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On the orbit structure of orthogonal actions with isotropy subgroups of maximal rank

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Some of the basic concepts and facts concerning compact Lie groups are naturally derived by applying results from the theory of compact transformation groups. In fact, if G is a compact semisimple Lie group and $\mathfrak{g}$ its Lie algebra then the adjoint action

$$\mathrm{Ad}\colon G\times\mathfrak{g}\to\mathfrak{g}$$

yields a natural setting for the introduction and study of such concepts as the Cartan subalgebras, the Weyl chambers and the Weyl group of G which in turn yield a description of the orbit structure of the adjoint action ([3] pp. 17—32). It will be shown below that an analogous procedure can be carried out in a more general setting. Actually, let

$$\alpha\colon G\times\mathbf{R}^n\to\mathbf{R}^n$$

be an orthogonal action of a compact connected semisimple Lie group G such that the isotropy subgroups of α are of maximal rank. Then concepts can be introduced concerning the action α which reduce to the Cartan subalgebras, the Weyl chambers and to the Weyl group of G in that special case when α is an adjoint action. Moreover, these general concepts yield such a description of the orbit structure of the action α which can be considered as an extension of the description of the orbit structure of the adjoint actions in terms of Weyl chambers and Weyl groups.

1. Some basic facts concerning orthogonal actions with isotropy subgroups of maximal rank

If $\alpha\colon G\times\mathbf{R}^n\to\mathbf{R}^n$ is an action of class C^∞ of a compact connected Lie group G, then the action α is said to be *orthogonal* provided that the transformation

$$\alpha_g\colon \mathbf{R}^n\to\mathbf{R}^n$$

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defined by $\alpha_g(x)=\alpha(g, x)$, $x\in\mathbf{R}^n$ is orthogonal for every $g\in G$. Let $z\in\mathbf{R}^n$ and consider for any point x of the orbit $G(z)$, which is an embedded submanifold of class C^∞ in $\mathbf{R}^n$, the orthogonal decomposition

$$T_x\mathbf{R}^n = N_x \oplus T_x G(z),$$

then $N(z)=\bigcup\{N_x | x\in G(z)\}$ is canonically a subbundle of class C^∞ in $T\mathbf{R}^n$ and it is called the *normal bundle* of the orbit $G(z)$. Consequently, the exponential map

$$\exp\colon T\mathbf{R}^n \to \mathbf{R}^n$$

restricted to $N(z)$ is a map $\varepsilon_z\colon N(z)\to\mathbf{R}^n$ of class C^∞. If $x\in G(z)$ then both N_x and $\varepsilon_z(N_x)$ are called the *normal subspace* to the orbit $G(z)$ at x. If, in particular, $G(z)$ is a principal orbit then the normal subspace $\varepsilon_z(N_x)$ of $G(z)$ for any $x\in G(z)$ intersects every orbit of α in consequence of the Principal Orbit Type Theorem. Moreover, in the special case when α is the adjoint action

$$\mathrm{Ad}\colon G\times\mathfrak{g} \to \mathfrak{g}$$

of a compact connected Lie group G and the orbit of $Z\in\mathfrak{g}$ is principal then the normal subspace to the orbit at any of its point X is equal to the uniquely defined Cartan subalgebra of $\mathfrak{g}$ containing X ([3] pp. 20—22).

Consider now an orthogonal action $\alpha\colon G\times\mathbf{R}^n\to\mathbf{R}^n$ such that its isotropy subgroups are of maximal rank and let $z\in\mathbf{R}^n$ be a point such that $G(z)$ is principal. If $x\in G(z)$ then the normal subspace N_x to $G(z)$ at x is the unique complement of $T_xG(z)$ in $T_x\mathbf{R}^n$ which is mapped onto itself by every transformation

$$T_x\alpha_g\colon T_x\mathbf{R}^n \to T_x\mathbf{R}^n, \quad g\in G_x$$

according to an earlier observation, where G_x is the isotropy subgroup of α at x [6]. Consequently, the observation yields that the action has a unique maximal slice at the point x. The following lemma is based on this observation.

1.1. Lemma. *Let $\alpha\colon G\times\mathbf{R}^n\to\mathbf{R}^n$ be an orthogonal action such that its isotropy subgroups are of maximal rank and $z\in\mathbf{R}^n$ a point such that $G(z)$ is a principal orbit. Then the set of elements $g\in G$ such that α_g maps the normal subspace $\varepsilon_z(N_z)$ of $G(z)$ onto itself is equal to the normalizer $N(G_z)$ of G_z in G.*

Proof. Consider first an element $g\in G$ such that α_g maps $\varepsilon_z(N_z)$ onto itself. If $h\in G_z$ then the transformation defined by $g^{-1}hg$ leaves every point of $\varepsilon_z(N_z)$ fixed since the orbit $G(z)$ is principal. Consequently, $g^{-1}hg\in G_z$ holds, but then g is an element of the normalizer of G_z. Consider secondly an element a of the normalizer $N(G_z)$. If now $h\in G_z$ then

$$a^{-1}ha = h'\in G_z$$

is valid and consequently the transformation $\alpha_h = \alpha_a \circ \alpha_{h'} \circ \alpha_a^{-1}$ maps the subspace $\alpha_a(\varepsilon_z(N_z))$ onto itself. But then $\alpha_a(\varepsilon_z(N_z))$ yields a maximal slice of the action α at z. Since by above mentioned observation α has a unique maximal slice at z, now

$$\alpha_a(\varepsilon_z(N_z)) = \varepsilon_z(N_z)$$

follows and consequently the element a has the property which is required by the lemma.

On account of the preceding lemma a concept can be introduced which has a basic role in deriving the subsequent results. Let $\alpha\colon G\times\mathbf{R}^n \to \mathbf{R}^n$ be an orthogonal action such that its isotropy subgroups are of maximal rank and $z\in\mathbf{R}^n$ a point such that $G(z)$ is a principal orbit. Then on account of the preceding lemma the restriction

$$N(G_z)\times\varepsilon_z(N_z) \to \varepsilon_z(N_z)$$

of the action α to the normal subspace $\varepsilon_z(N_z)$ of $G(z)$ can be considered. Since the orbit $G(z)$ is principal, the kernel of the restricted action, that is, the set of those elements $g\in N(G_z)$ for which the restriction of α_g to $\varepsilon_z(N_z)$ is the identity, is equal to G_z. Consequently the restricted action defines an effective action

$$v\colon A\times\varepsilon_z(N_z) \to \varepsilon_z(N_z)$$

of the group $A = N(G_z)/G_z$ on the normal space. Since G is compact and $G_z \subset G$ of maximal rank, the group A is finite by a basic result ([4] pp. 66—70) and the action v, being defined by the restriction of an orthogonal action to a subspace, is orthogonal. In the special case of the adjoint action

$$\mathrm{Ad}\colon G\times\mathfrak{g} \to \mathfrak{g}$$

of a compact connected semisimple Lie group G the group A obviously reduces to the Weyl group of G and the action v is equal to the canonical action of the Weyl group on the Cartan subalgebra $\mathfrak{h}\subset\mathfrak{g}$ to which the normal subspace $\varepsilon_z(N_z)$ reduces ([3] pp. 20—22).

Consider an orthogonal action $\alpha\colon G\times\mathbf{R}^n \to \mathbf{R}^n$ and an arbitrary point $z\in\mathbf{R}^n$. Since the orbit $G(z)$ is an embedded submanifold of class C^∞ in $\mathbf{R}^n$, the standard definition of cut points and focal points of submanifolds applies to $G(z)$. As it has been pointed out earlier if the isotropy subgroups of α are of maximal rank and if the orbit $G(z)$ is principal then the singular orbits of α are closely related to the focal locus of the orbit $G(z)$ [8]. The following lemma presents one of these earlier results referred to, which will be applied subsequently.

1.2. Lemma. *Let $\alpha\colon G\times\mathbf{R}^n \to \mathbf{R}^n$ be an orthogonal action such that its isotropy subgroups are of maximal rank and $z\in\mathbf{R}^n$ a point such that $G(z)$ is a principal orbit.*

Consider a unit vector $s \in N_z$ *and assume that* $x \in \mathbf{R}^n$ *is first focal point of the principal orbit* $G(z)$ *on the ray*

$$z + \tau\varepsilon(s), \quad \tau \geqq 0.$$

Then the orbit $G(x)$ *of the action* α *is a singular one.*

A proof of the above lemma was obtained by application of some results concerning the relation of focal points and some Jacobi fields [8].

2. The construction of the cut locus of a principal orbit as the union of some subspaces

Let $\alpha\colon G \times \mathbf{R}^n \to \mathbf{R}^n$ be an orthogonal action such that its isotropy subgroups are of maximal rank, $z \in \mathbf{R}^n$ a point such that the orbit $G(z)$ is principal and k the dimension of this orbit. Consider now a point x of the $(n-k)$-dimensional normal subspace $\varepsilon_z(N_z)$ such that $G(x)$ is principal too. It will be shown below that under an additional assumption the intersection of the cut locus of $G(x)$ with $\varepsilon_z(N_z)$ can be obtained as the union of a finite number of $(n-k-1)$-dimensional subspaces of $\mathbf{R}^n$. As subsequent observations exhibit under the additional assumption referred to, the intersection of the cut locus of the principal orbit $G(x)$ with the normal subspace $\varepsilon_z(N_z)$ is equal to the set of those points of $\varepsilon_z(N_z)$ which do not have principal orbits. Thus by studying the cut locus of a principal orbit, results concerning the orbit structure of the action α are to be obtained.

In studying the cut locus of a principal orbit first that case will be treated where a cut point is a first focal of the orbit. As it has been observed such focal points are conveniently described by some vectors which have been called critical vector- of the orbit [6], [7]. A derivation of these critical vectors is presented here in a somes what changed setting for sake of subsequent applications.

Let $\alpha\colon G \times \mathbf{R}^n \to \mathbf{R}^n$ be an orthogonal action with isotropy subgroups of maximal rank, $z \in \mathbf{R}^n$ a point such that $G(z)$ is principal and consider an arbitrary point $x \in \varepsilon_z(N_z)$. Since the orbit $G(x) \subset \mathbf{R}^n$ is an embedded submanifold of class C^∞, the second fundamental tensor

$$\omega_x\colon T_x G(x) \times T_x G(x) \to N_x$$

of $G(x)$ at x can be considered. Moreover, a simple argument yields that

$$\omega_x(T_x\alpha_g u, T_x\alpha_g v) = T_x\alpha_g\, \omega_x(u, v)$$

holds for $u, v \in T_x G(x)$ and $g \in G_x$. Therefore, if $s \in N_x$ is a unit vector which is left invariant by $T_x\alpha_g$ for a $g \in G_x$ then

$$\langle s, \omega_x(T_x\alpha_g u, T_x\alpha_g v)\rangle = \langle T_x\alpha_g s, T_x\alpha_g\, \omega_x(u, v)\rangle = \langle s, \omega_x(u, v)\rangle$$

holds for $u, v \in T_x G(x)$. In other words, the second fundamental form of $G(x)$ taken at x in the direction of s is left invariant by $T_x \alpha_g$ for the $g \in G_x$ considered. Assume now that $H \subset G_x$ is a subgroup such that every element of N_x is left invariant by $T_x \alpha_g$ for $g \in H$. Fix a unit vector $s \in N_x$ and consider the set

$$\{\lambda_i(s) | i = 1, \ldots, p\}$$

of eigenvalues of the second fundamental form of $G(x)$ taken at x in the direction of s. Then those eigenvectors of this second fundamental form which have $\lambda_i(s)$ as eigenvalue form a subspace $E_i(s) \subset T_x G(x)$ and consequently a decomposition into direct sum

$$T_x G(x) = \oplus \{E_i(s) | i = 1, \ldots, p\}$$

of mutually orthogonal subspaces is obtained. Owing to the above mentioned invariance of the second fundamental form, these subspaces $E_i(s)$, $i=1, \ldots, p$ are left invariant by the representation

$$T_x \alpha_g \colon T_x G(x) \to T_x G(x), \quad g \in H.$$

Consequently, the subspaces $E_i(s)$, $i=1, \ldots, p$ themselves are direct sums of irreducible subspaces of the above representation and thus a decomposition into direct sum

$$T_x G(x) = \oplus \{A_l | l = 1, \ldots, r\}$$

of irreducible subspaces of the considered representation is obtained. Assume now that the decomposition of $T_x G(x)$ into direct sum of irreducible subspaces of the considered representation is unique up to the order of the terms. Then the dependence of the eigenvalues $\lambda_i(s)$, $i=1, \ldots, p$ on the unit vector $s \in N_x$ can be easily described. In fact, fix an orthonormal base $(e_1, \ldots, e_k)$ of $T_x G(x)$ which is compatible with the above decomposition into direct sum of irreducible subspaces, and let $\mu_j(s)$ be the eigenvalue of the eigenvector e_j of the second fundamental form of $G(x)$ taken at x in the direction of s for $j=1, \ldots, k$. Then this second fundamental form in the chosen base is given by

$$\langle s, \omega_x(u, v) \rangle = \sum_{j=1}^{k} \mu_j(s) u^j v^j$$

for

$$u = \sum_{j=1}^{k} u^j e_j \quad \text{and} \quad v = \sum_{j=1}^{k} v^j e_j.$$

Moreover, fix an orthonormal base $(s_1, \ldots, s_{n-k})$ of N_x too and put $\mu_{jq} = \mu_j(s_q)$ for $j=1, \ldots, k$ and $q=1, \ldots, n-k$. Then as an obvious calculation shows the following is valid:

$$\mu_j(s) = \sum_{q=1}^{n-k} \mu_{jq} \tau^q \quad \text{where} \quad s = \sum_{q=1}^{n-k} \tau^q s_q.$$

Consider, therefore, those vectors $\bar{w}_j$, $j=1, \ldots, k$ which are defined as follows:

$$\bar{w}_j = \sum_{q=1}^{n-k} \mu_{jq} s_q \quad \text{for} \quad j = 1, \ldots, k.$$

As a simple calculation shows, the above vectors $\bar{w}_j$, $j=1, \ldots, k$ do not depend on the choice of the base $(s_1, \ldots, s_{n-k})$. Moreover, those vectors $\bar{w}_j$ for which the corresponding base vectors e_j are in one and the same irreducible subspace A_l are evidently equal. Consequently, the system of vectors $\bar{w}_j$, $j=1, \ldots, k$ reduces to a system of vectors w_l, $l=1, \ldots, r$. Restrict now to that special case when the point $x \in \varepsilon_z(N_z)$ is such that $G(x)$ is principal and when $H=G_x$. Then both of the above made two assumptions hold; in fact, every element of N_x is left fixed by $T_x\alpha_g$ for $g \in G_x$ since $G(x)$ is principal and the decomposition of $T_xG(x)$ into direct sum of irreducible subspaces of the representation

$$T_x\alpha_g \colon T_xG(x) \to T_xG(x), \quad g \in G_x$$

is unique up to the order of terms since the subgroup $G_x \subset G$ is of maximal rank [7]. The vectors w_l, $l=1, \ldots, r$ thus obtained are called the *critical vectors of the principal orbit* $G(x)$ at $x \in \varepsilon_z(N_z)$. In the special case of the adjoint action

$$\mathrm{Ad} \colon G \times \mathfrak{g} \to \mathfrak{g}$$

of a compact connected semisimple Lie group G the critical vectors of a principal orbit can be explicitely given in terms of the root vectors [7].

The second fundamental tensor of the principal orbit $G(x)$ at x can be expressed in terms of the critical vectors of the orbit as follows:

$$\omega_x(u, v) = \sum_{q=1}^{n-k} \langle s_q, \omega_x(u, v) \rangle s_q = \sum_{q=1}^{n-k} \left(\sum_{j=1}^{k} \mu_{jq} u^j v^j \right) s_q =$$

$$= \sum_{j=1}^{k} \bar{w}_j u^j v^j = \sum_{l=1}^{r} w_l \langle u_l, v_l \rangle$$

where u_l, v_l are respectively the projections of the vectors $u, v \in T_xG(x)$ on the subspace A_l for $l=1, \ldots, r$.

Let $x'=x+\varepsilon_z(t) \in \varepsilon_z(N_z)$ be such that $G(x')$ is principal, then the critical vectors of the principal orbit $G(x')$ at x' are given by those of $G(x)$ at x as follows:

$$w_l' = w_l + t, \quad l = 1, \ldots, r$$

owing to the expression of the second fundamental tensor in terms of second derivatives of a parameter representation of $G(x')$ and to the linearity of the orthogonal action α.

The intersection of the focal locus of the principal orbit $G(x)$ with the normal subspace $\varepsilon_z(N_z)$ can be conveniently described with the stand-by of the critical vectors of the orbit. In fact, let $s \in N_x$ be a unit vector, then the focal points of the orbit $G(x)$ on the line

$$x + \tau\varepsilon_x(s), \quad \tau \in \mathbf{R}$$

are attained by those values $\tau_1(s), \ldots, \tau_p(s)$ of τ which are given by the not vanishing eigenvalues $\lambda_i(s)$, $i=1, \ldots, p$ of the second fundamental form of $G(x)$ taken at x in the direction of s in the following way:

$$\tau_i(s) = \frac{1}{\lambda_i(s)} = \frac{1}{\langle \overline{w}_i, s \rangle} \quad \text{where} \quad i = 1, \ldots, p$$

(see e.g. [5] pp. 32—38). Thus the focal locus of $G(x)$ is completely determined by the critical vectors of the orbit. Consequently, the first focal point of $G(x)$ on the line considered is obviously attained by that value of τ which satisfies the following condition:

$$|\tau| = \min \left\{ \frac{1}{|\langle w_l, s \rangle|} \,\middle|\, l = 1, \ldots, r \right\}.$$

Consider now the unit sphere $S(x)$ of the normal subspace N_x centered at the origin and define the map $f_l\colon S(x) \to \varepsilon_x(N_x)$ for $l=1, \ldots, r$ as follows

$$f_l(s) = \frac{1}{\langle w_l, s \rangle} \varepsilon_x(s) \quad \text{where} \quad s \in S(x).$$

The image of f_l is obviously an $(n-k-1)$-dimensional flat F_l of $\mathbf{R}^n$ lying in the normal subspace $\varepsilon_z(N_z) = \varepsilon_x(N_x)$ which is orthogonal to $\varepsilon_x(w_l)$ and intersects the ray

$$x + \tau\varepsilon_x(w_l), \quad \tau \geqq 0$$

at the distance $|w_l|^{-1}$ from the point x. Consequently, the intersection of the focal locus of the principal orbit $G(x)$ with the normal subspace $\varepsilon_z(N_x) = \varepsilon_z(N_z)$ is given by the union of the $(n-k-1)$-flats F_l, $l=1, \ldots, r$. These $(n-k-1)$-flats do not depend on the choice of the point x in the normal subspace $\varepsilon_z(N_z)$ in consequence of the above already given dependence of the critical vectors w_l, $l=1, \ldots, r$ on the point x. Therefore the points of F_l, $l=1, \ldots, r$ are on singular orbits of α in consequence of 1.2. Lemma, since if $x \in \varepsilon_z(N_z)$ is appropriently chosen a point of F_l is first focal point of the principal orbit $G(x)$. Conversely, any point of $\varepsilon_z(N_z)$ which lies on a singular orbit is point of an F_l for some $l=1, \ldots, r$; in fact, at a point of a singular orbit of α a suitably chosen infinitesimal isometry of α vanishes and, since infinitesimal isometries are Jacobi fields, the well-known relation of Jacobi fields and focal points yields the assertion. The $(n-k-1)$-flats F_l, $l=1, \ldots, r$ are passing through the origin of $\mathbf{R}^n$. In fact, those homotheties of $\mathbf{R}^n$ which leave the origin fixed are equivariant with respect to the action α since this action is orthogonal. Therefore, these homotheties map singular orbits to singular orbits and consequently they map an element of the system F_l, $l=1, ..., r$ to another element of this system. Since this observation holds for any homothety of $\mathbf{R}^n$ leaving the origin fixed, the $(n-k-1)$-flats F_l, $l=1, ..., r$ are passing through the origin of $\mathbf{R}^n$. On account of the above observations the $(n-k-1)$-flats F_l, $l=1, ..., r$ are called the *singular* $(n-k-1)$*-dimensional subspaces* of the action α in the normal subspace $\varepsilon_z(N_z)$. In the special case of the adjoint action

$$\mathrm{Ad}\colon G\times\mathfrak{g}\to\mathfrak{g}$$

of a compact connected semisimple Lie group G the singular $(n-k-1)$-dimensional subspaces of the action in a Cartan subalgebra $\mathfrak{h}=\varepsilon_z(N_z)$ reduce to the walls of the Weyl chambers of G in this Cartan subalgebra ([3] pp. 17—23). Consequently the union of the singular $(n-k-1)$-dimensional subspaces is equal to the set of singular points of $\mathfrak{g}$ in the Cartan subalgebra $\mathfrak{h}$.

The following lemma yields an important property of the singular subspaces which has essential consequences for the subsequent results as well.

2.1. Lemma. *Let $\alpha\colon G\times\mathbf{R}^n\to\mathbf{R}^n$ be an orthogonal action such that its isotropy subgroups are of maximal rank, $z\in\mathbf{R}^n$ a point such that $G(z)$ is principal and k the dimension of this orbit. If F is a singular $(n-k-1)$-dimensional subspace of the action α in the normal subspace $\varepsilon_z(N_z)$ then there is an element g of $N(G_z)$ such that the restriction of α_g to $\varepsilon_z(N_z)$ is equal to the reflection of $\varepsilon_z(N_z)$ on F.*

Proof. Consider a point $c\in F$ which does not lie on the intersection of F with another singular $(n-k-1)$-dimensional subspace and the line L of $\mathbf{R}^n$ which lies in $\varepsilon_z(N_z)$ passes through c and is perpendicular to F. If $x\in L$ is sufficiently near to c then $G(x)$ is principal and x is a nearest point of $G(x)$ to c. Since c is a nearest point of $G(c)$ to $G(x)$, conversely x is a nearest point of $G(x)$ to $G(c)$ and therefore to c. Consider a point $\bar{c}$ of L such that $G(\bar{c})$ is principal and $xc\bar{c}$ is valid and let $\bar{x}$ be a nearest point of $G(x)$ to $\bar{c}$. Then $\bar{x}\in\varepsilon_z(N_z)$ holds since $\bar{c}$ is a nearest point of $G(\bar{c})$ to $\bar{x}$. Moreover, $\bar{x}$ cannot be on the same side of F in $\varepsilon_z(N_z)$ as x, since in that case the minimal segment $\bar{c}\bar{x}$ would contain a focal point of $G(\bar{x})$. If $\bar{c}$ converges to c on L then the corresponding points $\bar{x}$ have a point of accumulation x', which is a nearest point of $G(x)$ to c but is not on the same side of F in $\varepsilon_z(N_z)$ as x. Consider now an element $g\in G$ such that

$$x' = \alpha(g, x)$$

is valid. Then α_g maps $\varepsilon_z(N_z)=\varepsilon_x(N_x)=\varepsilon_{x'}(N_{x'})$ onto itself and consequently $g\in N(G_z)$ by 1.1. Lemma. Moreover α_g maps F onto a singular $(n-k-1)$-dimensional subspace of α in $\varepsilon_z(N_z)$; consequently, α_g maps F onto itself provided that

x sufficiently near to c and c is sufficiently far from the other singular $(n-k-1)$-dimensional subspaces in $\varepsilon_z(N_z)$. Thus x and x' have the same distance from F. But x and x' have the same distance from c. Therefore x' lies on L and α_g leaves c fixed. Assume now that the restriction of α_g to $\varepsilon_z(N_z)$ is not equal to the reflection of $\varepsilon_z(N_z)$ on F for every g satisfying the above conditions. In this case there is a sequence $\{c_i|i\in \mathbf{N}\}$ of different points of F and a sequence $\{x_i|i\in \mathbf{N}\}$ of different points of $\varepsilon_z(N_z)$ satisfying analogous conditions to those satisfied by c and x, and a corresponding sequence $\{g_i|i\in \mathbf{N}\}$ of elements of $N(G_z)$ such that the restriction of α_{g_i} to $\varepsilon_z(N_z)$ is not equal to the reflection of $\varepsilon_z(N_z)$ on F for $i\in \mathbf{N}$. Since the points $c_i\in F$ are arbitrary they can be selected so as to make the action of the elements of $\{g_i|i\in \mathbf{N}\}$ on $\varepsilon_z(N_z)$ different; in fact since α_g is not a reflection on F, the set of its fixed points in $\varepsilon_z(N_z)$ is a nowhere dense set, consequently c_i can be chosen so that it is not left fixed by α_{g_j} for $j=1, \dots, i-1$. Since the group $A=N(G_z)/G_z$ is finite by a former observation, a contradiction is obtained. Thus, there is a $g\in N(G_z)$ such that the restriction of α_g to $\varepsilon_z(N_z)$ is equal to the reflection of $\varepsilon_z(N_z)$ on F.

In the special case when α is the adjoint action of a compact connected semi-simple Lie group the preceding lemma reduces to the well-known fact that the Weyl group contains the reflections on the walls of the Weyl chambers ([2] pp. 17—23).

Consider now the general case of an orthogonal action $\alpha\colon G\times\mathbf{R}^n\to\mathbf{R}^n$ such that the isotropy subgroups of the action are of maximal rank, fix a point $z\in\mathbf{R}^n$ such that $G(z)$ is principal and let k be the dimension of this orbit. Let now $\hat{S}\subset N(G_z)$ be the set of those elements g for which α_g when restricted to $\varepsilon_z(N_z)$ yields reflection of $\varepsilon_z(N_z)$ on one of the singular $(n-k-1)$-dimensional subspaces of α lying in $\varepsilon_z(N_z)$; moreover let $\hat{W}\subset N(G_z)$ be the subgroup generated by $\hat{S}$. Put now $W=\hat{W}/G_z$ and $S=\hat{S}/G_z$ then (W, S) is obviously a Coxeter system ([1] pp. 72—89). The group W which is defined up to isomorphisms, will be called the *generalized Weyl group of the action* α. According to a basic result the group W admits a decomposition into a direct product $W=W_1\times\dots\times W_s$ and the vector space $\varepsilon_z(N_z)=N_z$ into a direct sum $N_z=T_0\oplus T_1\oplus\dots\oplus T_s$ of orthogonal subspaces such that the action of W on $T_1, \dots, T_s$ is irreducible and non-trivial ([1] pp. 81—83). The subgroups W_p, $p=1, \dots, s$ which are generated by reflections ([1] pp. 83—85) will be called the *irreducible factors of the generalized Weyl group of the action* α. According to a result of H. S. M. Coxeter these irreducible factors of the generalized Weyl group can be of the following types:

$$A_n,\, n\geqq 1;\; B_n,\, n\geqq 4;\; C_n,\, n\geqq 3;\; D_2^6;\; E_6, E_7, E_8;\; F_4;\; G_3;\; G_4.$$

The additional assumption referred to above can be given now as follows: The irreducible factors of the generalized Weyl groups of the action α are all of different type and no one of them is of type B_4.

Returning now to the study of the intersection of the cut locus of a principal orbit $G(x)$ with the normal subspace $\varepsilon_z(N_z)$ that case will be considered when the cut locus contains a point which is not a focal point of the principal orbit.

Consider therefore an orthogonal action $\alpha\colon G\times\mathbf{R}^n\to\mathbf{R}^n$ such that its isotropy subgroups are of maximal rank and the irreducible factors of its generalized Weyl group are all of different type and no one of them is of type B_4, let a point $z\in\mathbf{R}^n$ be such that $G(z)$ is principal and let k be the dimension of $G(z)$. Assume now that there is a point $x\in\varepsilon_z(N_z)$ such that $G(x)$ is an exceptional orbit and consider an element

$$g\in G_x-G_z.$$

Let $y\in\varepsilon_z(N_z)$ be such a point that $G(y)$ is principal. Then $y'=\alpha(g,y)$ is different from y and $y'\in\varepsilon_z(N_z)$ since G_x maps $\varepsilon_z(N_x)$ onto itself. Consider now the set F of those points of $\varepsilon_z(N_z)$ which have the same distance from y and y'. Since F is obviously an $(n-k-1)$-flat which contains the point x and the origin of $\mathbf{R}^n$, it is an $(n-k-1)$-dimensional subspace of $\mathbf{R}^n$. Anticipating some facts to be proved below, F is called the *$(n-k-1)$-dimensional exceptional subspace* of the action α in the normal subspace $\varepsilon_z(N_z)$ passing through the exceptional point x. If the point y is sufficiently near to x then y and y' are nearest points of the orbit $G(y)$ to x and consequently y and y' are nearest points of $G(y)$ to points of F which are sufficiently near to x. Thus a neighborhood of x in F is a subset of the cut locus of the principal orbit $G(y)$. Actually, the exceptional subspaces have but a seeming existence under the above assumptions as a subsequent result shows. In fact, they are introduced here in order to show the non-existence of exceptional orbits by contradiction.

The following lemma is a counterpart of the preceding one for the case of exceptional subspaces of an orthogonal action.

2.2. Lemma. *Let $\alpha\colon G\times\mathbf{R}^n\to\mathbf{R}^n$ be an orthogonal action such that its isotropy subgroups are of maximal rank, the irreducible factors of its generalized Weyl group are all of different type and no one of them is of type B_4. Let $z\in\mathbf{R}^n$ be such that $G(z)$ is principal and let F be an exceptional $(n-k-1)$-dimensional subspace of α in $\varepsilon_z(N_z)$. Then there is an element $g\in N(G_z)$ such that the restriction of α_g to $\varepsilon_z(N_z)$ is equal to the reflection of $\varepsilon_z(N_z)$ on F.*

Proof. Consider the decomposition $W=W_1\times\ldots\times W_s$ of the generalized Weyl group of α into direct product of irreducible factors and the decomposition $\varepsilon_z(N_z)=$ $=T_0\oplus T_1\oplus\ldots\oplus T_s$ of the normal space into direct sum of orthogonal subspaces already defined above. Let now $x\in\varepsilon_z(N_z)$ be such that the orbit $G(x)$ is exceptional. Then x is contained in the interior of a chamber C defined by the singular $(n-k-1)$-dimensional subspaces of α in $\varepsilon_z(N_z)$. Consider now an element g such that $g\in G_x-G_z$

holds. Then α_g maps $\varepsilon_z(N_z)$ onto itself and interchanges the singular $(n-k-1)$-dimensional subspaces of α in $\varepsilon_z(N_z)$ among themselves. Consequently α_g maps the chamber C onto itself. But then α_g induces an automorphism of the graph associated with the Coxeter system (W, S) which is the union of its connected components the graphs associated with the irreducible factors. Since the irreducible factors are of different types the automorphism of the graph is composed of automorphisms of its connected components. But these connected components can be only of the following types:

A_n, $n = 1$

B_n, $n = 4$

E_6

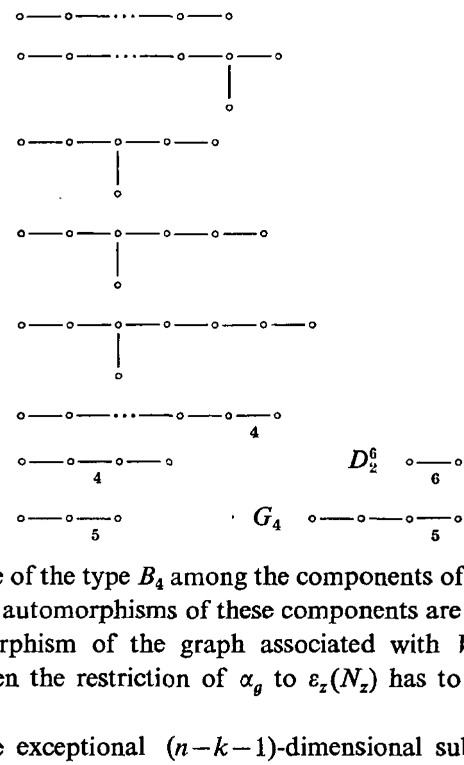
E_7

E_8

C_n

F_4

G_3



Since there is none of the type B_4 among the components of the graph associated with W the non-trivial automorphisms of these components are involutorious. Consequently the automorphism of the graph associated with W induced by α_g is involutorious. But then the restriction of α_g to $\varepsilon_z(N_z)$ has to be an involutorious isometry.

Consider now the exceptional $(n-k-1)$-dimensional subspace F of α lying in $\varepsilon_z(N_z)$ passing through x and defined by a point $y \in \varepsilon_z(N_z)$ such that $G(y)$ is principal. Then α_g interchanges the points y and $y''=\alpha(g, y)$ and consequently it maps F onto itself. Thus F interchanges the sides of F in $\varepsilon_z(N_z)$ as well. Assume now that the restriction of α_g to $\varepsilon_z(N_z)$ is not equal to the reflection of $\varepsilon_z(N_z)$ on F. Then there is a point $y^* \in \varepsilon_z(N_z)$ such that y^*, $\alpha(g, y^*)$ are different and the line of $\mathbf{R}^n$ passing through these two points is not perpendicular to F, moreover the point y^* can be chosen so near to x as to render the preceding requirements valid. Then the above construction applied to y^* yields an $(n-k-1)$-dimensional exceptional subspace F^* of α in $\varepsilon_z(N_z)$ which is different from F. Since by the above argument α_g maps F^* onto itself and interchanges the sides of F^* in $\varepsilon_z(N_z)$ a contradiction is obtained. Therefore, α_g restricted to $\varepsilon_z(N_z)$ is equal to the reflection on F.

The following obvious corollary of the preceding lemma has important conrequences which are given subsequently.

Corollary. *Let $\alpha\colon G\times\mathbf{R}^n\to\mathbf{R}^n$ be an orthogonal action such that its isotropy subgroups are of maximal rank, $z\in\mathbf{R}^n$ a point such that $G(z)$ is principal the irreducible factors of its generalized Weyl group are all of different type and no one of them is of type B_4, and k the dimension of this orbit. If F is an exceptional $(n-k-1)$-dimensional subspace of α in the normal subspace $\varepsilon_z(N_z)$ then, with the exception of those $(n-k-2)$-dimensional subspaces in which F intersects the singular $(n-k-1)$-dimensional subspaces of α in $\varepsilon_z(N_z)$, the points of F are on exceptional orbits of the action α.*

Proof. In consequence of the preceding lemma a point of F cannot be on a principal orbit of α. On the other hand if a point of F is on a singular orbit of α it is contained in a singular $(n-k-1)$-dimensional subspace of α by a previous observation.

The above corollary now justifies the anticipated terminology since it shows that the set of those points in $\varepsilon_z(N_z)$ which have exceptional orbits is included in the union of the exceptional $(n-k-1)$-dimensional subspaces of α in $\varepsilon_z(N_z)$.

Consider the adjoint action $\mathrm{Ad}\colon G\times\mathfrak{g}\to\mathfrak{g}$ of a compact connected semisimple Lie group G. The isotropy subgroups of this action Ad are all connected on account of some basic facts ([3] pp. 15—16). Consequently this action Ad has no exceptional orbits. The following theorem yields a generalization of this observation, a result, which has been stated already earlier without the additional assumption essential for the proof given here [6].

2.1. Theorem. *Let $\alpha\colon G\times\mathbf{R}^n\to\mathbf{R}^n$ be an orthogonal action such that its isotropy subgroups are of maximal rank the irreducible factors of its generalized Weyl group are all of different type and no one of them is of type B_4. Then the action α has no exceptional orbits.*

Proof. Since G is connected, α_g preserves the canonical orientation of $\mathbf{R}^n$ for every $g\in G$. In order to prove the theorem by an indirect argument assume that there is a point $x\in\mathbf{R}^n$ such that $G(x)$ is an exceptional orbit of α: On account of a result due to D. Montgomery ([2] pp. 188—189) the orbit $G(x)$ is orientable. Thus the canonical orientation of $\mathbf{R}^n$ induces an orientation of $G(x)$ too. Consider now the restriction

$$\alpha'\colon G\times G(x)\to G(x)$$

of the action α to the orbit $G(x)$. Then for every element $g \in G$ the corresponding transformation

$$\alpha'_g\colon G(x) \to G(x)$$

is orientation preserving since G is connected. Consider now a point $z \in \mathbf{R}^n$ such that $G(z)$ is principal and that $x \in \varepsilon_z(N_z)$ holds. Let k be the dimension of the orbit $G(z)$. Since $G(x)$ is exceptional, there is an exceptional $(n-k-1)$-dimensional subspare F of α in $\varepsilon_z(N_z)$ passing through the point x. Moreover, by 2.2. Lemma there exists a $g \in G$ such that the restriction of α_g to $\varepsilon_z(N_z)$ is equal to the reflection of $\varepsilon_z(N_z)$ through F. Thus, the restriction of α_g to $\varepsilon_z(N_z)$ is not an orientation preserving transformation; consequently, the restriction of α_g to $\exp\left(T_x G(x)\right)$ is not an orientation preserving transformation either. But then α_g cannot be an orientation preserving transformation of $G(x)$. Thus a contradiction is obtained which shows that α has no exceptional orbits.

The following theorem which is a consequence of preceding results yields a generalization of the well-known fact that Weyl group of a compact connected semisimple Lie group is generated by reflections on the walls of the Weyl chambers of the group ([3] pp. 17—23).

2.2. Theorem. *Let $\alpha\colon G \times \mathbf{R}^n \to \mathbf{R}^n$ be an orthogonal action such that its isotropy subgroups are of maximal rank the irreducible factors of its generalized Weyl group are all of different type and no one of them is of type B_4, $z \in \mathbf{R}^n$ a point such that $G(z)$ is principal and k the dimension of the orbit $G(z)$. Then the action*

$$v\colon A \times \varepsilon_z(N_z) \to \varepsilon_z(N_z)$$

of the group $A = N(G_z)/G_z$ on the normal subspace $\varepsilon_z(N_z)$ is generated by reflections of $\varepsilon_z(N_z)$ on the singular $(n-k-1)$-dimensional subspaces of α in $\varepsilon_z(N_z)$.

Proof. The reflections of $\varepsilon_z(N_z)$ on the singular $(n-k-1)$-dimensional subspaces of α in $\varepsilon_z(N_z)$ generate on account of 2.1. Lemma a subgroup of A which acts under the action v simply transitively on the set of chambers which are defined in $\varepsilon_z(N_z)$ by the singular $(n-k-1)$-dimensional subspaces ([1] pp. 72—74). Assume now that there is a $g \in N(G_z)$ such that the restriction of α_g to $\varepsilon_z(N_z)$ is not a product of reflections on singular $(n-k-1)$-dimensional subspaces. Since α_g maps the orbits of α onto themselves, it maps a chamber defined by the singular $(n-k-1)$-dimensional subspaces to such a chamber. Thus, the above indirect assumption implies that among these chambers there is one C which is mapped onto itself by α_g. But then there is an interior point x of C which is left invariant by α_g. Since x is an interior point of C, the orbit $G(x)$ cannot be singular according to previous results. But $G(x)$ cannot be principal, since g is not an element of G_z in consequence of its definition. By 2.1. Theorem $G(x)$ cannot be exceptional either. Thus a contradiction is obtained which shows that the action v is generated by the reflections of $\varepsilon_z(N_z)$ on the singular $(n-k-1)$-dimensional subspaces.

3. The orbit structure of orthogonal actions with isotropy subgroups of maximal rank

On account of the preceding results a description of the orbit structure of orthogonal actions with isotropy subgroups of maximal rank can be given. This description, provided by the following theorem, reduces in case of the adjoint action of a compact connected semisimple Lie group to the well-known result concerning the relation of the orbit space of the adjoint action to the Weyl chamber of the group ([3] pp. 17—23).

3.1. Theorem. *Let* $\alpha\colon G\times\mathbf{R}^n\to\mathbf{R}^n$ *be an orthogonal action such that its isotropy subgroups are of maximal rank, the irreducible factors of its generalized Weyl group are all of different type and no one of them is of type* B_4, $z\in\mathbf{R}^n$ *a point such that* $G(z)$ *is principal and* k *the dimension of this orbit. Let* C *be one of the chambers defined by the singular* $(n-k-1)$*-dimensional subspaces of* α *in the normal subspace* $\varepsilon_z(N_z)$ *and*

$$\lambda\colon \bar{C}\to\mathbf{R}^n/G$$

the map which renders to a point $x\in\bar{C}$ *its orbit* $G(x)$ *in the orbit space* $\mathbf{R}^n/G$ *of the action* α. *Then* λ *is a homeomorphism.*

Proof. The map λ is surjective. In fact, the normal subspace $\varepsilon_z(N_z)$ intersects every orbit of α in consequence of the Principal Orbit Type Theorem and consequently $\bar{C}$ intersects every orbit of α too on account of 2.2. Theorem and of the transitivity of v on the set of chambers in $\varepsilon_z(N_z)$. In order to show by an indirect argument that λ is injective, assume that there is a point $x\in\bar{C}$ and an element $g\in G$ such that $\alpha(g,x)=y\in\bar{C}$ holds and x, y are different points. It is sufficient to show the existence of such an element $g^*\in N(G_z)$ that $y=\alpha(g^*,x)$ is valid, since then by a basic result on reflection groups ([1] pp. 75—76) the points x, y coincide and thus a contradiction is obtained. If $G(x)$ is a principal orbit then α_g maps the normal subspace $\varepsilon_z(N_z)$ onto itself and therefore $g\in N(G_z)$ holds by 1.1. Lemma. Thus in this case the choice $g^*=g$ can be made. If $G(x)$ is singular then there is an $a\in G$ such that $y=\alpha(ag,x)$ and α_{ag} maps $\varepsilon_z(N_z)$ onto itself. Consequently, the choice $g^*=ag$ can be made in this case on account of 1.1. Lemma. Since by 2.1. Theorem the orbit $G(x)$ cannot be exceptional, all possibilities have been considered.

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