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The range of the transform of certain parts of a measure

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In this note we point out a very elementary condition which provides a uniform treatment for the results in [2, 4, 5] concerning the range of the transform of certain parts of a measure. We assume familiarity with the basic facts of [8].

Let G be a nondiscrete LCA group with character group Γ and let $M(G)$ denote the customary convolution algebra of bounded Borel measures on G. Denote by S the structure semi-group of $M(G)$ and let $\hat{S}$ denote the semi-characters of S; recall that $\hat{S}$ is the maximal ideal space of $M(G)$, see [8]. For $\mu \in M(G)$ let $\hat{\mu}$ denote the Gelfand transform defined on $\hat{S}$ by

$$\hat{\mu}(\chi) = \int_S \chi d\mu$$

where we have identified μ and the image of μ in $M(S)$; we will also let $\hat{}$ denote the usual Fourier—Stieltjes transformation. By $M_0(G)$ we mean the set of $\mu \in M(G)$ such that $\hat{\mu}$ vanishes at infinity, i.e. $\hat{\mu}$ is zero on $\bar{\Gamma} \setminus \Gamma$.

The main result of this paper is the theorem stated below; its proof is quite simple. After stating and proving our theorem, we present two examples which serve to indicate its scope. Example 1 is obtained by adapting the work of B. Host and F. Parreau [3]. In order to present Example 2, we prove a proposition by modifying an argument of I. Glicksberg and I. Wik [2]. Professor Glicksberg has kindly pointed out (private communication) that the proposition is also a consequence of the main result of [1].

Theorem. Let $h \in \bar{\Gamma} \setminus \Gamma$ *and* $E \setminus \Gamma$. *Then for every* $\mu \in M(G)$,

(1) $$(h\mu)\hat{}\,(\Gamma) \subset \hat{\mu}(\Gamma \setminus E)^-$$

if and only if

(2) $$h \in (\Gamma \setminus \gamma E)^- \quad \textit{for every} \quad \gamma \in \Gamma.$$

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Proof. Let E satisfy (2) with respect to some $h \in \bar{\Gamma} \setminus \Gamma$. Fix $\gamma_0 \in \Gamma$; since $h \in (\Gamma \setminus \gamma_0^{-1} E)^-$ there is a net $\langle \gamma_j \rangle \subset \Gamma$ such that $\gamma_j \to h$ and $\gamma_j \notin \gamma_0^{-1} E$ for all j. Observe that

$$(h\mu)\hat{}\,(\gamma_0) = \hat{\mu}(\gamma_0 h) = \lim_j \hat{\mu}(\gamma_0 \gamma_j)$$

because $\hat{\mu}$ is continuous on $\hat{S}$. Thus (2) implies (1).

Now let $h \in \bar{\Gamma} \setminus \Gamma$ and suppose for every $\mu \in M(G)$, $(h\mu)\hat{}\,(\Gamma) \subset \hat{\mu}(\Gamma \setminus E)^-$; we want to see that E satisfies (2) with respect to h. With this in mind fix $\gamma_0 \in \Gamma$ and let V be any open set of $\hat{S}$ containing $\{h\}$. It suffices to confirm that $V \cap (\Gamma \setminus \gamma_0 E)$ is not empty.

Let $W = \bar{\gamma}_0 V$. Then W is an open set containing $\{\bar{\gamma}_0 h\}$; by the definition of the Gelfand topology on $\hat{S}$ there exist measures $\mu_1, \ldots, \mu_n \in M(G)$ and $\varepsilon > 0$ such that

$$\bigcap_{i=1}^{n} \{\chi : |\hat{\mu}_i(\chi) - \hat{\mu}_i(\bar{\gamma}_0 h)| < \varepsilon\} \subseteqq W.$$

For $\mu \in M(G)$ put $\tilde{\mu}$ equal to the measure such that $(\tilde{\mu})\hat{} = \bar{\hat{\mu}}$ on Γ and let δ_0 be the identity measure in $M(G)$. Define auxiliary measures by:

$$\nu_i = \mu_i - \hat{\mu}_i(\bar{\gamma}_0 h)\delta_0 \quad \text{and} \quad \sigma_i = \nu_i * \tilde{\nu}_i; \quad i = 1, 2, \ldots, n.$$

Put $\sigma = \sum_{i=1}^{n} \sigma_i$; now, on the one hand, $\hat{\sigma}(h\bar{\gamma}_0) = 0$, while, on the other,

$$(h\sigma)\hat{}\,(\bar{\gamma}_0) = \hat{\sigma}(h\bar{\gamma}_0) \in \hat{\sigma}(\Gamma \setminus E)^-$$

by hypothesis.

We gather from all this that there is a net $(\gamma_\alpha) \subset \Gamma \setminus E$ such that $\hat{\sigma}(\gamma_\alpha) \to 0$. Now given $\varepsilon > 0$ choose α' such that for all $\alpha \geqq \alpha'$

$$|\hat{\sigma}(\gamma_\alpha)| < \varepsilon^2;$$

consequently for all $\alpha \geqq \alpha'$

$$\sum_{i=1}^{n} |\hat{\mu}_i(\gamma_\alpha) - \hat{\mu}_i(\bar{\gamma}_0 h)|^2 < \varepsilon^2.$$

Thus $|\hat{\mu}_i(\gamma_\alpha) - \hat{\mu}_i(\bar{\gamma}_0 h)| < \varepsilon$ for $\alpha \geqq \alpha'$, and so $\gamma_\alpha \in W$ for all $\alpha \geqq \alpha'$.

We have now proved that if $\alpha \geqq \alpha'$, $\gamma_0 \gamma_\alpha \in V \cap (\Gamma \setminus \gamma_0 E)$; thus $h \in (\Gamma \setminus \gamma_0 E)^-$ and this means that (1) implies (2).

Let G be an infinite compact abelian group; a subset $R \subset \Gamma$ is called a *Rajchman set* if whenever $\mu \in M(G)$ and supp $\hat{\mu} \subset R$ then $\mu \in M_0(G)$; here $\hat{}$ is the Fourier—Stieltjes transformation. Examples of Rajchman sets can be found in [7]; all the sets considered in [4, 5] are Rajchman sets.

Example I. If R is a Rajchman set then R satisfies (2) with respect to every idempotent $h \in \bar{\Gamma} \setminus \Gamma$; we point out that this fact is more or less implicit in [3]. To be explicit we need to reproduce some details from [3].

To confirm that R satisfies (2) with respect to every $h=h^2\in\bar{\Gamma}\setminus\Gamma$ we fix an $h_0^2=h_0\in\bar{\Gamma}\setminus\Gamma$ and suppose by way of contradiction that there is a $\gamma_0\in\Gamma$ such that $h_0\notin(\Gamma\setminus\gamma_0 R)^-$. Thus, there is an open set V_0 with $h_0\in V_0$ such that $V_0\cap(\Gamma\setminus\gamma_0 R)$ is empty and $1\notin V_0$. For the remainder of the proof, $^-$ is complex conjugation.

By the definition of the Gelfand topology on $\hat{S}$ there exist measures $\mu_1, \dots, \mu_n\in \in M(G)$ and $\varepsilon>0$ so that $\bigcap_{i=1}^{n}\{\chi\colon |\hat{\mu}_i(\chi)-\hat{\mu}_i(h_0)|<\varepsilon\}$ is open and contained in V_0. Put $A_i=\{z\in\mathbf{C}\colon |z-\hat{\mu}_i(h_0)|<\varepsilon\}$ and consider the open set $\bigcap_{i=1}^{n}\{(h_0\mu_i)\hat{}\}^{-1}(A)$; since $h_0=h_0^2$ it follows that $h_0\in\bigcap_{i=1}^{n}\{(h_0\mu_i)\hat{}\}^{-1}(A_i)\}$ and therefore

$$W_1=\left\{\bigcap_1^n \hat{\mu}_i^{-1}(A_i)\right\}\cap\left\{\bigcap_1^n (h_0\mu_i)\hat{}^{-1}(A_i)\right\}$$

is an open set about h_0. Put $W_1^*=\{\chi\colon \chi\in\overline{W}_1\}$ and define $V_1=W_1\cap W_1^*$; since $h_0=\bar{h}_0$ we see that $V_1\subset V_0$ and V_1 is an open set about h_0. Choose $\beta_1\in\Gamma$ such that $\beta_1,\beta_1^{-1}\in V_1$. Next define $B_1=\{\beta_1,\beta_1^{-1},1\}$; let

$$W_2=\{\chi\colon \chi\in\{(\beta\mu_i)\hat{}\}^{-1}(A_i) \text{ for all } i \text{ and all } \beta\in B_1\}\cap$$
$$\cap\{\chi\colon \chi\in\{(\beta h_0\mu_i)\hat{}\}^{-1}(A_i) \text{ for all } i \text{ and all } \beta\in B_1\}$$

and $V_2=W_2\cap W_2^*$; evidently $V_2\subset V_1$ and $h_0\in V_2$. Since V_2 is open and B_1 is finite we select $\beta_2\in\Gamma$ such that $\beta_2\in V_2\setminus B_1$.

Put $B_2=\{\beta=\prod_{i=1}^{2}\beta_i^{\delta_i}\colon \delta_i\in\{-1,0,1\}\}\cup\{1\}$; let

$$W_3=\{\chi\colon\chi\in\{(\beta\mu_i)\hat{}\}^{-1}(A_i) \text{ for all } i \text{ and all } \beta\in B_2\}\cap$$
$$\cap\{\chi\colon\chi\in\{(\beta h_0\mu_i)\hat{}\}^{-1}(A_i) \text{ for all } i \text{ and all } \beta\in\beta_2\}$$

and $V_3=W_3\cap W_3^*$; evidently $V_3\subset V_2$ and $h_0\in V_3$. Since V_3 is open and B_2 is finite we select $\beta_3\in\Gamma$ such that $\beta_3\in V_3\setminus B_2$. Continuing in this manner we inductively construct a sequence of distinct characters $\langle\beta_j\rangle_1^\infty$ such that $\beta=\prod_{i=1}^{j}\beta_i^{\delta_i}$, $\delta_i\in\{-1,0,1\}$ and $\delta_i\neq 0$ for some i, then $\beta\in V_0$; since $\beta\in V_0\cap\Gamma$, this means that $\beta\gamma_0^{-1}\in R$ for all β of the form $\beta=\prod_{i=1}^{j}\beta_i^{\delta_i}$, $\delta_i\in\{-1,0,1\}$. As shown in [3] (see Theorem 2.8 of [6, p. 21]) there is a dissociate sequence $\langle\omega_p\rangle_1^\infty$ with the property that if ω is of the form $\omega=\prod_{i=1}^{k}\omega_i^{\delta_i}$, $\delta_i\in\{-1,0,1\}$, then ω is also of the form $\omega=\prod_{j=1}^{n}\beta_j^{m_j}$, $m_j\in\{-1,0,1\}$.

Since $\langle\omega_p\rangle_1^\infty$ is dissociate we may now construct a Riesz product $\lambda\in M(G)$ such that $\operatorname{supp}\hat{\lambda}\subset R$ and $\lambda\notin M_0(G)$; this contradicts the fact that R is a Rajchman set and so our discussion is complete.

The above example is not the only one we know: Let **R** denote the additive group of real numbers and let $\varphi : \Gamma \to \mathbf{R}$ be a nontrivial homomorphism. A measure $\mu \in M(G)$ is said to *vanish at infinity in the direction of* φ if whenever $\varphi(\gamma_j) \to +\infty$ then $\hat{\mu}(\gamma_j) \to 0$; denote the set of all measures vanishing at infinity in the direction of φ by $M_\varphi(G)$. A subset $R \subset \Gamma$ is said to be *φ-Rajchman* if for $\mu \in M(G)$ and $\operatorname{supp} \hat{\mu} \subset R \Rightarrow \mu \in M_\varphi(G)$. Then it can be shown that if E is φ-Rajchman, E satisfies (2) with respect to various h's. Notice that in general there are φ-Rajchman sets which are not Rajchman sets; let $\Gamma = \{m + n\sqrt{2} : m, n \in \mathbf{Z}\}$ and let φ be the identity homomorphism of Γ into **R**. Then the set $\{x \in \Gamma : x \geqq 0\}$ is φ-Rajchman but not Rajchman.

Although φ-Rajchman sets and Rajchman sets are the same for the additive group of integers **Z**, there do exist non-Rajchman subsets of **Z** which determine the range of the transform of certain parts of a measure. For the circle group **T** put $\mu = \mu_d + \mu_c$ where $\mu \in M(\mathbf{T})$, μ_d is discrete and μ_c continuous.

Let $\beta(\mathbf{Z})$ denote the Bohr compactification of **Z** and for $E \subset \mathbf{Z}$ let $\bar{E}$ be the closure of E in $\beta(\mathbf{Z})$. Our result is then:

Proposition. *If $E \subset \mathbf{Z}$ and $\mathbf{Z} \setminus \bar{E}$ is dense in $\beta(\mathbf{Z})$ then for $\mu \in M(\mathbf{T})$*

$$\hat{\mu}_d(\mathbf{Z}) \subset \hat{\mu}(\mathbf{Z} \setminus E)^-.$$

Proof. For $\mu \in M(\mathbf{T})$ write $\mu = \mu_d + \mu_c$; fix $0 < \varepsilon < 1$ and $m_0 \in \mathbf{Z} \setminus \bar{E}$. We see from [2] that there is an infinite sequence $\langle m_i \rangle_1^\infty$ of distinct integers satisfying

$$|\hat{\mu}_c(m_0 + m_n - m_j)| < \frac{\varepsilon}{2} \quad \text{for} \quad j < n. \tag{2.1}$$

Put $H = \langle m_i \rangle_1^\infty$ and consider $\bar{H}$ where the closure is of course taken in $\beta(\mathbf{Z})$. Since $\operatorname{card} H = \infty$, there is an $x \notin \mathbf{Z}$ and a net $m_\alpha \in H$, $\alpha \in \Lambda$ such that $m_\alpha \to x \in \beta(\mathbf{Z})$.

Inasmuch as $m_0 \in \mathbf{Z} \setminus \bar{E}$ it follows that there is an $\alpha_0 \in \Lambda$ such that for all α and β greater than α_0

$$m_0 + m_\alpha - m_\beta \notin E \tag{2.2}$$

and

$$|\hat{\mu}_d(m_0) - \hat{\mu}_d(m_0 + m_\alpha - m_\beta)| < \frac{\varepsilon}{2}. \tag{2.3}$$

Notice that (2.3) is valid since $\hat{\mu}_d$ is a continuous function on $\beta(\mathbf{Z})$. As a consequence of (2.2) and (2.3) there is a $k \geqq 1$ and an $r > k$ such that $m_0 + m_r - m_k \notin E$ and $|\hat{\mu}_d(m_0) - \hat{\mu}_d(m_0 + m_r - m_k)| < \frac{\varepsilon}{2}$. Since

$$|\hat{\mu}_d(m_0) - \hat{\mu}(m_0 + m_r - m_k)| \leqq |\hat{\mu}_d(m_0) - \hat{\mu}_d(m_0 + m_r - m_k)| + |\hat{\mu}_c(m_0 + m_r - m_k)|,$$

and $r>k$, we gather from (2.1) that

$$|\hat{\mu}_d(m_0)-\hat{\mu}(m_0+m_r-m_k)| \leqq \varepsilon.$$

Thus $\hat{\mu}_d(\mathbf{Z}\setminus\bar{E})\subset\hat{\mu}(\mathbf{Z}\setminus E)^-$ and since $\mathbf{Z}\setminus\bar{E}$ is dense in $\beta(\mathbf{Z})$ we obtain $\hat{\mu}_d(\mathbf{Z})\subset\subset\hat{\mu}(\mathbf{Z}\setminus E)^-$. The proof is complete.

Example II. Let $\mathbf{N}$ be the natural numbers and for each $n\in\mathbf{N}$ put $E_n=\{m : m=\sum_{i=1}^{n}\delta_i 5^i,\ \delta_i\in\{-1,0,1\}\}$; set $E=\bigcup_{1}^{\infty}E_n$. Let $\mathbf{D}=\{e^{2\pi ik/5^j} : k\in\mathbf{Z}, j\in\mathbf{N}\}$ and consider E as a subset of $\hat{\mathbf{D}}$ where $\mathbf{D}$ is given the discrete topology. Now the integer accumulation points of E in $\hat{\mathbf{D}}$ belong to E so it follows that E is a closed subset of $\mathbf{Z}$ in the relative tolopogy of $\beta(\mathbf{Z})$. Notice that E has natural density zero so by Wiener's Theorem it follows that if supp $\hat{\varrho}\subset E$ then ϱ is continuous and this in turn implies that $\mathbf{Z}\setminus E=\mathbf{Z}\setminus\bar{E}$ is dense in $\beta(\mathbf{Z})$. Clearly E is not a Rajchman set since it contains the spectrum of an infinite Riesz product.

Remark. An easy application of Theorem 1 and Corollary 2 of [5, p. 2] establishes the following assertion: *Let $E\subset\Gamma$ satisfy* (2) *with respect to some* $h=h^2\in\in\bar{\Gamma}\setminus\Gamma$ *and let S be an infinite Sidon subset of Γ; then $E\cup S$ satisfies* (2) *with respect to h.*

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