Acta Sci. Math., 43 (1981), 383—387

A Mal'cev condition for compact congruences to be principal

PAVO ZLATOŠ

The aim of this note is to show that the following property of a variety (equational class) **K** of algebras is equivalent to a Mal'cev condition:

For any algebra $\mathfrak{A}\in\mathbf{K}$ *each compact congruence relation* Φ *on* $\mathfrak{A}$ *is principal.* We shall refer to the property above as "$\mathfrak{A}$ has (PCC)" (principal compact congruences). As usual, a class **K** of algebras has (PCC) iff every $\mathfrak{A}\in\mathbf{K}$ has (PCC). Though the fact that (PCC) defines a Mal'cev class of varieties could be easily proved using the general results on Mal'cev conditions (see [1], [8], [9] or [5, Appendix 3]), we prefer to describe this condition explicitly.

In [5] and [7] a wide use of algebras enjoying (PCC) is made, in particular, they are employed in the proof of the characterization theorem of congruence lattices. In [6], to every algebra $\mathfrak{A}$ an algebra $\overline{\mathfrak{A}}$ with an isomorphic congruence lattice having (PCC) is constructed (see also [5, Exercise 2.30]). In view of this result when studying lattice-theoretical properties of congruence lattices, it is sufficient to deal with algebras having (PCC) since the principal congruences can be better described. The authors in [6] raised the question to characterize those classes **K** of similar algebras having (PCC). We shall solve this problem in the case when **K** is a variety.

Throughout the paper the standard notation and terminology of [5] is used. For the reader's sake, we summarize all the results needed in the following four easy lemmas which will be stated without proof. The first one is actually [5, Lemma 10.6].

Lemma 1. *A congruence relation* Φ *on an algebra* $\mathfrak{A}$ *is compact if and only if it can be represented as a finite join of principal congruences.*

Thus, particularly, principal congruences are compact.

The second lemma is a modified version of [5, Theorem 10.4], describing the smallest congruence $\theta(\mathbf{H})$ on an algebra $\mathfrak{A}$ containing the subset $\mathbf{H}\subseteqq\mathbf{A}\times\mathbf{A}$ (i.e., the binary relation **H** on **A**; see also [5, Theorem 10.3] and the final note in [4]).

Received July 4, 1980.

Lemma 2. *Let $\mathfrak{A}$ be an algebra,* $\mathbf{H}$ *a symmetric binary relation on* $\mathbf{A}$, *and* $x, y \in \mathbf{A}$. *Then*

$$(x, y) \in \theta(\mathbf{H})$$

if and only if for some natural number $m \geqq 1$ *there exist a sequence of pairs* $(a_0, b_0), (a_1, b_1), \ldots, (a_m, b_m) \in \mathbf{H}$ *and a sequence* $g_0, g_1, \ldots, g_m$ *of unary algebraic functions on* $\mathfrak{A}$ *such that for* $i = 0, 1, \ldots, m-1$ *the following algebraic identities hold in* $\mathfrak{A}$:

$$x = g_0(z), \quad g_m(z) = y \quad \textit{for each} \quad z \in \mathbf{A},$$

$$g_i(a_i) = g_{i+1}(a_{i+1}) \quad \textit{for } i \textit{ even},$$

$$g_i(b_i) = g_{i+1}(b_{i+1}) \quad \textit{for } i \textit{ odd}.$$

The third lemma gives in view of Lemma 1 an immediate characterization of (PCC) for single algebras.

Lemma 3. *For every algebra* $\mathfrak{A}$ *the following two conditions are equivalent:*

(i) $\mathfrak{A}$ *has* (PCC);

(ii) *for all* $a_0, a_1, b_0, b_1 \in \mathbf{A}$ *there are* $c, d \in \mathbf{A}$ *such that*

$$\theta(a_0, b_0) \vee \theta(a_1, b_1) = \theta(c, d).$$

The last lemma is rather technical, enabling to state the final result in a "nicer" form.

Lemma 4. *Let* $\mathfrak{A}$ *be an algebra generated by a set* $\mathbf{S} \subseteqq \mathbf{A}$. *For every algebraic function* $g\colon \mathbf{A}^n \to \mathbf{A}$ *on* $\mathfrak{A}$ *there is a natural number* m, *a* $m+n$*-ary polynomial* p *and elements* $s_0, \ldots, s_{m+1}$ *from* $\mathbf{S}$ *such that for all* $a_0, \ldots, a_{n-1} \in \mathbf{A}$ *holds*

$$g(a_0, \ldots, a_{n-1}) = p(s_0, \ldots, s_{m-1}, a_0, \ldots, a_{n-1}).$$

Now, everything is ready to state the promised Mal'cev condition.

Theorem. *For any variety* $\mathbf{K}$ *of algebras the following four conditions are equivalent:*

(i) *There are quaternary polynomials* r *and* s *such that for each algebra* $\mathfrak{A} \in \mathbf{K}$ *and all* $a_0, a_1, b_0, b_1 \in \mathbf{A}$ *holds*

$$\theta(a_0, b_0) \vee \theta(a_1, b_1) = \theta\big(r(a_0, a_1, b_0, b_1),\ s(a_0, a_1, b_0, b_1)\big).$$

(ii) $\mathbf{K}$ *has* (PCC).

(iii) *The free algebra over* $\mathbf{K}$ *with four generators* $\mathbf{F}_{\mathbf{K}}(4)$ *has* (PCC).

(iv) *For some natural numbers* $m \geqq 1$, $n \geqq 1$ *there are quaternary polynomials* r, s, *quinternary polynomials* $p_0^0, p_0^1, p_1^0, p_1^1, \ldots, p_m^0, p_m^1$, $q_0, q_1, \ldots, q_n$ *and a* *function* f: $\{0, 1, ..., n\} \to \{0, 1\}$ *such that for* $i=0, 1, j=0, 1, ..., m-1$ *and* $k=0, 1, ..., n-1$ *the following identities hold in* $\mathbf{K}$:

$$x_i = p_0^i(x_0, x_1, y_0, y_1, z), \quad p_m^i(x_0, x_1, y_0, y_1, z) = y_i,$$

$$p_j^i(x_0, x_0, y_0, y_1, r) = p_{j+1}^i(x_0, x_1, y_0, y_1, r) \quad \textit{for } j \textit{ even},$$

$$p_j^i(x_0, x_1, y_0, y_1, s) = p_{j+1}^i(x_0, x_1, y_0, y_1, s) \quad \textit{for } j \textit{ odd},$$

$$r = q_0(x_0, x_1, y_0, y_1, z), \quad q_n(x_0, x_1, y_0, y_1, z) = s,$$

$$q_k(x_0, x_1, y_0, y_1, x_{f(k)}) = q_{k+1}(x_0, x_1, y_0, y_1, x_{f(k+1)}) \quad \textit{for } k \textit{ even},$$

$$q_k(x_0, x_1, y_0, y_1, y_{f(k)}) = q_{k+1}(x_0, x_1, y_0, y_1, y_{f(k+1)}) \quad \textit{for } k \textit{ odd}.$$

Proof. Applying Lemmas 1—4 we can easily establish the implications (i)⇒(ii)⇒(iii) and (iv)⇒(i). It suffices to prove (iii)⇒(iv). According to Lemma 3, there exist elements $r, s \in \mathbf{F_K}(4)$ (i.e. polynomials in variables x_0, x_1, y_0, y_1, the latter being the free generators of $\mathbf{F_K}(4)$) such that

$$\theta(x_0, y_0) \vee \theta(x_1, y_1) = \theta(r, s).$$

This equality of congruences is equivalent to the conjunction of the following three conditions:

$$(x_i, y_i) \in \theta(r, s) \quad (i = 0, 1)$$

and

$$(r, s) \in \theta(x_0, y_0) \vee \theta(x_1, y_1) = \theta(\{(x_0, y_0), (x_1, y_1)\}).$$

Then, combining Lemmas 2 and 4, these conditions yield the identities.

Finally, we shall present three examples of known varieties enjoying (PCC).

Example 1. The variety of residuated lattices (and hence also the varieties of Heyting algebras and Boolean algebras) has (PCC).

A *residuated lattice* $\mathfrak{L} = (\mathbf{L}; \wedge, \vee, \cdot, \to, 0, 1)$ is an algebra of type (2, 2, 2, 2, 0, 0) such that $(\mathbf{L}; \vee, \wedge, 0, 1)$ is a bounded lattice, $(\mathbf{L}; \cdot, 1)$ is a commutative monoid and the identities

$$x \leqq y \to xy, \quad (x \to y)x \leqq y$$

hold in $\mathfrak{L}$. If in addition the identity

$$(*) \quad x \cdot x = x$$

is satisfied, then $\mathfrak{L}$ is a Heyting algebra. (Note that $(*)$ is equivalent to $xy = x \wedge y$ in the variety of residuated lattices.) Similarly, a residuated lattice satisfying the identity

$$x \vee (x \to 0) = 1$$

(which already implies $(*)$) is a Boolean algebra. For closer discussion see [3].

Let us introduce the polynomial

$$x \leftrightarrow y = (x \vee y) \rightarrow (x \wedge y) = (x \rightarrow y) \wedge (y \rightarrow x).$$

Then the Theorem applies with $m=2$, $n=3$, $f(1)=0$, $f(2)=1$ and

$$r(x_0, x_1, y_0, y_1) = (x_0 \leftrightarrow y_0) \wedge (x_1 \leftrightarrow y_1),$$

$$s(x_0, x_1, y_0, y_1) = 1,$$

$$p_1^i(x_0, x_1, y_0, y_1, z) = (x_i(r \leftrightarrow z) \vee y_i) \wedge (x_i \vee y_i z) \quad (i = 0, 1),$$

$$q_1(x_0, x_1, y_0, y_1\} z) = ((x_0 \leftrightarrow y_0) \vee (z \leftrightarrow y_0)) \wedge (x_1 \leftrightarrow y_1),$$

$$q_2(x_0, x_1, y_0, y_1, z) = (x_1 \leftrightarrow y_1) \vee (x_1 \leftrightarrow z).$$

Example 2. A variety $\mathbf{D}$ is said to be a *discriminator variety* iff there is a ternary polynomial t which is the *ternary discriminator*, i.e.

$$t(x, y, z) = \begin{cases} z & \text{if} \quad x = y, \\ x & \text{if} \quad x \neq y, \end{cases}$$

on every subdirectly irreducible algebra $\mathfrak{A} \in \mathbf{D}$ (see [10]). Assuming $\mathbf{D}$ to be a discriminator variety, let us introduce the following polynomials:

$$T(x, y, z_0, z_1) = t(t(x, y, z_0),\ t(x, y, z_1),\ z_1),$$

and

$$d(x, y, z) = T(x, y, x, z) = t(x, t(x, y, z), z).$$

Hence, T becomes the *normal transform* or *quaternary discriminator*

$$T(x, y, z_0, z_1) = \begin{cases} z_0 & \text{if} \quad x = y, \\ z_1 & \text{if} \quad x \neq y, \end{cases}$$

and d becomes the *dual discriminator*

$$d(x, y, z) = \begin{cases} x & \text{if} \quad x = y, \\ z & \text{if} \quad x \neq y, \end{cases}$$

on every subdirectly irreducible $\mathfrak{A} \in \mathbf{D}$. Applying the Theorem for $m=2$, $n=3$, $f(1)=0$, $f(2)=1$, again, and

$$r(x_0, x_1, y_0, y_1) = t(x_0, y_0, y_1), \quad s(x_0, x_1, y_0, y_1) = t(y_0, x_0, x_1),$$

$$p_1^0(x_0, x_1, y_0, y_1, z) = d(x_0, y_0, z),$$

$$p_1^1(x_0, x_1, y_0, y_1, z) = T(x_0, y_0, t(x_1, z, y_1),\ T(x_0, z, x_1, y_1)),$$

$$q_1(x_0, x_1, y_0, y_1, z) = t(d(x_0, z, y_0),\ t(x_0, z, y_0), y_1),$$

$$q_2(x_0, x_1, y_0, y_1, z) = t(y_0, x_0, z),$$

the fact that $\mathbf{D}$ has (PCC) follows immediately (cf. also [10, Theorem 2.2]).

Example 3. For fundamentals on *lattice-ordered groups* we refer to [2]. We shall use the additive notation without requiring the l-groups to be commutative. Let us introduce the polynomial

$$|x| = -x \vee x.$$

The well known fact that l-groups have (PCC) (see [2, Theorem XIII. 18]) follows then from our Theorem by putting $m=n=3, f(1)=0, f(2)=1$ and

$$r(x_0, x_1, y_0, y_1) = |x_0-y_0|+|x_1-y_1|, \quad s(x_0, x_1, y_0, y_1) = 0,$$

$$\left.\begin{aligned} p_1^i(x_0, x_1, y_0, y_1, z) &= (z\wedge(x_i-y_i))+y_i, \\ p_2^i(x_0, x_1, y_0, y_1, z) &= (z\wedge(y_i-x_i))+x_i, \end{aligned}\right. \quad (i = 0,1)$$

$$q_1(x_0, x_1, y_0, y_1, z) = |z-y_0|+|x_1-y_1|,$$

$$q_2(x_0, x_1, y_0, y_1, z) = |x_1-z|.$$

References

[1] J. T. Baldwin and J. Berman, A model-theoretic approach to Mal'cev conditions, *J. Symbolic Logic*, **42** (1977), 277—288.

[2] G. Birkhoff, *Lattice Theory*, 3rd edition, American Mathematical Society (Providence, Rhode Island, 1967).

[3] R. P. Dilworth and M. Ward, Residuated lattices, *Trans. Amer. Math. Soc.*, **45** (1939), 334—354.

[4] G. Grätzer, Two Mal'cev-type theorems in universal algebra, *J. Comb. Theory*, **8** (1970), 334—342.

[5] G. Grätzer, *Universal Algebra*, 2nd edition, Springer-Verlag (New York—Heidelberg—Berlin, 1979).

[6] G. Grätzer and E. T. Schmidt, On inaccessible and minimal congruence relations I, *Acta Sci. Math.*, **21** (1960), 337—342.

[7] G. Grätzer and E. T. Schmidt, Characterization of congruence lattices of abstract algebras, *Acta Sci. Math.*, **24** (1963), 34—59.

[8] W. Neumann, On Mal'cev conditions, *J. Austral. Math. Soc.*, **17** (1974), 376—384.

[9] W Taylor, Characterizing Mal'cev conditions, *Alg. Univ.*, **3** (1973), 351—397.

[10] H. Werner, Discriminator Algebras, *Studien zur Algebra und ihre Anwendungen* **6**. Akademie-Verlag (Berlin, 1978).

KATEDRA ALGEBRY A TEÓRIE ČÍSEL
PRÍRODOVEDECKÁ FAKULTA UNIVERZITY KOMENSKÉHO
MATEMATICKÝ PAVILÓN, MLYNSKÁ DOLINA
81631 BRATISLAVA, CZECHOSLOVAKIA