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On center-valued states of von Neumann algebras

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Center-valued states are projections of norm one onto the centre of the algebra. This concept is the natural extension of the notion of the (scalar-valued) state. The space of normal states is sequentially complete and the same can be said about the space of normal center-valued states with respect to the pointwise weak convergence.

We remark that center-valued states are central-linear maps. Central-linear maps (or module homomorphisms) onto the centre were studied extensively also in [4] and in [8].

On a von Neumann algebra $\mathscr{A}$ each normal state φ has the representation $\varphi(A)=\sum_{i=1}^{\infty}\langle Ax_i, x_i\rangle$ where $\sum_{i=1}^{\infty}\|x_i\|^2=1$. In section 2 we prove a similar formula for center-valued states: if $\int\oplus\mathscr{A}(z)\,d\mu(z)$ is the central decomposition of $\mathscr{A}$ in the Hilbert space $\mathfrak{H}$, then any central-valued state τ has the form

$$\tau(A)=\int\oplus\sum_{i=1}^{\infty}\langle A(z)x_i(z), x_i(z)\rangle I(z)\,d\mu(z)\quad \Big(A=\int\oplus A(z)\,d\mu(z)\Big)$$

where $x_i\in\mathfrak{H}$ $(i\in\mathbf{N})$.

In the last section we use the above representation theorem to obtain an alternative proof of a result of H. HALPERN [5] and S. STRĂTILĂ—L. ZSIDÓ [8] concerning central ranges for elements of von Neumann algebras (here on separable spaces).

0. Preliminaries. We only consider separable Hilbert spaces $\mathfrak{H}$. $\mathscr{A}$ will always denote a von Neumann algebra on $\mathfrak{H}$, and $\mathscr{A}_1$ its closed unit ball.

For the reduction theory of von Neumann algebras we refer to [3] and [7].

In this paper Z always means a separable metric space and μ a positive Borel measure on Z. If

$$\mathfrak{H}=\int_Z\oplus\mathfrak{H}(z)\,d\mu(z)$$

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(cf. [3], chap. II, § 1, def. 3) then $\{x_i\}_{i=1}^{\infty}$ will be a dense sequence in $\mathfrak{H}$, for which we may assume that, for all $z\in Z$, $\{x_i(z)\}_{i=1}^{\infty}$ is dense in $\mathfrak{H}(z)$ and the map $z\mapsto \|x_i(z)\|$ is bounded.

If $\mathscr{B}\subset\mathscr{A}$ is bounded and $B\in\mathscr{B}$ then

$$V(n, m) = \left\{T\in\mathscr{B}\colon |\langle (T-B)x_i, x_j\rangle| \leqq \frac{1}{m},\ i, j\leqq n\right\} \quad (n, m\in \mathbf{N})$$

is a neighbourhood base of B in $\mathscr{B}$, endowed with the weak operator topology Consequently, $\mathscr{A}_1$ endowed with the weak operator topology can be metrized with the metric ϱ defined by

$$\varrho(A, B) = \sum_{i,j\in\mathbf{N}} |\langle (A-B)x_i, x_j\rangle|\cdot 2^{-i-j}.$$

1. Center-valued states. In this section we introduce the notion of center-valued state and establish some properties. (See also [4] and [5].)

1.1. Definition. Let $\mathscr{A}$ be a von Neumann algebra with center $\mathscr{C}$. By a *center-valued state* we mean a linear mapping τ from $\mathscr{A}$ into $\mathscr{C}$ such that

(i) $\tau(C\cdot A) = C\tau(A)$ $(A\in\mathscr{A},\ C\in\mathscr{C})$

(ii) $\tau(I) = I$

(iii) if $A\geqq 0$ then $\tau(A)\geqq 0$ $(A\in\mathscr{A})$.

1.2. Proposition. *Let $\mathscr{A}$ be a von Neumann algebra with center $\mathscr{C}$. The linear mapping $\tau\colon \mathscr{A}\to\mathscr{C}$ is a center-valued state if and only if the following conditions are fulfilled:*

(a) $\|\tau\| = 1$, (b) $\tau(C) = C$ $(C\in\mathscr{C})$.

Proof. Let τ be a center-valued state. If $A\geqq 0$ then $0\leqq\tau(A)\leqq\tau(\|A\|\cdot I) = \|A\|\cdot I$ so $\|\tau(A)\|\leqq\|A\|$. For an arbitrary $A\in\mathscr{A}$ the Schwarz-inequality gives that $\|\tau(A)\|^2 = \|\tau(A)^*\tau(A)\| = \|\tau(A^*)\tau(A)\|\leqq\|\tau(A^*A)\|\leqq\|A^*A\| = \|A\|^2$. Hence $\|\tau\|\leqq 1$, and (a) and (b) follow.

The converse is a special case of a well-known result of Tomiyama [10] on projections of norm one.

1.3. Definition. If $\mathscr{A}$ is a von Neumann algebra then the set of all center-valued states on $\mathscr{A}$ will be denoted by $\Sigma(\mathscr{A})$ and by Σ if $\mathscr{A}$ is fixed. We endow Σ with the topology of pointwise weak convergence.

1.4. Proposition. *$\Sigma(\mathscr{A})$ is compact.*

Proof. Let $X = \Pi\{X_A\colon A\in\mathscr{A}\}$ where X_A is $\|A\|\cdot\mathscr{C}_1$ with the compact weak operator topology. So X is compact. Define $e\colon \Sigma\to X$ by the formula $pr_A e(\tau) = \tau(A)$.

e is a topological embedding and we want to show that the range of e is closed. Set

$$\begin{aligned} H_1(A, B) &= \{\tau\in X:\ pr_{A+B}\tau = pr_A\tau + pr_B\tau\}, \\ H_2(A, \lambda) &= \{\tau\in X:\ pr_{\lambda A}\tau = \lambda pr_A\tau\}, \\ H_3(C) &= \{\tau\in X:\ pr_C\tau = C\}. \end{aligned}$$

These sets are closed for any $A, B\in\mathscr{A}$, $C\in\mathscr{C}$ and $\lambda\in\mathbf{C}$. Since $\|pr_A\tau\|\leqq\|A\|$ for any $A\in\mathscr{A}$ and $\tau\in X$, according to point 1.2,

$$e(\Sigma) = \bigcap_{A, B\in\mathscr{A}} H_1(A, B)\cap \bigcap_{\substack{A\in\mathscr{A}\\ \lambda\in\mathbf{C}}} H_2(A, \lambda)\cap \bigcap_{C\in\mathscr{C}} H_3(C)$$

that is the range of e is closed.

1.5. Proposition. *For a center-valued state τ on the von Neumann algebra $\mathscr{A}$ the following conditions are equivalent:*

(i) *τ is σ-weakly continuous,*
(ii) *τ is weakly continuous on the unit ball,*
(iii) *τ is strongly continuous on the unit ball,*
(iv) *$\tau^{-1}(0)$ is σ-weakly closed,*
(v) *τ is normal.*

Proof. We obtain the assertion by applying a theorem of TOMIYAMA [10] for the case of projections of norm onto the center.

1.6. Example. Assume that the von Neumann algebra $\mathscr{A}$ in the Hilbert space $\mathfrak{H}$ is expressed as a direct integral of factors, $\mathscr{A}=\int_Z\oplus\,\mathscr{A}(z)\,d\mu(z)$, and let $\mathfrak{H}=\int_Z\oplus\,\mathfrak{H}(z)\,d\mu(z)$ be the corresponding decomposition of $\mathfrak{H}$. If $x\in\mathfrak{H}$ such that $\|x(z)\|=1$ for μ-a.e. on Z, then

$$\tau:\ A\mapsto \int_Z \oplus\langle A(z)x(z), x(z)\rangle I(z)\,d\mu(z) \quad \Big(A=\int_Z\oplus A(z)\,d\mu(z)\Big)$$

is a normal center-valued state. (Here $I(z)$ stands for the identity operator on the space $\mathfrak{H}(z)$.)

The center of $\mathscr{A}$ consists of the diagonal operators and the verifications of (a) and (b) in 1.2 is easy. By Prop. 1.5 it remains only to prove the strong operator continuity of τ on the unit ball of $\mathscr{A}$.

Assume that $A_n\in\mathscr{A}$, $\|A_n\|\leqq 1$ and $A_n\xrightarrow{so}0$. In order to prove that $\tau(A_n)\xrightarrow{so}0$ it suffices to show that $\|\tau(A_n)u\|\to 0$ for every $u\in\mathfrak{H}$ such that $\|u(z)\|$ is bounded on Z (cf. [3], chap. II, § 1, prop. 7). But, setting $K=\sup\{\|u(z)\|:\ z\in Z\}$ we have by the Schwarz inequality

$$\|\tau(A_n)u\|^2 = \langle \tau(A_n)^* \tau(A_n)u, u\rangle \leqq \langle \tau(A_n^* A_n)u, u\rangle =$$
$$= \int \langle A_n(z)^* A_n(z), x(z), x(z)\rangle \langle u(z), u(z)\rangle \, d\mu(z) \leqq$$
$$\leqq K^2 \int \langle A_n(z)^* A_n(z) x(z), x(z)\rangle \, d\mu(z) = K^2 \|A_n x\|^2 \to 0.$$

1.7. Definition. $\Sigma^n(\mathscr{A})$ denotes the set of all normal center-valued states on the von Neumann algebra $\mathscr{A}$ endowed with the topology of pointwise convergence in the weak operator topology.

1.8. Proposition. *$\Sigma^n(\mathscr{A})$ is sequentially complete.*

Proof. It is sufficient to see that Σ^n is sequentially closed in Σ. Suppose that $\tau_n \to \tau$ and $\tau_n \in \Sigma^n$, $\tau \in \Sigma$. Let f be a normal linear functional on $\mathscr{C}$. Then $f \circ \tau_n$ is normal linear functional on $\mathscr{A}$. $f \circ \tau_n(A) \to f \circ \tau(A)$ for every $A \in \mathscr{A}$ and so $f \circ \tau$ is normal (see [1] Cor. III.3). Since $f \circ \tau$ is normal for every normal f on $\mathscr{C}$, τ is also normal.

2. Decomposition of center-valued states. In this section we show that if the von Neumann algebra $\mathscr{A}$ is expressed as a direct integral of von Neumann algebras then any normal center-valued state of $\mathscr{A}$ is decomposable concerning the integral.

2.1. Lemma. *Assume that $\mathscr{A} = \int_Z \oplus \mathscr{A}(z)\, d\mu(z)$. Then there exists a countable family $\mathscr{T}$ in $\mathscr{A}_1$ such that*

(i) *$\mathscr{T}$ is strongly dense in $\mathscr{A}_1$,*

(ii) *$\mathscr{T}(z) = \{T(z)\colon T \in \mathscr{T}\}$ is strongly dense in $\mathscr{A}(z)_1$, μ-a.e. on Z.*

(Here $T = \int_Z \oplus T(z)\, d\mu(z)$.)

Proof. By the definition of the direct integral of von Neumann algebras there is a sequence $A_n = \int_Z \oplus A_n(z)\, d\mu(z)$ $(n \in \mathbf{N})$ such that $\mathscr{A}(z)$ is the von Neumann algebra generated by $\{A_n(z)\colon n \in \mathbf{N}\}$ μ-a.e. on Z and we may assume that $\mathscr{A}$ is generated by $\{A_n\colon n \in \mathbf{N}\}$. Let $\mathscr{K}$ be the *-algebra over the complex rationals generated by $\{A_n\colon n \in \mathbf{N}\}$. Take

$$\mathscr{T} = \Big\{ \int_Z \oplus \tilde{T}(z)\, d\mu(z)\colon T = \int_Z \oplus T(z)\, d\mu(z) \in \mathscr{K} \Big\}, \quad \tilde{A} = \begin{cases} A & \text{if } \|A\| \leqq 1, \\ A \cdot \|A\|^{-1} & \text{if } \|A\| > 1. \end{cases}$$

$\mathscr{T}$ is countable and by Kaplansky's density theorem it satisfies (i)—(ii).

2.2. Theorem. *Let $\mathscr{A} = \int_Z \oplus \mathscr{A}(z)\, d\mu(z)$ and τ be a normal center-valued state on $\mathscr{A}$. Then for almost every $z \in Z$ there is a normal center-valued state τ_z on $\mathscr{A}(z)$ such that for every $A = \int_Z \oplus A(z)\, d\mu(z) \in \mathscr{A}$ the operator field $z \mapsto \tau_z A(z)$ is μ-meas-*

urable and

$$\tau(A)=\int_Z \oplus \tau_z A(z)\,d\mu(z).$$

Proof. Using the lemma we have two countable families $\mathscr{S}$ and $\mathscr{T}$ such that
(i) $\mathscr{T}(z)\subset\mathscr{A}(z)_1$ and $\zeta(z)\subset\mathscr{C}(z_1)$ μ-a.e. on Z,
(ii) $\mathscr{T}(\mathscr{T}(z))$ is strongly dense in $\mathscr{A}_1$ (in $\mathscr{A}(z)_1$ μ-a.e. on Z),
(iii) $\mathscr{S}$ $(\mathscr{S}(z))$ is strongly dense in $\mathscr{C}_1$ (in $\mathscr{C}(z)_1$ μ-a.e. on Z).
Let

$$\mathscr{R}=\left\{\sum_{i=1}^{k}\alpha_i S_i T_i\colon\ k\in\mathbf{N};\ S_i\in\mathscr{S},\ T_i\in\zeta,\ \alpha_i \text{ is complex rational } (i\geqq k)\right\}.$$

If τ is a normal center-valued state then for $z\in Z$ we define $\hat{\tau}_z$ by the formula

$$\hat{\tau}_z\left(\sum_{i=1}^{k}\alpha_i S_i(z)T_i(z)\right)=\sum_{i=1}^{k}\alpha_i S_i z)\tau(T_i)(z)$$

where $\sum_{i=1}^{k}\alpha_i S_i T_i\in\mathscr{R}$. We will show that $\hat{\tau}_z$ is well-defined μ-a.e. on Z.

Take $R_1, R_2\in\mathscr{R}$ $\left(R_1=\sum_{i=1}^{k}\alpha_i S_i T_i,\ R_2=\sum_{j=1}^{l}\beta_j S_j T_j\right)$
and put

$$H(R_1, R_2)=\left\{z\in Z\colon\ R_1(z)=R_2(z),\ \sum_{i=1}^{k}\alpha_i S_i(z)\tau(T_i)(z)\neq\sum_{j=1}^{l}\beta_j S_j(z)\tau(T_j)(z)\right\}.$$

This set is measurable and its characteristic function χ belongs to $\mathscr{C}$. Hence

$$\chi\tau(R_1)=\tau(\chi R_1)=\tau(\chi R_2)=\chi\tau(R_2).$$

So $\tau(R_1)(z)=\tau(R_2)(z)$ for μ-a.e. $z\in H(R_1, R_2)$. Since $\sum_{i=1}^{k}\alpha_i S_i(z)\,\tau(T_i)(z)=\tau(R_1)(z)$ and $\sum_{j=1}^{l}\beta_j S_j(z)\tau(T_j)(z)=\tau(R_2)(z)$ μ-a.e., we have obtained that $\mu(H(R_1, R_2))=0$. Since $\mathscr{R}$ is countable, if follows

$$\mu\Big(\bigcup_{R_1, R_2\in\mathscr{R}} H(R_1, R_2)\Big)=0.$$

Let

$$S=\{z\in Z\colon\ \hat{\tau}_z|\mathscr{R}(z)_1 \text{ is not weak operator continuous at } 0\},$$

where $\mathscr{R}(z)_1=\{R(z)\colon\ R\in\mathscr{R},\ \|R(z)\|\leqq 1\}$.

We claim that $\mu(S)=0$. For $A, B\in\mathscr{A}(z)$ define

$$\varrho_z(A, B)=\sum_{i,j\in\mathbf{N}}|\langle(A-B)x_j(z), x_i(z)\rangle|2^{-i-j}.$$

So ϱ_z is a measurable field of metrics metrizing the unit ball of $\mathscr{A}(z)$ endowed with the weak operator topology (see 0). The set

$$H(k, l, \varepsilon, \delta) = \{z\in Z\colon \text{ there is } R\in\mathscr{R} \text{ such that } \|R(z)\| < 1,$$
$$\varrho_z(R(z), 0) < \delta \text{ and } |\langle \hat{\tau}_z(R(z))x_l(z), x_k(z)\rangle| > \varepsilon\}$$

is measurable and

$$S = \bigcup_{l=1}^{\infty}\bigcup_{k=1}^{\infty}\bigcup_{i=1}^{\infty}\bigcap_{j=0}^{\infty} H(k, l, \varepsilon_i, \delta_j)$$

provided that $\varepsilon_i \searrow 0$ and $\delta_j \searrow 0$. Hence S is measurable.

Suppose that $\mu(S)>0$. Then we have $K\subset S$, $\varepsilon>0$, and $k, l\in\mathbf{N}$ such that

(iv) $\mu(K) > 0$

and for $z\in K$ and $j\in\mathbf{N}$ there is an $R_z^j\in\mathscr{R}$ with the properties

(v) $\|R_z^j(z)\| < 1$,

(vi) $\varrho_z(R_z^j(z), 0) < \delta_j$,

(vii) $|\langle \hat{\tau}_z(R_z^j(z))x_l(z),\ x_k(z)\rangle| > \varepsilon$.

By Lusin's lemma we may assume that K is compact and the functions

(viii) $z \mapsto \|R(z)\|$,

(ix) $z \mapsto \varrho_z(R(z), 0)$,

(x) $z \mapsto \langle(\tau R)(z)x_l(z), x_k(z)\rangle$

are continuous on K for any $R\in\mathscr{R}$. In this case the inequalities (v)—(vii) are fulfilled on an open set in K. For any $j\in\mathbf{N}$ a compactness argument gives a measurable partition $\{H_i^j\colon i\leqq p(j)\}$ of K and operators $R_i^j\in\mathscr{R}$ $(i\leqq p(j))$ such that for $z\in H_i^j$ $R_i^j(z)$ satisfies (v)—(vii). Let χ_i^j be the characteristic function of H_i^j $(j\in\mathbf{N},\ i\leqq p(j))$ and define

$$R^j(z) = \sum_{i=1}^{P(j)} \chi_i^j(z)R_i^j(z)e_i^j(z) \quad \text{where} \quad e_i^j(z) = \overline{\operatorname{Arg}\langle\hat{\tau}_z R_i^j(z)x_l(z), x_k(z)\rangle},$$

and for $0\neq\lambda\in\mathbf{C}$ set $\operatorname{Arg}\lambda = \lambda\cdot|\lambda|^{-1}$.

Taking $R^j = \int_Z^{\oplus} R^j(z)\,d\mu(z)$ we have $R^j\in\mathscr{A}_1$ and

$$\varrho(R^j, 0) \leqq \int_Z \varrho_z(R^j(z), 0)\,d\mu(z) = \sum_{i=1}^{P(j)} \int_{H_i^j} \varrho_z(R_i^j(z), 0)\,d\mu(z) \leqq \mu(K)\delta_j;$$

moreover,

$$\langle\tau(R^j)x_l, x_k\rangle = \int_Z \langle\tau R^j(z)x_l(z), x_k(z)\rangle\,d\mu(z) =$$
$$= \sum_{i=1}^{P(j)} \int_{H_i^j} |\langle\hat{\tau}_z R_i^j(z)x_l(z), x_k(z)\rangle|\,d\mu(z) \geqq \mu(K)\varepsilon.$$

This contradicts the continuity of τ. Hence $\mu(S)=0$ so $\hat{\tau}_z|\mathscr{R}(z)_1$ is weak operator continuous μ-a.e. on Z. It is then also uniformly continuous with respect to the uniformity defined by the metric ϱ_z.

Now extend $\hat{\tau}_z|\mathscr{R}(z)_1$ by uniform continuity with respect to the compact metrizable weak operator topology to $\mathscr{A}(z)_1$ and then by the homogeneity to $\mathscr{A}(z)$. So we get a linear τ_z such that $\|\tau_z\|\leqq 1$ and $\tau_z|\mathscr{C}(z)$ is the identity. Hence τ_z is a center-valued state μ-a.e. on Z.

We want to check that $\tau(A)=\int_Z \oplus\, \tau_z A(z)\,d\mu$ if $A=\int_Z \oplus\, A(z)\,d\mu(z)$. We may assume that $\|A\|\leqq 1$. In this case there is a sequence $T_n\in\mathscr{T}$ $(n\in\mathbf{N})$ such that $T_n\xrightarrow{s}A$. Then for a subsequence T_{n_k} we have $T_{n_k}(z)\xrightarrow{s}A(z)$ for μ-a.e. $z\in Z$ (cf. [3] chap. II. § 2. prop. 4). According to the weak continuity of τ_z now ${}_zT_{n_k}(z)\xrightarrow{w}\tau_z A(z)$. Consequently $\tau(T_{n_k})=\int_Z \oplus\, \tau_z T_{n_k}(z)\,d\mu(z)\xrightarrow{w}\int_Z \oplus\, \tau_z A(z)\,d\mu(z)$, On the other hand $\tau(T_{n_k})\xrightarrow{w}\tau_z(A)$ so $\tau(A)=\int_Z \oplus\, \tau_z A(z)\,d\mu(z)$.

2.3. Theorem. *Let $\mathscr{A}$ be a von Neumann algebra acting on the Hilbert space $\mathfrak{H}=\int\oplus\,\mathfrak{H}(z)\,d\mu(z)$ and suppose that $\mathscr{A}$ is decomposable as a direct integral of factors $\int\oplus\,\mathscr{A}(z)\,d\mu(z)$. Then $\tau\colon \mathscr{A}\to\mathscr{C}$ is a normal center-valued state if and only if it has the form*

$$\tau(A)=\int\oplus\sum_{i=1}^{\infty}\langle A(z)u_i(z),u_i(z)\rangle I(z)\,d\mu(z)\quad \left(A=\int\oplus A(z)\,d\mu(z)\right)$$

where $u_i\in\mathfrak{H}$ $(i\in\mathbf{N})$ and $\sum_{i=1}^{\infty}\|u_i(z)\|^2=1$ μ-a.e. on Z.

Proof. If $\tau\colon \mathscr{A}\to\mathscr{C}$ has the form described above then τ is a center-valued state since $\mathscr{C}$ consists of the diagonal operators and it follows from 1.6 that τ is normal.

Now assume that τ is a normal center-valued state. By Theorem 2.2, $\tau=\int_Z\oplus\,\tau_z\,d\mu(z)$. Let $H_n=\{z\in Z\colon \dim\mathfrak{H}(z)=n\}$ $(n=1,2,\ldots,\infty)$ and put $\tau_n=\int_{H_n}\oplus\,\tau_z\,d\mu(z)$. So $\tau=\oplus\tau_n$ and it suffices to prove the theorem for τ_n $(n=1,2,\ldots,\infty)$. Hence we may identify each $\mathfrak{H}(z)$ with a fixed Hilbert space $\mathfrak{H}_0$.

Let Y be the unit ball of $\mathfrak{H}_0\oplus\mathfrak{H}_0\oplus\ldots$ endowed with the weak topology. So Y is a compact metrizable space.

Let $\mathscr{T}\subset\mathscr{A}$ be a countable family with the properties (i)—(ii) in 2.1. Define

$$H(T)=\left\{(z,y_1,y_2,\ldots)\in Z\times Y\colon \tau_z T(z)=\sum_{i=0}^{\infty}\langle T(z)y_i,y_i\rangle I(z)\right\}$$

$H(T)$ is a Borel set in $Z\times Y$ and so is $H=\bigcap\{H(T)\colon T\in\mathscr{T}\}$.

We will use the principle of measurable choice (see [7] p. 35). H is analytic and for $z\in Z$ there is a normal state φ_z on $\mathscr{A}(z)$ such that $\tau_z=\varphi_z\cdot I(z)$. Hence,

$$\tau_z(S) = \sum_{i=1}^{\infty}\langle Su_z^i, u_z^i\rangle I(z)$$

for every $S\in\mathscr{A}(z)$ and for some $u_z^i\in\mathfrak{H}_0$ ($i\in\mathbf{N}$). Consequently $(z, u_z^1, u_z^2, \ldots)\in H$. By the principle of measurable choice there exists a μ-measurable function $\Phi\colon Z\to Y$ such that $\Phi(z)\in H$ μ-a.e. on Z. If $\Phi(z)=(u_1(z), u_2(z), \ldots)$ then $u_i\in\mathfrak{H}$ ($i\in\mathbf{N}$). We have obtained that

$$\tau_z T(z) = \sum_{i=1}^{\infty}\langle T(z)u_i(z), u_i(z)\rangle I(z)$$

for any $T\in\mathscr{T}$, μ-a.e. on Z. Since $\mathscr{T}$ is dense in $\mathscr{A}_1$ and τ is continuous we have

$$\tau(A) = \int_Z\oplus\sum_{i=1}^{\infty}\langle A(z)u_i(z), u_i(z)\rangle I(z)\,d\mu(z)$$

for every $A\in\mathscr{A}$. Moreover, $\sum_{i=1}^{\infty}\|u_i(z)\|^2=1$ μ-a.e. on Z because $\tau(I)=I$. This completes the proof.

3. An application. In this section we use 2.2 in order to give an alternative proof for an extension, given in [8] and [5], of a result of J. B. Conway.

3.1. We introduce some notations. If A belongs to the algebra $\mathscr{A}$ then let $C_0(A)$ be the convex hull of the set $\{UAU^*\colon U$ is a unitary in $\mathscr{A}\}$. Moreover, let $C(A)=\overline{C_0(A)}^w\cap\mathscr{C}$ and $\overline{W}(A)$ be the closed numerical range of A. The following proposition was proved by J. B. Conway [2].

3.2. Proposition. *If $\mathscr{A}$ is a type* III *factor and $A\in\mathscr{A}$ then $C(A)=\overline{W}(A)=$ $=\Sigma(A)$.*

3.3. Proposition. *If $\mathscr{A}=\int_Z\oplus\mathscr{A}(z)\,d\mu(z)$ and $A=\int_Z\oplus A(z)\,d\mu(z)$ then $C(A)=\int_Z\oplus C(A(z))\,d\mu(z)$.*

The latter assertion was proved by S. Komlósi [5] and it means that $B=\int_Z\oplus B(z)\,d\mu(z)\in C(A)$ if and only if $B(z)\in C(A(z))$ μ-a.e. on Z.

3.4. Lemma. *Let $\mathscr{A}=\int_Z\oplus\mathscr{A}(z)\,d\mu(z)$ and $A=\int_Z\oplus A(z)\,d\mu(z)\in\mathscr{A}$. Assume that U is a weak operator neighbourhood of the diagonal operator $B=\int_Z\oplus f(z)I(z)\,d\mu(z)$*

and $f(z)\in\overline{W}(A(z))$ *for* $z\in Z$. *Then there is a* $u\in\mathfrak{H}$ *such that*

(i) $\int\limits_Z \oplus\langle A(z)u(z), u(z)\rangle d\mu(z)\in U$,

(ii) $\|u(z)\|=1$ μ-a.e. on Z.

Proof. Take a sequence $\{e_n\}\subset\mathfrak{H}$ such that $\|e_n(z)\|=1$ and $\{e_n(z)\}$ is dense in $\{s\in\mathfrak{H}(z): \|s\|=1\}$ μ-a.e. on Z. Suppose that U is determined by $\varepsilon>0$ and $y_i\in\mathfrak{H}$ $(i\leqq m)$ that is

$$U=\{T\in\mathscr{A}: |\langle (T-B)y_i: y_j\rangle|<\varepsilon \ \ (i,j\leqq m)\}.$$

Choose a compact $K\subset Z$ with the properties

(a) $z\mapsto\langle A(z)e_i(z), e_j(z)\rangle$ is continuous on K $(i,j\in\mathbf{N})$,

(b) $\mu(Z\setminus K)<\delta$,

(c) $\int\limits_{Z\setminus K}|\langle y_i(z), y_j(z)\rangle|\,d\mu(z)<\delta \quad (i,j\leqq m)$,

(d) f is continuous on K.

(δ is arbitrary but fixed). We can find $x_z\in\{e_n\}$ and an open G_z containing z such that

$$|f(v)-\langle A(v)x_z(v), x_z(v)\rangle|<\delta \quad (v\in G_z)$$

for $z\in K$. Using compactness one has a measurable partition $\{H_i: i\leqq k\}$ of K such that $H_i\subset G_{z_i}$ for some $z_i\in K$ $(i\leqq k)$. Let χ_i (χ) be the characteristic function of H_i $(Z\setminus K)$ and define u by

$$u(z)=\chi(z)e_1(z)+\sum_{i=1}^{k}\chi_i(z)x_{z_i}(z) \quad (z\in Z).$$

$\|u(z)\|=1$ is fulfilled evidently for μ-a.e. $z\in Z$. An easy estimation gives

$$\left|\left\langle\left(\int\limits_Z\oplus\langle A(z)u(z), u(z)\rangle I(z)\,d\mu(z)-B\right)y_i,\ y_j\right\rangle\right|\leqq\delta(\|A\|+\|B\|+\|y_i\|\cdot\|y_j\|).$$

So if δ is small enough then (i) is satisfied.

3.5. Theorem. *If* $\mathscr{A}$ *is a type* III *von Neumann algebra and* $A\in\mathscr{A}$ *then*

$$C(A)=\overline{\Sigma^n(A)}^w.$$

Proof. We express $\mathscr{A}$ as a direct integral of type III factors: $\int\limits_Z\oplus\mathscr{A}(z)\,d\mu(z)$. If $B=\int\limits_Z\oplus B(z)d\mu(z)\in C(A)$ then $B(z)\in C(A(z))$ μ-a.e. on Z by 3.3. According to 3.2 $B(z)\in\overline{W}(A(z))$ and we can use 3.4. For every weak neighbourhood U of B

there is a $u\in\mathfrak{H}$ such that

$$\int_Z \oplus A\langle (z)u(z), u(z)\rangle I(z)\,d\mu(z)\in U.$$

However, $T\mapsto \int_Z \oplus \langle T(z)u(z), u(z)\rangle I(z)\,d\mu(z)$ defines a normal center-valued state (cf. 1.6) hence $C(A)\subset \overline{\Sigma^n(A)}^w$.

Conversely, for any $\tau\in\Sigma^n$, $\tau=\int_Z \oplus \tau_z\,d\mu(z)$ follows from 2.2 and $\tau_z(A(z))\in C(A(z))$ from 3.2. By 3.3 we have $\int_Z \oplus \tau_z(A(z))\,d\mu(z)\in \int_Z \oplus C(A(z))\,d\mu(z)=C(A)$. So $\tau(A)\in C(A)$ and we have obtained that $\Sigma^n(A)\subset C(A)$. Since $C(A)$ is closed, it follows $\overline{\Sigma^n(A)}^w\subset C(A)$.

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