\tilde{T}$ is a noncanonical c.e. of T in some Hilbert space $\tilde{\mathfrak{H}}\supset\mathfrak{H}$, and if

$$\tilde{T} = \begin{pmatrix} \hat{T} & C \\ B & A \end{pmatrix}$$

with respect to the decomposition $\tilde{\mathfrak{H}} = \mathfrak{H}\oplus\mathfrak{L}$, then the operator function T_z in (6) can be written as

$$T_z = \hat{T} + zC(I-zA)^{-1}B.$$

Hence $T_{1/z}$ is the transfer function of the node $(A, B, C, \hat{T}, \mathfrak{L}, \mathfrak{H})$ in the sense of [7].

An analogue of the theorem above can be formulated for a dissipative operator. In this form it has applications to the spectral theory of canonical differential operators, which will be considered elsewhere.

Added in proof. An extension of the Theorem to dual pairs of contractions will appear in: Proceedings of the 6th Conference on Operator Theory in Timisoara and Herculane, 1981, Birkhäuser (Basel—Boston—Stuttgart, 1982).

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