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Generalized Hausdorff matrices bounded on l^p and c

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Necessary and sufficient conditions for an infinite matrix A to belong to $B(c)$, the algebra of bounded linear operators on c, the space of convergent sequences, have been known since the early 1900's. Necessary and sufficient conditions for $A \in B(l)$ were established by Knopp and Lorentz in 1949. In both cases the conditions can be verified by examining only the entries of A. Necessary and sufficient conditions do not exist for a general matrix $A \in B(l^p)$ for $p>1$, in terms involving only the entries of A, and it is doubtful that conditions, analogous to the Silverman—Toeplitz conditions, will ever be found. However, considerable progress has been made for certain classes of Hausdorff matrices.

A Hausdorff matrix is a lower triangular matrix with entries $h_{nk}=\binom{n}{k}\Delta^{n-k}\mu_k$, where $\binom{n}{k}$ denotes the ordinary binomial coefficient, and Δ is the forward difference operator defined by $\Delta\mu_k=\mu_k-\mu_{k+1}$, $\Delta^n\mu_k=\Delta(\Delta^{n-1}\mu_k)$. H is called totally regular if $\{\mu_n\}$ has the representation $\mu_n=\int_0^1 t^n\,d\beta(t)$, $\beta(t)\in BV[0,1]$, satisfying $\beta(0+)=\beta(0)=0$, $\beta(1)=1$, and nonnegative and nondecreasing over $[0,1]$. The best known example is C, the Cesàro matrix of order one, obtained by setting $\mu_n=(n+1)^{-1}$.

For a sequence $\{a_n\}$ let $b_n=H_n(a)=\sum_{k=0}^{n} h_{nk}a_k$. In 1934 HARDY [5] established the following result. If $\{a_n\}$ is a nonnegative sequence in l^p, $p>1$, H totally regular, then $\sum b_n^p<K(p)\sum a_n^p$, where $K(p)=(\int_0^1 t^{-1/p}\,d\beta(t))^p$, unless $a_n=0$ for all n, or H is the identity transformation. The value of $K(p)$ is best possible.

In 1965 BROWN, HALMOS, and SHIELDS [1] showed that C is a bounded operator on l^2, with norm 2, and is hyponormal. In 1970 KRIETE and TRUTT [8] established the fact that C is subnormal. In 1971 [11] the author showed that the existence of

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the integral $\int_0^1 t^{-1/p}\,d\beta(t)$, for totally regular Hausdorff matrices, implies $H\in B(l^p)$, with norm $K(p)$. Some specific Hausdorff methods, such as the Cesàro, Hölder, and Euler methods of positive order, were shown to be in $B(l^p)$ and their norms were computed. In 1972 LEIBOWITZ [9], independently, showed that $C\in B(l^p)$ and computed the point spectrum of its adjoint. The following year [10] he determined the spectra of those Hausdorff matrices in $B(l^p)$ with absolutely continuous mass functions β. SHARMA [12] observed that all the Hausdorff matrices in $B(l^2)$ are subnormal. In 1974, JAKIMOVSKI, RHOADES, and TZIMBALARIO [7] obtained *necessary* and *sufficient* conditions for totally regular generalized Hausdorff matrices to belong to $B(l^p)$. The generalized Hausdorff matrices considered are those with entries $h_{nk}^{(\alpha)}=\binom{n+\alpha}{n-k}\Delta^{n-k}\mu_k$, $\alpha\geqq 0$. In 1977, GHOSH, RHOADES, and TRUTT [4] proved that the generalized Hausdorff matrix generated by $\mu_n=\int_0^1 t^{n+\alpha}dt$, for positive integer α, is subnormal. Thus, for each positive integer α, the corresponding algebra of generalized Hausdorff matrices in $B(l^2)$ is subnormal. Using some of the results of SHIELDS and WALLEN [13], DEDDENS [3] described formally the spectrum of each Hausdorff matrix in $B(l^2)$ and also computed the norms of the Cesàro, Hölder, and Euler matrices.

In this paper *necessary* conditions are established for a generalized Hausdorff matrix to belong to $B(l^p)$, without the assumption of total regularity. Necessary and sufficient conditions are obtained for those generalized Hausdorff matrices in $B(c)$ to belong to $B(l^2)$. Let $|H|$ denote the matrix whose entries are $|h_{nk}|$. In Theorem 7 it is shown that $|H|\in B(l^p)$ if and only if $H^{(-1/q)}\in B(l)$. This result, along with Theorem 2 shows how close one is to establishing the conjecture that $H\in B(l^p)$ if and only if $|H|\in B(l^p)$.

Throughout this paper α denotes an arbitrary nonnegative real number. The case $\alpha=0$ corresponds to ordinary Hausdorff summability.

Let $C^{(\alpha)}$ denote the generalized Hausdorff matrix generated by $\mu_n=\int_0^1 t^{n+\alpha}\,dt$. A routine calculation verifies that the nonzero entries of the nth row of $C^{(\alpha)}$ are $(n+\alpha+1)^{-1}$. Let $*$ denote the adjoint, $1/p+1/q=1$.

Lemma 1. *$I-2C^{*(\alpha)}/q\in B(l^q)$ and has simple eigenvectors of the form*

$$x_n = x_0\prod_{j=1}^{n}\left(1-\frac{1/\lambda}{j+\alpha}\right),\quad \text{where}\quad x_0\in\mathbf{C},\ \operatorname{Re}(1/\lambda)>1/q. \tag{1}$$

Proof. From [8, Theorem 1], $C^{(\alpha)}\in B(l^p)$, so that $I-2C^{(\alpha)}/q\in B(l^p)$, and hence $I-2C^{*(\alpha)}/q\in B(l^q)$.

Suppose $(I-2C^{*(\alpha)}/q)x=\zeta x$. Then, as in the proofs of [1, Theorem 2] or [10, Theorem 1], one obtains (1), where $1/\lambda=2/q(1-\zeta)$. As in [1], it is readily verified that $\{x_n\}\in l^q$ for $\operatorname{Re}(1/\lambda)>1/q$. From (1) it is clear that each of the eigenvectors is simple.

Let $\sigma_p(A)$, $\sigma(A)$, and $\varrho(A)$ denote, respectively, the point spectrum, and resolvent sets for an operator A. Let $D=\{z\in \mathbf{C}\,|\,|z|<1\}$, $\bar{D}$ the closure of D.

Lemma 2. *$\sigma_p(I-2C^{*(\alpha)}/q)$ contains D and $\sigma(I-2C^{(\alpha)}/q)=\bar{D}$.*

The first result is immediate, since, from Lemma 1, every point of D is in the point spectrum of $I-2C^{*(\alpha)}/q$. To prove the second result it will be sufficient to show that $|\zeta|>1$ implies $\zeta\in\varrho(I-2C^{(\alpha)}/q)$. The generating sequence for the generalized Hausdorff method corresponding to $\zeta I-I+2C^{(\alpha)}/q$ is $\mu_n=\zeta-1+ +2/q(n+\alpha+1)$. Let $\varepsilon_n=1/\mu_n$. Then

$$\varepsilon_n=\frac{1}{\zeta-1}\left[1-\frac{2/q}{n+\alpha+1+2/q(\zeta-1)}\right].$$

If $H_\varepsilon^{(\alpha)}$ denotes the corresponding generalized Hausdorff matrix, then $H_\varepsilon^{(\alpha)}= =(H_\mu^{(\alpha)})^{-1}$, and

$$\|H_\varepsilon^{(\alpha)}\|_p\leqq\frac{1}{|\zeta-1|}\left[1+\frac{2}{q}\|H_\delta^{(\alpha)}\|_p\right],$$

where $\delta_n=\big(n+\alpha+1+2/q(\zeta-1)\big)^{-1}$. It suffices to show that $H_\delta^{(\alpha)}\in B(l^p)$. As an $H^{(\alpha)}$ matrix, δ_n has the representation

$$\delta_n=\int_0^1 t^{n+\alpha}\,d\beta(t),\quad \text{where}\quad \beta(t)=\frac{t^{1+2/q(\zeta-1)}}{1+2/q(\zeta-1)}.$$

Since $|\zeta|>1$ implies $1-1/p+\operatorname{Re}\big(2/q(\zeta-1)\big)>0$,

$$\int_0^1 t^{-1/p}|d\beta(t)|=\int_0^1 t^{-1/p+\operatorname{Re}(2/q(\zeta-1))}\,dt<\infty.$$

From [7, Theorem 1] $H_\delta^{(\alpha)}\in B(l^p)$ and the proof is finished.

Lemma 3. *Let $A, B\in B(l^p)$, $p>1$. If α is a simple eigenvalue for A with corresponding eigenvector x, and if B commutes with A, then x is an eigenvector for B.*

Proof. Let α and x be as in the Lemma. $Bx=B\left(\frac{1}{\alpha}Ax\right)=\frac{1}{\alpha}B(Ax)= =\frac{1}{\alpha}(BA)x=\frac{1}{\alpha}(AB)x=A\left(\frac{1}{\alpha}Bx\right)$. Since $x\in l^p$, $A, B\in B(l^p)$ guarantee the associativity of the multiplication.

Thus $A(Bx)=\alpha(Bx)$; i.e., Bx is also an eigenvector for A corresponding to the value α. Since the eigenvalues of A are simple, $Bx=\delta x$ for some scalar δ; i.e., x is an eigenvector for B.

A special case of Lemma 3 appears as Theorem 1 of [10]. Lemma 3 can obviously be generalized, but the present form is sufficient for our purposes.

Theorem 1. *Let* $H^{(\alpha)}\in B(l^p)$. *Then*

$$\|H^{(\alpha)}\|_p \geqq \sup_{\mathrm{Re}\,\delta>1/q}\left|\sum_{n=k}^{\infty}\binom{n+\alpha-\delta}{n-k}\Delta^{n-k}\mu_k\right|.$$

Proof. It is known that $H^{(\alpha)}$ commutes with $C^{(\alpha)}$. Therefore $H^{*(\alpha)}$ commutes with $C^{*(\alpha)}$, and hence commutes with $I-2C^{*(\alpha)}/q$. Let $x=\{x_n\}$ be defined as in (1). Since x is a simple eigenvector for $I-2C^{*(\alpha)}/q$, x is an eigenvector for $H^{*(\alpha)}$ by Lemma 3. $H^{(\alpha)}\in B(l^p)$ implies $H^{*(\alpha)}\in B(l^q)$, so that $H^{*(\alpha)}x\in l^q$. Moreover,

$$(H^{*(\alpha)}x)_n=\sum_{k=n}^{\infty}h_{nk}^{*(\alpha)}x_k=\sum_{k=n}^{\infty}h_{kn}^{(\alpha)}x_k=$$

$$=\sum_{k=n}^{\infty}\binom{k+\alpha}{n-k}\Delta^{k-n}\mu_n x_k=\sum_{r=0}^{\infty}\binom{n+r+\alpha}{r}\Delta^r\mu_n x_{n+r}.$$

Note that we may write $x_{n+r}=x_n\prod_{j=n+1}^{n+r}(1-\delta/(j+\alpha))$, where $\delta=1/\lambda$. Therefore

$$(H^{*(\alpha)}x)_n=x_n\sum_{r=0}^{\infty}\binom{n+r+\alpha}{n-k}\Delta^r\mu_n\prod_{j=n+1}^{n+r}\left(\frac{j+\alpha-\delta}{j+\alpha}\right)=$$

$$=x_n\sum_{r=0}^{\infty}\binom{n+r+\alpha-\delta}{r}\Delta^r\mu_n=x_n\sum_{k=n}^{\infty}\binom{k+\alpha-\delta}{k-n}\Delta^{k-n}\mu_n.$$

Since x is an eigenvector for H^*, it follows that $\sum_{k=n}^{\infty}\binom{k+\alpha-\delta}{k-n}\Delta^{k-n}\mu_n=c(\delta)$, where c is independent of n. Also,

$$\infty>\|H^{(\alpha)}\|_p=\|H^{*(\alpha)}\|_q\geqq\|H^{*(\alpha)}x\|_q/\|x\|_q\geqq\left|\sum_{k=n}^{\infty}\binom{k+\alpha-\delta}{k-n}\Delta^{k-n}\mu_n\right|,$$

so that

$$\|H^{(\alpha)}\|_p\geqq\sup_{\mathrm{Re}(\delta)>1/q}\left|\sum_{k=n}^{\infty}\binom{k+\alpha-\delta}{k-n}\Delta^{k-n}\mu_n\right|.$$

The above result has shown that each of the column sums of the matrix $H^{(\alpha-\delta)}$ is equal to $c(\delta)$. More is true.

Theorem 2. *Under the conditions of Theorem* 1, *the columns of* $H^{(\alpha-\delta)}$ *belong to* l.

Proof. $$\sum_{n=k}^{\infty}|h_{nk}^{(\alpha-\delta)}| = \sum_{n=k}^{\infty}\left|\binom{n+\alpha-\delta}{n-k}\right||\Delta^{n-k}\mu_k| =$$

$$= \sum_{n=k}^{\infty}\frac{\left|\binom{n+\alpha-\delta}{n-k}\right|}{\binom{n+\alpha}{n-k}}\binom{n+\alpha}{n-k}|\Delta^{n-k}\mu_k| = \frac{\Gamma(k+\alpha+1)}{|\Gamma(k+\alpha+1-\delta)|}\sum_{n=k}^{\infty}\frac{|\Gamma(n+\alpha+1-\delta)|}{\Gamma(n+\alpha+1)}|h_{nk}^{(\alpha)}|.$$

Since $H^{(\alpha)}\in B(l^p)$, the columns of $H^{(\alpha)}$ are uniformly in l^p. If

$$\{|\Gamma(n+\alpha+1-\delta)/\Gamma(n+\alpha+1)|\}\in l^q,$$

the result follows by Hölder's inequality. Since $|\Gamma(n+\alpha+1-\delta)|/\Gamma(n+\alpha+1)\sim \sim n^{-\mathrm{Re}(\delta)}$ and $\mathrm{Re}(\delta)>1/q$, we have $\{|\Gamma(n+\alpha+1-\delta)/\Gamma(n+\alpha+1)|\}\in l^q$.

Let c denote the space of convergent sequences. Condition $H^{(\alpha)}\in B(c)$ implies that $\{\mu_n\}$ has the representation

(2) $$\mu_n = \int_0^1 t^{n+\alpha}d\beta(t), \quad n \geqq 0,\ \beta(t)\in BV[0, 1].$$

Theorem 3. *Let $H^{(\alpha)}\in B(l^p)\cap B(c)$. If, in addition,*

(3) $$\int_0^1 t^{-1/p}|d\beta(t)| < \infty,$$

then

(4) $$\sup_{\mathrm{Re}(\delta)>1/q}\left|\int_0^1 t^{\delta-1}d\beta(t)\right| \leqq \|H^{(\alpha)}\|_p \leqq \int_0^1 t^{-1/p}|d\beta(t)|.$$

Proof. $H^{(\alpha)}\in B(c)$ implies $\{\mu_n\}$ has the representation (2).

$$\sum_{n=k}^{\infty}\binom{n+\alpha-\delta}{n-k}\Delta^{n-k}\mu_k = \sum_{n=k}^{\infty}\binom{n+\alpha-\delta}{n-k}\int_0^1 t^{k+\alpha}(1-t)^{n-k}d\beta(t) =$$

$$= \int_0^1 t^{k+\alpha}\sum_{n=k}^{\infty}\binom{n+\alpha-\delta}{n-k}(1-t)^{n-k}d\beta(t) =$$

$$= \int_0^1 t^{k+\alpha}[1-(1-t)]^{-(k+1+\alpha-\delta)}d\beta(t) = \int_0^1 t^{\delta-1}d\beta(t),$$

the interchange of integration and summation being justified by condition (3). The left inequality now follows from Theorem 1. The right inequality is Theorem 1 of [7].

Theorem 4. *Let $H^{(\alpha)}\in B(l^2)$. Then there exists a unique bounded analytic function $\hat{f}$ defined on D such that*

$$H^{(\alpha)}=\hat{f}(I-C^{(\alpha)}). \tag{5}$$

$\hat{f}(D)$ is a nonempty open set, $\sigma(H^{(\alpha)})=$closure of $\hat{f}(D)$, and $\sigma_p(H^{(\alpha)})$ contains the set $\hat{f}(D)^-$, where $^-$ denotes complex conjugation. If $\{\mu_n\}$ are the diagonal elements of $H^{(\alpha)}$, then

$$\mu_n=\hat{f}(1-(n+\alpha+1)^{-1}). \tag{6}$$

Assuming the existence of such an $\hat{f}$ satisfying (5), its uniqueness follows from (6). From Lemma 2, $\sigma_p(I-C^{*(\alpha)})\supseteqq D$ and $\sigma(I-C^{(\alpha)})=\bar{D}$. The spectral results of the theorem then follow from the spectral mapping theorem, since $\hat{f}$ is analytic in D.

To prove (5) it will be sufficient to construct a Hilbert space H of complex valued functions defined on D, with the usual addition of functions and multiplication by scalars, which satisfies the following four axioms of [13, p. 782]:

(a) Point evaluations are bounded linear functionals on H. Hence, to each $\zeta\in D$, there corresponds a function $\hat{k}_\zeta$ in H such that $\hat{f}(\zeta)=(\hat{f},\hat{k}_\zeta)$ for all $\hat{f}\in H$.

(b) The operator M_z of multiplication by z on H maps H into itself and is a contraction.

(c) The functions $\hat{k}_\zeta$ are simple eigenfunctions of the operator M_z^*.

(d) The functions in H are analytic in D.

From Lemma 1, each $\zeta\in D$ is a simple eigenvalue of $I-C^{*(\alpha)}$, with corresponding eigenvector f_ζ, whose components are defined by (1) with $x_0=1$. Define $k_\zeta=f_{\bar{\zeta}}$. Then $(I-C^{*(\alpha)})k_\zeta=\bar{\zeta}k_\zeta$. The vectors $\{k_\zeta\}$, $\zeta\in D$ span l^2. To see this, let $\{e_n\}$ denote the standard orthonormal basis for l^2, i.e., $e_n(k)=\sigma_{nk}$, $n, k\geqq 0$. Define a sequence of real numbers $\{\zeta_r\}$ by $\zeta_r=(\alpha+r)/(\alpha+r+1)$, $r=0, 1, 2, \dots$, and denote the corresponding sequence of eigenvectors by $\{f_r\}$. A straightforward calculation verifies that $f_0=e_0$, and that $\sum_{k=0}^{r}\binom{r}{k}(-1)^k f_k=e_r r!/(1+\alpha)\dots(r+\alpha)$ for $r>0$. Therefore $\{f_r\}$ spans l^2, so that, a fortiori, $\{k_\zeta\}$, $\zeta\in D$ spans l^2.

As in [14], l^2 can be transformed into a Hilbert space of complex valued functions. For $f\in l^2$, define its transform $\hat{f}$ by

$$\hat{f}(\zeta)=(f,k_\zeta),\quad \zeta\in D. \tag{7}$$

Let H denote the set of all such functions $\hat{f}$, with the usual addition of functions and scalar multiplication, and with inner product defined by $(\hat{f},\hat{g})=(f,g)$. Then H is a Hilbert space, and the mapping U: $l^2\to H$, defined by $Uf=\hat{f}$, is a unitary transformation of l^2 onto H. Also $U(I-C^{(\alpha)})=M_z$, where M_z denotes multiplication by z on H. Since $\|\hat{k}_\zeta\|_2$ is uniformly bounded on compact subsets of H, from (7), $|\hat{f}(\zeta)|\leqq\|f\|_2\|k_\zeta\|_2$, and each $\hat{f}$ in H is bounded over D.

To show that each f is analytic, it will be sufficient to show that H contains a dense subset of analytic functions. The $\{e_n\}$ in l^2 are transformed as follows:

$$\hat{e}_0(\zeta) = (e_0, k) = 1, \ \hat{e}_n(\zeta) = (e_n, k_\zeta) = \frac{(-1)^n w(w-1)\dots(w-n+1)\,\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)} \quad (n \geqq 1), \ \zeta \in D$$

where $w+\alpha=\zeta/(1-\zeta)$. These transforms are rational functions whose only pole is at $\zeta=1$. Thus, their finite linear combinations are analytic in D.

That the functions $\hat{k}_\zeta$ are simple eigenfunctions of M_z^* follows from Lemma 1 and the argument of [13, p. 782].

It remains to show that M_z is a contraction, or equivalently, that $\|I-C^{(\alpha)}\|_2 \leqq 1$. If it can be shown that $C^{(\alpha)}$ is hyponormal, then, from [15, Theorem 1], its norm is equal to its spectral radius. From Lemma 2 this value is 1, so that $\|I-C^{(\alpha)}\|_2=1$.

Lemma 4. *$C^{(\alpha)}$ is hyponormal.*

For α a nonnegative integer this is a known result since, from [4, Theorem 2], $C^{(\alpha)}$ is subnormal, hence hyponormal.

It is easy to verify that

$$(C^{*(\alpha)}C^{(\alpha)}-C^{(\alpha)}C^{*(\alpha)})_{nk} = \begin{cases} \beta_n+\alpha\gamma_{nk}, & n > k \\ \beta_k+\alpha\gamma_{nk}, & n \leqq k \end{cases}$$

where

$$\beta_n = \sum_{j=n}^{\infty}\frac{1}{(j+\alpha+1)^2}-\frac{1}{n+\alpha+1}, \quad \gamma_{nk} = 1/(n+\alpha+1)(k+\alpha+1).$$

To show that $C^{(\alpha)}$ is hyponormal we must show that $C^{*(\alpha)}C^{(\alpha)}-C^{(\alpha)}C^{*(\alpha)}$ is a positive operator; i.e., that $D_n \geqq 0$ for each n, where

$$D_n = \begin{vmatrix} \beta_0+\alpha\gamma_{00} & \beta_1+\alpha\gamma_{01} & \dots & \beta_{n-1}+\alpha\gamma_{0,n-1} \\ \beta_1+\alpha\gamma_{10} & \beta_1+\alpha\gamma_{11} & \dots & \beta_{n-1}+\alpha\gamma_{1,n-1} \\ \vdots & \vdots & & \vdots \\ \beta_{n-1}+\alpha\gamma_{n-1,0} & \beta_{n-1}+\alpha\gamma_{n-1,1} & \dots & \beta_{n-1}+\alpha\gamma_{n-1,n-1} \end{vmatrix}.$$

D_n can be written as the sum of two determinants, where the first column of the first determinant contains the β_i, the first column of the second determinant consists of $\alpha\gamma_{i0}$, and the remaining columns of the two determinants are identical. Each of these determinants can, in turn, be written as the sum of two determinants, by decomposing their second columns. Thus one has $D_n = D_n^{(1)}+D_n^{(2)}+D_n^{(3)}+D_n^{(4)}$.

In $D_n^{(4)}$ the entries in the i-th row of the first two columns are $\alpha\gamma_{i0}$ and $\alpha\gamma_{i1}$, respectively. If one factors $1/(\alpha+1)$ from the first column and $1/(\alpha+2)$ from the second column, then the first two columns of $D_n^{(4)}$ are identical, so $D_n^{(4)}=0$.

Exploiting this idea, $D_n^{(3)}$ becomes

$$D_n^{(3)} = \begin{vmatrix} \alpha\gamma_{00} & \beta_1 & \cdots & \beta_{n-1} \\ \alpha\gamma_{10} & \beta_1 & \cdots & \beta_{n-1} \\ \vdots & \vdots & & \vdots \\ \alpha\gamma_{n-1,0} & \beta_{n-1} & \cdots & \beta_{n-1} \end{vmatrix}.$$

In a similar manner one may write

$$D_n^{(2)} = \begin{vmatrix} \beta_0 & \alpha\gamma_{01} & \cdots & \beta_{n-1} \\ \beta_1 & \alpha\gamma_{11} & \cdots & \beta_{n-1} \\ \vdots & \vdots & & \vdots \\ \beta_{n-1} & \alpha\gamma_{n-1,1} & \cdots & \beta_{n-1} \end{vmatrix}.$$

One may write

$$D_n^{(1)} = \begin{vmatrix} \beta_0 & \beta_1 & \beta_2 & \beta_3+\alpha\gamma_{03} & \cdots & \beta_{n-1}+\alpha\gamma_{0,n-1} \\ \beta_1 & \beta_1 & \beta_2 & \beta_3+\alpha\gamma_{13} & \cdots & \beta_{n-1}+\alpha\gamma_{1,n-1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ \beta_{n-1} & \beta_{n-1} & \beta_{n-1} & \beta_{n-1}+\alpha\gamma_{n-1,3} & \cdots & \beta_{n-1}+\alpha\gamma_{n-1,n-1} \end{vmatrix} +$$

$$+ \begin{vmatrix} \beta_0 & \beta_1 & \alpha\gamma_{02} & \beta_3+\alpha\gamma_{03} & \cdots & \beta_{n-1}+\alpha\gamma_{0,n-1} \\ \beta_1 & \beta_1 & \alpha\gamma_{12} & \beta_3+\alpha\gamma_{03} & \cdots & \beta_{n-1}+\alpha\gamma_{0,n-1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ \beta_{n-1} & \beta_{n-1} & \alpha\gamma_{n-1,2} & \beta_{n-1}+\alpha\gamma_{n-1,3} & \cdots & \beta_{n-1}+\alpha\gamma_{n-1,n-1} \end{vmatrix}.$$

As before, the second determinant becomes

$$\begin{vmatrix} \beta_0 & \beta_1 & \alpha\gamma_{02} & \beta_3 & \cdots & \beta_{n-1} \\ \beta_1 & \beta_1 & \alpha\gamma_{12} & \beta_3 & \cdots & \beta_{n-1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ \beta_{n-1} & \beta_{n-1} & \alpha\gamma_{n-1,2} & \beta_{n-1} & \cdots & \beta_{n-1} \end{vmatrix}.$$

Continuing in this manner, one may write $D_n = \sum_{i=1}^n E_n^{(i)}$, where

$$E_n^{(n)} = \begin{vmatrix} \beta_0 & \beta_1 & \cdots & \beta_{n-1} \\ \beta_1 & \beta_1 & \cdots & \beta_{n-1} \\ \vdots & \vdots & & \vdots \\ \beta_{n-1} & \beta_{n-1} & \cdots & \beta_{n-1} \end{vmatrix},$$

and $E_n^{(i)}$, for $0 \leqq i < n$, is the result of replacing the i-th column of $E_n^{(n)}$ with $(\alpha\gamma_{ji})_{j=0}^{n-1}$.

It will now be shown that each determinant is nonnegative. To accomplish this it will be sufficient to show that, for each n,

(i) β_n is monotone decreasing, and (ii) $\begin{vmatrix} \beta_{n-1}-\beta_n & \gamma_{n-1,n} \\ \beta_n-\beta_{n+1} & \gamma_{n,n+1} \end{vmatrix} > 0.$

For (i), $\beta_n-\beta_{n+1}=1/(n+\alpha+1)^2(n+\alpha+2)>0$. Expanding the determinant in (ii), and using (i), yields.

$$\frac{1}{(n+\alpha)(n+\alpha+1)^2(n+\alpha+2)}\left(\frac{1}{n+\alpha}-\frac{1}{n+\alpha+1}\right)>0.$$

$E_n^{(n)}$ is an L-shaped determinant, which has been shown in [1, p. 131] to be nonnegative, since β_n is monotone decreasing.

To evaluate $E_n^{(i)}$ for $1<i<n$, subtract column 1 from column 0. Then subtract column 2 from column 1. Continue in this way through column $i-2$. Then $E_n^{(i)}$ takes the form

$$\frac{\alpha}{(i+\alpha+1)}\begin{vmatrix}\beta_0-\beta_1 & \beta_1-\beta_2 & \cdots & \beta_{i-2}-\beta_{i-1} & \beta_{i-1} & 1/(\alpha+1) & \beta_{i+1} & \cdots & \beta_{n-1}\\ 0 & \beta_1-\beta_2 & & & & 1/(\alpha+2) & & \cdots & \beta_{n-1}\\ \vdots & & & & & & & & \vdots\\ 0 & 0 & & & & 1/(\alpha+n) & \beta_{n-1} & \cdots & \beta_{n-1}\end{vmatrix}.$$

Columns zero through $i-2$ of $E_n^{(i)}$ have all zeros below the main diagonal, and the diagonal entries are $\beta_j-\beta_{j-1}$, $0\leqq j<i-1$, which are positive by (i). To show that $E_n^{(i)}$ is positive, it is sufficient to show that

$$\begin{vmatrix}\beta_{i-1} & 1/(\alpha+1) & \beta_{i+1} & \cdots & \beta_{n-1}\\ \beta_i & 1/(\alpha+i+1) & \beta_{i+1} & \cdots & \beta_{n-1}\\ \vdots & \vdots & & & \vdots\\ \beta_{n-1} & 1/(\alpha+n) & \beta_{n-1} & \cdots & \beta_{n-1}\end{vmatrix}>0.$$

Subtract row 1 from row 0, then row 2 from row 1, etc., to obtain

$$\begin{vmatrix}\beta_{i-1}-\beta_i & 1/(\alpha+i)(\alpha+i+1) & 0 & \cdots & 0\\ \beta_i-\beta_{i+1} & 1/(\alpha+i+1)(\alpha+i+2) & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ \beta_{n-1} & 1/(\alpha+n) & \beta_{n-1} & \cdots & \beta_{n-1}\end{vmatrix}. \tag{8}$$

The above determinant has all zeros above the main diagonal, beginning with column 2. The corresponding diagonal entries are $\beta_j-\beta_{j+1}$, except for the last one, which is β_{n-1}. Expanding yields a positive number times the determinant of (ii).

To evaluate $E_n^{(1)}$, subtract row 1 from row 0, row 2 from row 1, etc., to obtain a determinant with the same property as (6). Expanding then gives a positive number times the determinant

$$\begin{vmatrix}\alpha(\gamma_{00}-\gamma_{10}) & 0\\ \alpha(\gamma_{10}-\gamma_{20}) & \beta_1-\beta_2\end{vmatrix},$$

which is easily seen to be positive.

To evaluate $E_n^{(0)}$, factor $\alpha/(1+\alpha)$ from column 0. Then subtract row 1 from row 0, row 2 from row 1, etc., to obtain a determinant of the same form as (8).

We shall now verify equation (6). First we shall show that (6) is true for $\hat{f}(z)=z^r$. The result is trivially true for $r=0$. Assume the induction hypothesis. Then $\hat{f}(I-C^{(\alpha)})=(I-C^{(\alpha)})^{r+1}=(I-C^{(\alpha)})(I-C^{(\alpha)})^r$, so that

$$(\hat{f}(I-C^{(\alpha)}))_{nk} = \sum_{j=k}^{n} (I-C^{(\alpha)})_{nj}(I-C^{(\alpha)})^r_{jk}.$$

In particular,

$$\mu_n = (\hat{f}(I-C^{(\alpha)}))_{nn} = (I-C^{(\alpha)})_{nn}(I-C^{(\alpha)})^r_{nn} =$$

$$= \left(1-\frac{1}{n+\alpha+1}\right)\left(1-\frac{1}{n+\alpha+1}\right)^r = \left(1-\frac{1}{n+\alpha+1}\right)^{r+1} = \hat{f}(1-(n+\alpha+1)^{-1}).$$

If $\hat{f}$ is an arbitrary analytic function in D, then $\hat{f}(z)=\sum_{k=0}^{\infty} a_k z^k$, so that

$$\mu_n = (\hat{f}(I-C^{(\alpha)}))_{nn} = \left(\sum_{k=0}^{\infty} a_k (I-C^{(\alpha)})^k_{nn}\right) =$$

$$= \sum_{k=0}^{\infty} a_k (1-(n+\alpha+1)^{-1})^k = \hat{f}(1-(n+\alpha+1)^{-1}).$$

Theorem 5. *Let $H^{(\alpha)} \in B(l^2) \cap B(c)$. Then*

$$\|H^{(\alpha)}\|_2 = \sup_{|1-\lambda|<1} \left|\int_0^1 t^{-1+1/\lambda} d\beta(t)\right| = \sup_{\mathrm{Re}(z)>-1/2} \left|\int_0^1 t^z d\beta(t)\right|,$$

where μ_n is defined by (2).

Proof. From Theorem 4 there exists a bounded analytic function $\hat{f}$ on D such that $H^{(\alpha)}=\hat{f}(I-C^{(\alpha)})$. From [13],

$$\|H^{(\alpha)}\|_2 = \|\hat{f}(I-C^{(\alpha)})\|_\infty = \sup_{|z|<1} |\hat{f}(z)|.$$

To obtain an explicit representation of the norm, it is necessary to determine the particular analytic function $\hat{f}$ which is associated with $H^{(\alpha)}$. Equation (6) says that $\hat{f}$ is determined by the μ_n. Since $H \in B(c)$, μ_n satisfies (2). Therefore

$$\hat{f}(1-(n+\alpha+1)^{-1}) = \int_0^1 t^{n+\alpha} d\beta(t).$$

Writing $z=1-(n+\alpha+1)^{-1}$ we obtain

$$\hat{f}(z) = \int_0^1 t^{\frac{z}{1-z}} d\beta(t).$$

Note that $z/(1-z)=-1+1/(1-z)$. With $w=1-z$, then $|z|<1$ gets mapped into $|1-w|<1$, so that

$$\hat{f}(z) = \int_0^1 t^{\frac{1}{1-z}-1} d\beta(t) = \int_0^1 t^{\frac{1}{w}-1} d\beta(t).$$

For the second representation of the norm, note that $|1-w|<1$ is equivalent to $\mathrm{Re}\,(1/w)>1/2$, i.e., $\mathrm{Re}\left(\frac{1}{w}-1\right)>-1/2$. Now set $z=\frac{1}{w}-1$.

Theorem 6. *Let $H^{(\alpha)}\in B(l^p)\cap B(c)$, $p>1$. If $\beta(t)$ is a totally monotone mass function, then*

$$\sup_{\mathrm{Re}(z)>1/p} \left|\int_0^1 t^z d\beta(t)\right| = \int_0^1 t^{-1/p} d\beta(t).$$

Proof. Let $\psi(z)=\int_0^1 t^z d\beta(t)$. Then $\psi(z)$ is analytic for $\mathrm{Re}\,(z)>-1/p$ and continuous for $\mathrm{Re}\,(z)=-1/p$. Since β is totally monotone,

$$\sup_{\mathrm{Re}(z)>-1/p} |\psi(z)| = \sup_{y\in R} \left|\int_0^1 t^{iy-1/p} d\beta(t)\right| \geqq \int_0^1 t^{-1/p} d\beta(t) \geqq 0.$$

The conclusion follows from (4).

Corollary 1. *Let $H^{(\alpha)}\in B(l^2)\cap B(c)$ with $\beta(t)$ totally monotone. Then $\|H^{(\alpha)}\|_2=\hat{f}(-1)$, where $\hat{f}$ satisfies* (5).

Proof. From Theorem 6, the supremum occurs at $-1/2$, which corresponds to $w=2$, which corresponds to $z=-1$.

Let $C=C^{(0)}$; i.e., C is the Cesàro matrix of order 1. If one sets $H=\{\psi|\psi$ is a bounded analytic function on $|z-1|<1\}$ and makes the association $H=\psi(C)$ for each Hausdorff matrix in $B(l^2)$, then, for each Hausdorff matrix with a totally monotone mass function β, $\|H\|_2=\psi(2)$ from Corollary 1. This result has been verified for several particular Hausdorff matrices by Deddens [3].

Let $|H|$ denote the matrix whose entries are $|h_{nk}|$.

Theorem 7. *Let $p>1$. Then $|H|\in B(l^p)$ if and only if $H^{-1/q}\in B(l)$.*

Proof. From the proof of [7, Theorem 2], $|H|\in B(l^p)$ implies

$$\sup_n \sum_{k=0}^{n} \binom{n+1/p}{n-k} |\Delta^{n-k}\mu_k| < \infty;$$

i.e., $H^{(1/p)} \in B(c)$. Since $\binom{n}{k} \leqq \binom{n+1/p}{n-k}$ for $p>0$, $H \in B(c)$. Therefore there exists a function $\beta(t) \in BV[0, 1]$ such that

$$\mu_n = \int_0^1 t^n d\beta(t).$$

From [8, Lemma 1], $\int_0^1 t^{-1/p} |d\beta(t)|$ exists. We may write

$$\mu_n = \int_0^1 t^{n+1-1/q}(t^{1/q-1} d\beta(t)) = \int_0^1 t^{n+1-1/q} d\gamma(t),$$

where $\gamma(t) = \int_0^t u^{-1/p} d\beta(u)$. Since $\int_0^1 t^{-1/p} |d\beta(t)|$ exists, $\gamma \in BV[0, 1]$.

Now, from [6, Theorem 16.3], $H^{(-1/q)} \in B(l)$. This implies $H^{(-1/q)} \in B(l)$

$$\sup_n \sum_{n=k}^{\infty} \binom{n-1/q}{n-k} |\Delta^{n-k} \mu_k| < \infty.$$

From [6, Theorem 16.2], there exists a function $\beta(t) \in BV[0, 1]$ such that

$$\mu_n = \int_0^1 t^{n+1-1/q} d\beta(t), \; n-1/q > 0. \tag{9}$$

Define $\mu_0 = \int_0^1 t^{1-1/p} d\beta(t)$, which exists, since $\beta(t) \in BV[0, 1]$. Then (9) is true for all $n \geqq 0$, which implies $H^{1/p)} \in B(c)$ and so $H \in B(c)$. Thus there exists a function $\gamma(t) \in BV[0, 1]$ such that

$$\mu_n = \int_0^1 t^n d\gamma(t).$$

$H^{(1/p)} \in B(c)$ implies

$$\sup_n \sum_{k=0}^{n} \binom{n+1/p}{n-k} |\Delta^{n-k} \mu_k| < \infty.$$

From [7, Lemma 1], $\int_0^1 t^{-1/p} |d\gamma(t)|$ exists. By [7, Corollary 1], $|H| \in B(l^p)$.

A result similar to Theorem 7 is true for $H^{(\alpha)}$ with $\alpha > 0$.

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