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Carleman and Korotkov operators on Banach spaces

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This paper is an attempt to use basic theory of vector measures as an approach to study certain types of classical integral operators and their generalizations. The first section begins with a look at the classical Carleman integral operators from L_2 to L_2. In the course a connection, which seems to us to be heretofore unnoticed, is established with operators whose truncates on large sets is compact into L_∞. This is then generalized to the consideration of operators from any Banach space into the spaces $L_p(\mu)$ $0 \leqq p \leqq \infty$. The second section is devoted to an application to a "folklore theorem" about weak compactness in $L_\infty(\mu)$. Next specialization is made of the general results of section 1 to integral operators from one L_p space to another in section 3. A class of operators we call Korotkov operators are studied in section 4. The last section is devoted to extensions to general function spaces.

1. Carleman operators

We start with a description of the classical situation as motivation for the current work. The book of HALMOS and SUNDER [6] may be consulted for more details. Let (Ω, Σ, μ) be a finite separable measure space and let T be a linear operator from $L_2(\mu)$ to $L_2(\mu)$. The operator T is called an *integral operator* if there is a $\mu \times \mu$-measurable function k such that

(i) $k(s, \cdot)f(\cdot) \in L_1(\mu)$ for μ-almost all s and all $f \in L_2(\mu)$,

(ii) $\int_\Omega k(\cdot, t)f(t)\,d\mu(t) \in L_2(\mu)$ for all f in $L_2(\mu)$, and

(iii) $Tf(s) = \int_\Omega k(s, t)f(t)\,d\mu(t)$ almost everywhere for all f in $L_2(\mu)$.

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It is an old theorem of BANACH [1] that such an operator is automatically continuous. An integral operator is called a *Carleman integral operator* if its kernel additionally satisfies

(iv) $k(s, \cdot)\in L_2(\mu)$ for μ-almost all s.

Following Halmos—Sunder, Korotkov and probably many others, note that the last condition can be interpreted to mean that there exists a function $g\colon \Omega\to L_2(\mu)$ such that $g(s)(\cdot)=k(s, \cdot)$ for almost all s in Ω. By the Riesz representation theorem we may regard g as taking values in $(L_2(\mu))^*(=L_2(\mu))$ and read condition (ii) as saying $\langle g(\cdot), f\rangle\in L_2(\mu)$ for all f in $L_2(\mu)$ and read (iii) as saying that $Tf(\cdot)=\langle g(\cdot), f\rangle$ for all f in $L_2(\mu)$. Moreover since $\langle g(\cdot), f\rangle\in L_2(\mu)$ for all f in $L_2(\mu)$ and since $L_2(\mu)^*=L_2(\mu)$ we see that g is a weakly measurable function into $L_2(\mu)$. Since $L_2(\mu)$ is separable here, PETTIS's measurability theorem [3, II. 2.2] shows that g is (strongly) measurable.

Conversely if $g\colon \Omega\to L_2(\mu)$ is measurable and if $\langle g(\cdot), f\rangle\in L_2(\mu)$ for all f in $L_2(\mu)$, then a theorem of DUNFORD and PETTIS [4, III. 11.17] produces a $\mu\times\mu$-measurable function k such that $k(s, \cdot)=g(s)(\cdot)$ for almost all s in Ω with the property that

$$\langle g(\cdot), f\rangle=\int_\Omega k(\cdot, t)f(t)\,d\mu(t)$$

almost everywhere. Thus the operator on $L_2(\mu)$ to $L_2(\mu)$ defined by $f\to\langle g(\cdot), f\rangle$ is a Carleman integral operator.

This proves the first theorem and sets the perspective of most of this paper.

Theorem 1. *Let (Ω, Σ, μ) be a finite separable measure space. A linear operator $T\colon L_2(\mu)\to L_2(\mu)$ is a Carleman integral operator if and only if there exists a measurable $g\colon \Omega\to L_2(\mu)$ such that for each f in $L_2(\mu)$ the equality $(Tf)(\cdot)=\langle g(\cdot), f\rangle$ obtains almost everywhere.*

Throughout the remainder of the paper let (Ω, Σ, μ) be an arbitrary finite measure space; let $L_p(\mu)$, $0<p\leqq\infty$ be the usual Lebesgue spaces and let $L_0(\mu)$ be the space of all measurable (equivalence classes of) functions on Ω under the topology of convergence in measure.

Definition 2. Let $0\leqq p\leqq\infty$ and let X be a Banach space. A linear operator $T\colon X\to L_p(\mu)$ is called a *Carleman operator* if there is a measurable function $g\colon \Omega\to X^*$ such that $(Tx)(s)=\langle g(s), x\rangle$ almost everywhere for all x in X. In this case g is called the *kernel* of T.

A quick application of the closed graph theorem shows that a Carleman operator is necessarily continuous.

Recall that [3, Chap. III] a Banach space X has the *Radon—Nikodým property* (RNP) if for any finite measure space (Ω, Σ, μ) and for any bounded linear operator $T\colon L_1(\mu)\to X$ there is a bounded measurable function $g\colon \Omega\to X$ such that

$$Tf = \int_\Omega fg\,d\mu,$$

as a Bochner integral, for all f in $L_1(\mu)$.

The first theorem is a generalization of a theorem of KOROTKOV [10, 11] and is closely related to a theorem of WONG [17].

Theorem 3. *Let* $0\leqq p\leqq\infty$ *and let* X *be a Banach space whose dual* X^* *has* RNP. *A bounded linear operator* $T\colon X\to L_p(\mu)$ *is a Carleman operator if and only if there is a measurable function* Φ *on* Ω *such that*

$$|(Tx)(s)| \leqq \|x\|\,\Phi(s)$$

almost everywhere for each x *in* X.

Proof. Suppose T is a Carleman operator with kernel g. Then, for s in Ω, we have

$$|(Tx)(s)| = |g(s)x| \leqq \|g(s)\|_{X^*}\|x\|_X.$$

Since g is measurable, so is $\|g(\cdot)\|_{X^*}$. Taking $\Phi(s)=\|g(s)\|_{X^*}$ proves the necessity.

Conversely, suppose there is a measurable function $\Phi\colon \Omega\to\mathbf{R}$ such that $|(Tx)(s)|\leqq\|x\|\,\Phi(s)$ almost everywhere for all $x\in X$. Without loss of generality, we may assume that $\Phi(w)$ is everywhere finite. For $n\geqq 1$, define the sets $E_n=[n-1\leqq|\Phi|<n]$ and note that Φ is bounded on each E_n. Define the operator E_n on every L_p by

$$E_n f = f\chi_{E_n}.$$

By hypothesis, $E_n\circ T$ has its range in $L_\infty(\mu)$; moreover an application of the closed graph theorem shows that

$$E_n\circ T\colon X\to L_\infty(\mu)$$

is continuous. Define $S_n\colon L_\infty(\mu)^*\to X^*$ to be the adjoint of $E_n\circ T$ and consider its restriction to $L_1(\mu)$. Since X^* has RNP, there is a measurable $g_n\colon \Omega\to X^*$ such that

$$S_n f = \int_\Omega fg_n\,d\mu \quad \text{for all} \quad f\in L_1(\mu).$$

Next observe that $((E_n\circ T)x)(\cdot)=\langle g_n(\cdot), x\rangle$ a.e. by computing for f in $L_1(\mu)$ and x in X the integral

$$\int_\Omega (E_nT)xf\,d\mu = \langle S_n(f), x\rangle = \Big\langle \int_\Omega fg_n\,d\mu, x\Big\rangle = \int_\Omega f\langle g_n, x\rangle\,d\mu$$

where the last equality follows from the "commuting" of the Bochner integral with bounded linear operators [3, II. 2.9].

In particular, $\langle g_n(\cdot), x\rangle$ vanishes almost everywhere outside E_n for each x in X. Without loss of generality, we may take $g_n(\omega)=0$ for $\omega\notin E_n$. Then g_n is still measurable and $((E_n\circ T)x)(\cdot)=\langle g_n(\cdot), x\rangle$ for all x in X.

Now define $g\colon \Omega\to X^*$ by $g(s)=g_n(s)$ for s in E_n (recall that the E_n are disjoint and exhaustive). Then g is measurable and it remains only to show that $(Tx)(s)=\langle g(s), x\rangle$ almost everywhere for x in X.

To this end, note that if $h\in L_p(\mu)$, then

$$h = \lim_m h\chi_{F_m} \quad \text{in measure, where} \quad F_m = \bigcup_{n=1}^m E_n.$$

Hence if $x\in X$, then

$$Tx = \lim_m (Tx)\chi_{F_m} = \lim_m \sum_{n=1}^m (E_n\circ T)x = \lim_m \sum_{n=1}^m \langle g_n(\cdot), x\rangle =$$

$$= \lim_m \langle g(\cdot), x\rangle\chi_{F_m} = \langle g(\cdot), x\rangle,$$

where all limits are taken in measure. This completes the proof.

The first corollary is implicit in STEGALL [16]. Its converse is also true, but we shall not prove it here because it is the main theme of Stegall's paper.

Corollary 4. *If X^* has* RNP, *then every continuous linear operator from X into $L_\infty(\mu)$ is a Carleman operator.*

Proof. If $T\colon X\to L_\infty(\mu)$ is a continuous linear operator, then $|Tx|(\cdot)\leqq \|T\|\,\|x\|$ a.e.; apply Theorem 3.

Corollary 5. *A weakly compact operator from an arbitrary Banach space into $L_\infty(\mu)$ is a Carleman operator.*

Proof. Let X be an arbitrary Banach space and let $T\colon X\to L_\infty(\mu)$ be a weakly compact operator. Then $T^*\colon L_\infty(\mu)\to X^*$ is weakly compact as is its restriction to $L_1(\mu)$. By a classical theorem of DUNFORD, PETTIS, and PHILLIPS [3, III. 2.12], there exists a measurable $g\colon \Omega\to X^*$ such that

$$T^*f = \int fg\,d\mu, \quad f\in L_1(\mu).$$

Now, by the proof of Theorem 3, we see that

$$(Tx)(\cdot) = \langle g(\cdot), x\rangle$$

for all $x\in X$, so that T is indeed a Carleman operator.

The next theorem characterizes Carleman operators from X to $L_p(\mu)$ in terms of compactness of the operator into $L_\infty(\mu)$. No RNP assumptions need be made on X or its dual. This theorem appears to be new even in the classical case of Carleman integral operators from $L_2(\mu)$ to $L_2(\mu)$. Recall that to each set E in Σ there is the associated operator $E\colon L_p(\mu)\to L_p(\mu)$ defined by $Ef=f\chi_E$. Call a linear operator $T\colon X\to L_p(\mu)$ *almost weakly (or norm) compact into* $L_\infty(\mu)$ if for each $\varepsilon>0$ there is a set $E\in\Sigma$ with $\mu(\Omega\setminus E)<\varepsilon$ such that $E\circ T$ is a weakly (or norm) compact operator X into $L_\infty(\mu)$.

Theorem 6. *Let $0\leqq p\leqq\infty$ and let X be a Banach space. If a continuous linear operator $T\colon X\to L_p(\mu)$ is almost weakly compact into $L_\infty(\mu)$, then T is a Carleman operator. Conversely, if T is a Carleman operator, then T is almost norm compact into $L_\infty(\mu)$.*

Proof. Suppose $T\colon X\to L_p(\mu)$ is almost weakly compact into $L_\infty(\mu)$. Then there is a disjoint sequence (E_n) in Σ with $\bigcup_{n=1}^{\infty} E_n=\Omega$ such that $E_n\circ T\colon X\to L_\infty(\mu)$ is weakly compact. By Corollary 5 each $E_n\circ T$ is a Carleman operator and thus is given by $(E_n\circ T)x=g_n(\cdot)x$ where $g_n\colon \Omega\to X^*$ is a measurable function supported on E_n. Define $g(s)=g_n(s)$ for $s\in E_n$ and proceed as in the proof of Theorem 3 to prove $Tx=g(\cdot)x$. Thus T is a Carleman operator.

On the other hand, suppose $T\colon X\to L_p(\mu)$ is a Carleman operator with kernel g. Since g is measurable, there is a sequence (g_n) of measurable simple functions from Ω to X^* such that $\lim_m \|g_n(\cdot)-g(\cdot)\|_X=0$ almost everywhere. Fix $\varepsilon>0$ and use Egorov's theorem to obtain a set $E\in\Sigma$ such that $\mu(\Omega\setminus E)<\varepsilon$ and $\lim_m \|g_n(\cdot)-g(\cdot)\|_X=0$ uniformly on E. Since g must be bounded on E, $E\circ T$ maps X into $L_\infty(\cdot)$. Define the finite rank operators $T_n\colon X\to L_\infty(\mu)$ by $T_n x=\chi_E g_n(\cdot)x$. Then

$$\lim_n \sup_{\|x\|\leqq 1} \|E\circ T(x)-E\circ T_n(x)\|_\infty=\lim_n \sup_{\|x\|\leqq 1} \|\chi_E g(\cdot)x-\chi_E g_n(\cdot)x\|_\infty\leqq$$

$$\leqq \lim_n \sup_{\|x\|\leqq 1} \sup_{s\in E} |g(s)x-g_n(s)x| = \lim_n \sup_{s\in E} \|g(s)-g_n(s)\|_{X^*}=0.$$

Thus, $E\circ T\colon X\to L_\infty(\mu)$ is the uniform limit of finite rank operators and is consequently a compact operator.

At this point, we are again getting close to the theme of STEGALL [16]. In his recent study of the RNP in dual spaces, Stegall effectively works with Carleman operators from X to $L_\infty(\mu)$ and proves that X^* has RNP if and only if all operators from X to $L_\infty(\mu)$ are Carleman operators. The connection is as follows: An operator $T\colon X\to L_\infty(\mu)$ is a Carleman operator if and only if the restriction of the adjoint T^* on $L_1(\mu)$ is representable in the sense of [3, Chap. II].

2. Weakly compact sets in $L_\infty(\mu)$

This section is a bit of a digression to some well-established but not so well-known facts about weakly compact sets in $L_\infty(\mu)$. Probably this section should have been included in [3, Chap. VIII]; it certainly could have been. The subject of this section is a folklore theorem which explains the interchange between norm and weak compactness in $L_\infty(\mu)$ and thus explains the interchange in Theorem 6. Probably all of this section should be attributed to GROTHENDIECK [5], but unfortunately the theorem which we are about to prove does not seem to be generally known. In keeping with the terminology above, a subset W of $L_\infty(\mu)$ will be called (relatively) *almost norm compact* if for each $\varepsilon>0$ there exists E in Σ with $\mu(\Omega\setminus E)<\varepsilon$ such that $\chi_E W$ is (relatively) norm compact in $L_\infty(\mu)$.

Theorem 7. (Folklore) *If W is a relatively weakly compact subset of $L_\infty(\mu)$, then W is relatively almost norm compact in $L_\infty(\mu)$.*

Proof. Let W be a relatively weakly compact subset of $L_\infty(\mu)$. By the factorization theorem of DAVIS, FIGIEL, JOHNSON, and PELCZYŃSKI [2] there is a reflexive Banach space R and a bounded linear operator T: $R\to L_\infty(\mu)$ such that $W\subseteqq T(B_R)$, where B_R is the closed unit ball of R. By Corollary 5, the weakly compact operator T is Carleman. By Theorem 6, given $\varepsilon<0$ there is $E\in\Sigma$ with $\mu(\Omega\setminus E)<\varepsilon$ such that E T: $R\to L_\infty(\mu)$ is a compact operator. Thus $\chi_E W\subseteqq E\circ T(B_R)$ is relatively norm compact. This completes the proof.

Note that the converse of Theorem 7 is not true. Indeed, if X^* has the RNP, then any operator T: $X\to L_\infty(\mu)$ is a Carleman operator by Corollary 4. Thus by Theorem 6, the operator T is almost norm compact into $L_\infty(\mu)$; i.e. $T(B_X)$ is relatively almost norm compact in $L_\infty(\mu)$ as in the conclusion of Theorem 7. But, if X is not reflexive, then T need not be a weakly compact operator; so, $T(B_X)$ need not be weakly compact in $L_\infty(\mu)$.

Theorem 7 has an easy corollary which seems to have been known to Grothendieck. To our best knowledge it was first stated explicitly by PERESSINI [14, Prop. 5]. Weaker versions of it were proved by ZOLEZZI [18] and KHURANA [8]. (See KHURANA [9] for an interesting generalization to the vector-valued case.)

Corollary 8. *If (Ω, Σ, μ) is a finite measure space, then a weakly convergent sequence in $L_\infty(\mu)$ converges almost everywhere.*

3. Carleman operators as classical integral operators

The definition of a Carleman operator was motivated by that of a Carleman integral operator [6] from $L_2(\mu)$ to $L_2(\mu)$. The definition of Carleman integral operator has a natural extension to the other $L_p(\mu)$ spaces and it will be shown here that the natural extension coincides with the definition of Carleman operators on $L_p(\mu)$. Throughout this section $(S, \mathscr{F}, \lambda)$ and (Ω, Σ, μ) will be finite measure spaces. In addition p and r will be numbers such that $0 \leqq p \leqq \infty$ and $1 < r < \infty$. The number s will be conjugate to r in the sense that $r^{-1}+s^{-1}=1$.

Theorem 9. *A continuous linear operator $T\colon L_r(\lambda) \to L_p(\mu)$ is a Carleman operator if and only if there exists a $\lambda \times \mu$-measurable function $k\colon S \times \Omega \to \mathbf{R}$ such that $k(\cdot, y)$ in $L_s(\lambda)$ for μ-almost all y in Ω and such that*

$$(Tf)(\cdot) = \int_S f(x)\, k(x, \cdot)\, d\lambda(x)$$

for all f in $L_r(\lambda)$ μ-almost everywhere.

Proof. Suppose $T\colon L_r(\lambda) \to L_p(\mu)$ is a Carleman operator with kernel $g\colon \Omega \to (L_r(\lambda))^* = L_s(\lambda)$ so that $Tf(\cdot) = \langle g(\cdot), f\rangle$. In order to produce a $\lambda \times \mu$-measurable function k such that $k(x, y) = (g(y))(x)$ a.e., consider $g(\cdot)/\|g(\cdot)\|_s$. This is a bounded $L_s(\lambda)$-valued measurable function and is therefore μ-Bochner integrable. An appeal to a theorem of Dunford and Pettis [4, III. 11.17] produces a $\lambda \times \mu$-measurable function k_1 such that

$$k_1(x, y) = ((g(y))(x))/\|g(y)\|_s$$

for μ-almost all y in Ω. Now set $k(x, y) = \|g(y)\|_s k_1(x, y)$, note that k is $\lambda \times \mu$-measurable. Now observe that for f in $L_r(\lambda)$ and μ-almost all y in Ω, we have

$$T(f)(y) = \langle g(y), f\rangle = \int_S (g(y))(x) f(x)\, d\lambda(x) = \int_S f(x)\, k(x, y)\, d\lambda(x).$$

Conversely, suppose there is a jointly measurable function $k\colon S \times \Omega \to \mathbf{R}$ such that $k(\cdot, y) \in L_s(\lambda)$ for μ-almost all y in Ω and such that, for f in $L_r(\lambda)$, we have

$$(Tf)(\cdot) = \int_S f(x)\, k(x, \cdot)\, d\lambda(x)$$

almost everywhere. Without loss of generality we assume that $k(\cdot, y) \in L_s(\lambda)$ for all y in Ω. Define $g_1\colon \Omega \to L_s(\lambda)$ by $g_1(y) = k(\cdot, y)$. Since

$$\langle g_1(y), f\rangle = \int f(x)(g_1(y))(x)\, d\lambda(x) = \int f(x)\, k(x, y)\, d\lambda(x) = (Tf)(y)$$

for all $f \in L_r(\lambda)$ for μ-almost all y, we see that g_1 is weakly measurable. But any weakly measurable function with values in a reflexive space (or WCG space, for that matter) is equivalent to a (strongly) measurable function [3, p. 88]; viz. there is a measurable $g\colon \Omega\to L_s(\lambda)$ such that for all $x^*\in L_r(\lambda)^*$ we have $\langle x^*, g\rangle=\langle x^*, g_1\rangle$ μ-almost everywhere. Thus $\langle g_1(y), f\rangle=\langle g(y), f\rangle$ for almost all $y\in\Omega$ and all $f\in L_s(\lambda)$. Consequently, for each f in $L_r(\lambda)$, we have

$$(Tf)(y) = \int_S f(x)k(x, y)\,d\lambda(x) = \langle g_1(y), f\rangle = \langle g(y), f\rangle$$

for μ-almost all y. This proves that T is a Carleman operator.

The sufficiency proof of Theorem 9 also shows how to replace kernels that are not jointly measurable by ones that are, since joint measurability of k is not used in that argument. The requirement of joint measurability in the definition of integral operator is not necessary for most of the work but is usually assumed so as to guarantee that each integral operator has a unique kernel. The question of whether every operator determined by a nonjointly measurable kernel can be induced from a jointly measurable kernel appears to be open (see [6, § 8] for a discussion), but is easily settled for the Carleman operators.

Corollary 10. *Let $T\colon L_r(\lambda)\to L_p(\mu)$ be a continuous linear operator. If there exists a (possibly) nonjointly measurable function $k\colon S\times\Omega\to\mathbf{R}$ such that $k(\cdot, y)\in L_s(\lambda)$ for μ-almost all y in Ω and such that*

$$(Tf)(y) = \int_S f(x)k(x, y)\,d\lambda(x)$$

for all $f\in L_r(\lambda)$ and μ-almost all y in Ω, then T is a Carleman operator (and hence given also by a jointly measurable kernel).

4. Compactness and Korotkov operators

It follows directly that if $T\colon X\to L_\infty(\mu)$ is a Carleman operator, then $T(B_X)$ is compact in every $L_p(\mu)$ for $0\leqq p<\infty$; indeed Theorem 6 guarantees that for a set E of large measure $E\circ T\colon X\to L_\infty(\mu)$ is compact. Thus, if $T\colon X\to L_\infty(\mu)$ is Carleman and $\lim\limits_{\mu(E)\to 0}\int_E |Tx|^p\,d\mu=0$ uniformly in $\|x\|\leqq 1$, then T maps bounded sets into relatively compact subsets of $L_p(\mu)$. Moreover, since $0<r<p<\infty$ implies that $\lim\limits_{\mu(E)\to 0}\int_E |f|^r\,d\mu=0$ uniformly for any bounded set of $L_p^*(\mu)$, it follows that if $T\colon X\to L_p(\mu)$ is a Carleman operator, then T maps bounded subsets of X into $L_r(\mu)$-relatively compact sets. This line of reasoning holds up for a class of operators which strictly includes the Carleman operators.

Definition 11. A bounded linear operator $T: X \to L_p(\mu)$ is called a *Korotkov operator* if there is a measurable real function Φ on Ω such that $|(Tx)(\cdot)| \leqq \|x\| \Phi(\cdot)$ for all x in X μ-almost everywhere (cf. KOROTKOV [10, 11]).

Recall that Theorem 3 shows that every Carleman operator is a Korotkov operator while the converse evidently requires that X^* have the RNP. It is, therefore, not surprising that a representation for the Korotkov operators can be found which differs only in its measurability requirements.

Lemma 12. *Let X be a separable Banach space and let $0 \leqq p \leqq \infty$. If $T: X \to L_p(\mu)$ is a Korotkov operator then there exists a weak*-measurable function $g: \Omega \to X^*$ such that $(Tx)(\cdot) = \langle g(\cdot), x\rangle$ for all $x \in X$ almost everywhere.*

Proof. We refer to the sufficiency part of the proof of Theorem 3, where we have the bounded linear operators $(E_m \circ T)^*: L_1(\mu) \to X^*$. Replace the hypothesis that X^* has RNP by the hypothesis that X is separable. Standard arguments ([4, VI. 8.6], [3, p. 79]) for representing such operators yield $g_n: \Omega \to X^*$ vanishing off E_n such that

$$((E_m \circ T)^* f)(x) = \int_{E_n} g_m(\cdot) x \, f \, d\mu$$

for all $x \in X$ and $f \in L_1(\ )$. Piece g together as in that proof to get that $g(\cdot)x$ is measurable for each x and that $Tx = \langle g, x\rangle$ for all $x \in X$.

It is possible to drop the separability assumption in Lemma 12. This, however, depends on much deeper arguments than those used — viz. the existence of liftings. With this tool it is possible to prove [7] that if X is an arbitrary Banach space and $S: L_1(\mu) \to X^*$ is any continuous linear operator, then there exists a bounded weak*-measurable function $h: \Omega \to X^*$ such that $((Sf)x)(\cdot) = \int_\Omega h(\cdot) x f d\mu$ for all $x \in X$ and $f \in L_1(\mu)$. This would generalize Lemma 12 to non separable spaces; since, however, we can manage without this deep theorem we shall not use it.

Definition 13. Let $0 < p < \infty$. A bounded subset K of $L_p(\mu)$ is called *equi-integrable* if

$$\lim_{\mu(E) \to 0} \int_E |f|^p \, d\mu = 0$$

uniformly in $f \in K$. A subset of $L_0(\mu)$ is called *equi-integrable* if it is relatively compact.

Recall that a subset M of a Banach space it is called *weakly conditionally compact* if every sequence in M has a weak Cauchy subsequence.

Theorem 14. *Let $0 < p < \infty$ and X be an arbitrary Banach space. A Korotkov operator $T: X \to L_p(\mu)$ with the property that $T(B_X)$ is equi-integrable in $L_p(\mu)$ maps weakly conditionally compact sets into norm compact sets.*

Proof. Let W be a weakly conditionally compact subset of X and let (y_n) be a sequence in W. It must be shown that $(T(y_n))$ has an $L_p(\mu)$-convergent subsequence. Let y be the (separable) closed subspace of X determined by $\{y_n\}_{n=1}^{\infty}$. The restriction of T to y, still denoted by T, is a Korotkov operator from y to $L_p(\mu)$. According to Lemma 12 there is a function $g\colon \Omega\to y^*$ such that $Ty=\langle g, y\rangle$ for all y in Y. Since W is weakly conditionally compact, the sequence (y_n) has a weak Cauchy subsequence (y_{n_j}). Since g has B values in y^*, it follows that $(T(y_{n_j}))= =(\langle g, y_{n_j}\rangle)$ is a pointwise convergent sequence in an equi-integrable set. Vitali's convergence theorem guarantees that $(T(y_{n_j}))$ is $L_p(\mu)$ convergent. This completes the proof.

Corollary 15. *Let $0\leqq r<p\leqq\infty$ and X be an arbitrary Banach space. A Korotkov operator from X into $L_p(\mu)$ maps weakly conditionally compact subsets of X into relatively compact subsets of $L_r(\mu)$. A Korotkov operator from X into $L_0(\mu)$ maps weakly conditionally compact subsets of X into relatively compact subsets of $L_0(\mu)$.*

Proof. For $p>0$, the Hölder inequality can be used to show that $L_p(\mu)$-bounded sets are equi-integrable in $L_r(\mu)$ so that Theorem 14 applies.

For $p=0$, glance at the proof of Theorem 14 and remember that pointwise convergence implies convergence in measure for sequences.

ROSENTHAL's characterization [15] of Banach spaces containing copies of l_1 and Corollary 15 give the next corollary.

Corollary 16. *Let $0\leqq p>\infty$. Let X be a Banach space containing no copy of l_1. A Korotkov operator from X to L_p that maps bounded sets into equi-integrable sets maps bounded sets into relatively compact sets. Consequently, if $0\leqq r<p\leqq\infty$, then a Korotkov operator from X to $L_p(\mu)$ maps bounded sets into $L_r(\mu)$-relatively compact sets; and a Korotkov operator from X into $L_0(\mu)$ maps bounded sets into relatively compact sets.*

5. Extensions to Banach function spaces and other function spaces

Let (Ω, Σ, μ) be any finite measure space and $Y(\mu)$ be any linear topological space (not necessarily locally convex) of (equivalence classes of μ-measurable functions on Ω. For a Banach space X, say that a continuous linear operator $T\colon X\to Y(\mu)$ is a Carleman operator if there is a measurable $g\colon \Omega\to X^*$ such that

$$Tx = \langle g, x\rangle$$

for all x in X. If $Y(\mu)$ has the property that φ in $Y(\mu)$ implies $\varphi\chi_E \in Y(\mu)$ for all E in Σ, then a check of the proofs of the theorems of Section 1 shows that they remain true if $L_p(\mu)$ is replaced by $Y(\mu)$.

Theorem 9 generalizes readily to a wide class of Banach function spaces. (For the basic definitions and results used here see LUXEMBURG & ZAANEN [13] and LUXEMBURG [12].) We give a brief summary. Start with a measure space (Ω, Σ, μ), which for simplicity we assume is a finite measure space. Let M be the set of all measurable scalar functions and M^+ the nonnegative members of M. The order on M is pointwise and functions differing only on a null set are identified. A *function norm* is a function $\varrho\colon M^+ \to [0, \infty]$ that is positive homogeneous, subadditive, takes the value zero if and only if the function is zero almost everywhere, and preserves order (viz. $u \leqq v$; $v \in M^+ \Rightarrow \varrho(u) \leqq \varrho(v)$). The function norm is extended to all of M by $\varrho(f) = \varrho(|f|)$. We denote by L_ϱ the set of all $f \in M$ satisfying $\varrho(f) < \infty$. The result is an ordered normed vector space. We assume that $L_\varrho(\mu)$ is norm complete. Without loss of generality we also assume that ϱ is *saturated* i.e. there are no ϱ-unfriendly sets (a set $E \subset \Omega$ such that $\varrho(\chi_F) = \infty$ for every $F \subset E$ with $\mu(F) > 0$). The associate norm is defined by

$$\varrho'(g) = \sup\left\{\left|\int fg\, d\mu\right| \varrho(f) \leqq 1\right\};$$

and, of course, Hölder's inequality $\left|\int fg\, d\mu\right| \leqq \varrho(f)\varrho'(g)$ obtains.

This last step gives a function norm whose corresponding $L_{\varrho'}(\mu)$ is a Banach space. Finally a function in L_ϱ is of *absolutely continuous norm* whenever $\varrho(f_n) \downarrow 0$ for every $(f_n) \subseteqq L_\varrho$ such that $|f| \geqq f_1 \geqq f_2 \geqq \ldots \downarrow 0$. We call the collection of all such functions L_ϱ^α. At this point the extension of Theorem 9 to the context of Banach function spaces goes right through. Let $(\mathscr{S}, f, \lambda)$ and (Ω, Σ, μ) be finite measure spaces. Let $Y(\mu)$ be as above; let ϱ_1 be a function norm such that L_{ϱ_1} is reflexive (this is equivalent to $L_\varrho = L_\varrho^\alpha$, $L_{\varrho'} = L_\varrho^\alpha$, and $\varrho(f_n) \uparrow \varrho(f)$ whenever $0 \leqq f_n \uparrow f$). A continuous linear operator $T\colon L_{\varrho_1}(\lambda) \to Y(\mu)$ is a Carleman operator if and only if there exists a $\lambda \times \mu$-measurable function $k\colon \mathscr{S} \times \Omega \to \mathbf{R}$ such that $k(\cdot, y) \in L_{\varrho_1'}(\lambda)$ for μ-almost all $y \in \Omega$ with

$$(Tf)(y) = \int_{\mathscr{S}} f(x)\, k(x, y)\, d\lambda(x) \quad \text{g.e.}$$

for all $f \in L_{\varrho_1}(\lambda)$.

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