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Some propositions on analytic matrix functions related to the theory of operators in the space $\Pi_\varkappa$

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It is well known that certain classes of analytic functions play a useful role in the theory of hermitian and selfadjoint operators in Hilbert space. On the other hand, sometimes, general propositions from the spectral theory of operators yield simple solutions of problems in complex function theory. This is especially true for the theory of selfadjoint and unitary operators in spaces with indefinite metric.

In this note we prove some consequences of the theory of Q-functions and characteristic functions of hermitian and isometric operators in the space $\Pi_\varkappa$, as developed in [1] and [2], for scalar and matrix valued analytic functions of a complex variable. It seems rather unexpected to us that in this way we get new results [1]) also for the so-called Nevanlinna or R-functions (mappings of the upper half-plane into itself) so well studied in different contexts during the last 50 years.

There are now several papers (see, e.g., [5]) which generalize the well known theorem of Rouché to matrix or operator functions. In these papers, however, it is assumed that the boundary of the domain considered consists of regular points only. Here we show that our methods permit a generalization of Rouché's theorem to the case of matrix functions of the class $\mathscr{D}^{n\times n}$ (see § 4 below) over the unit disc. Instead of the unit disc more general domains with sufficiently smooth Jordan boundaries may be considered. For the case of scalar functions this generalization was proved [2]) by V. M. ADAMJAN, D. Z. AROV and M. G. KREIN in [6] and has found essential applications in the theory of Hankel operators with scalar kernel. Theorem 4.2 below can be used in the investigation of Hankel operators with matrix kernel.

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[1]) Some of the results were proved by a different method in [3] (cf. also [4]).

[2]) However in a less complete form, without counting the generalized poles of negative type (see the remark after Theorem 4.2).

§ 1. Basic propositions

1. An $(n\times n)$-matrix function K, defined on a nonempty set $Z\times Z$, is said to have $\varkappa$ negative squares (on Z) if it has the following two properties:
1) $K(z,\zeta)=K(\zeta,z)^*$ $(z,\zeta\in Z)$,
2) for any positive integer k, any $z_1,\dots,z_k\in Z$ and n-vectors $\xi_1,\dots,\xi_k\in\mathfrak{C}^n$ [1]) the matrix

$$\left(K(z_\nu,z_\mu)\xi_\nu,\xi_\mu\right)_{\nu,\mu=1,2,\dots,k}$$

has at most $\varkappa$ negative eigenvalues and for at least one choice of k, $z_1,\dots,z_k$, and $\xi_1,\dots,\xi_k$, it has exactly $\varkappa$ negative eigenvalues.

In this note the following three classes of analytic $(n\times n)$-matrix functions will play an important role.

a) $\mathbf{N}_\varkappa^{n\times n}$ is the set of all $(n\times n)$-matrix functions Q which are meromorphic on $\mathfrak{C}_+$ and such that the kernel N_Q:

$$N_Q(z,\zeta):=\frac{Q(z)-Q(\zeta)^*}{z-\zeta^*}\quad (z,\zeta\in\mathfrak{D}_Q)$$

has $\varkappa$ negative squares ($\mathfrak{D}_Q\subset\mathfrak{C}_+$ denotes the domain of holomorphy of Q).

b) $\mathbf{C}_\varkappa^{n\times n}$ is the set of all $(n\times n)$-matrix functions F which are meromorphic on $\mathfrak{D}$ and such that the kernel C_F:

$$C_F(z,\zeta):=\frac{F(z)+F(\zeta)^*}{1-z\zeta^*}\quad (z,\zeta\in\mathfrak{D}_F)$$

has $\varkappa$ negative squares.

c) $\mathbf{S}_\varkappa^{n\times n}$ is the set of all $(n\times n)$-matrix functions θ which are meromorphic on $\mathfrak{D}$ and such that the kernel S_θ:

$$S_\theta(z,\zeta):=\frac{I-\theta(\zeta)^*\theta(z)}{1-z\zeta^*}\quad (z,\zeta\in\mathfrak{D}_\theta)$$

has $\varkappa$ negative squares.

In the special case $n=1$ these classes (of scalar valued functions) were studied in [7]. In the more general case where the values of the functions Q and θ are bounded linear operators on a Hilbert space, the corresponding classes were introduced in [1] and [2].

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[1]) $\mathfrak{C}$ is the set of all complex numbers, $\mathfrak{C}_+(\mathfrak{C}_-)$ the open upper (lower) half-plane, $\overline{\mathfrak{C}}_+$ the closure of $\mathfrak{C}_+$ in $\mathfrak{C}$. Furthermore, $\mathfrak{D}$ $(\overline{\mathfrak{D}})$ denotes the open (closed) unit disc and $\partial\mathfrak{D}$ the boundary of $\mathfrak{D}$. The usual scalar product and norm in $\mathfrak{C}^n$ are denoted by $(.,.)$ and $\|\cdot\|$. If A is an $n\times n$ matrix, $\|A\|$ denotes the norm of the operator induced by A on $\mathfrak{C}^n$. If $z\in\mathfrak{C}$, then z^* denotes the complex conjugate of z.

We mention that these classes can be defined in a different way (cf. [1] and [2]). For instance, an $(n\times n)$-matrix function Q_0 which is defined and continuous on some open set $\mathfrak{D}'\subset\mathfrak{C}_+$ and for which the kernel N_{Q_0} has $\varkappa$ negative squares on $\mathfrak{D}'$ can be extrapolated in a unique way to a function $Q\in\mathbf{N}_\varkappa^{n\times n}$. Further, a function $Q\in\mathbf{N}_\varkappa^{n\times n}$ can be extrapolated to a function $\tilde{Q}$ locally meromorphic on $\mathfrak{C}_+\cup\mathfrak{C}_-$ by the formula

$$\tilde{Q}(z):=\begin{cases}Q(z), & z\in\mathfrak{D}_Q,\\ Q(z^*)^*, & z^*\in\mathfrak{D}_Q.\end{cases}$$

Then the kernel $N_{\tilde{Q}}$ has $\varkappa$ negative squares on $\mathfrak{D}_Q\cup\mathfrak{D}_Q^*$. In a similar way, $F\in\mathbf{C}_\varkappa^{n\times n}$ can be extrapolated to the complement of the closed unit disc by setting $\tilde{F}(z^{-1}):=-F(z^*)^*$ $(z^*\in\mathfrak{D}_F)$.

The classes $\mathbf{N}_\varkappa^{n\times n}$ and $\mathbf{C}_\varkappa^{n\times n}$ are very closely related. Namely, if φ is a linear fractional mapping of $\mathfrak{D}$ onto $\mathfrak{C}_+$, then the formula $F=iQ\circ\varphi$ $(Q\in\mathbf{N}_\varkappa^{n\times n})$ establishes a one-to-one correspondence between $\mathbf{N}_\varkappa^{n\times n}$ and $\mathbf{C}_\varkappa^{n\times n}$. Hence the statements about the class $\mathbf{C}_\varkappa^{n\times n}$ given below can easily be transferred to the class $\mathbf{N}_\varkappa^{n\times n}$.

Proposition 1.1. *Let* $F\in\mathbf{C}_\varkappa^{n\times n}$ *and* $\alpha\in\mathfrak{C}$, $\operatorname{Re}\alpha>0$. *Then the function* θ *defined by*

(1.1) $$\theta(z):=\big(F(z)-\alpha^*I\big)\big(F(z)+\alpha I\big)^{-1}$$

belongs to the class $\mathbf{S}_\varkappa^{n\times n}$.

Proof. First we show that for each α, $\operatorname{Re}\alpha>0$, we can find a $z_0\in\mathfrak{D}_F$ such that $\big(F(z_0)+\alpha I\big)^{-1}$ exists. Otherwise for some fixed α, $\operatorname{Re}\alpha>0$, and each $z\in\mathfrak{D}_F$ there would exist an n-vector $\xi(z)\neq 0$ such that $F(z)\xi(z)=-\alpha\xi(z)$. It follows

(1.2) $$(1-z\zeta^*)^{-1}\big((F(z)+F(\zeta)^*)\xi(z),\xi(\zeta)\big)=-2\operatorname{Re}\alpha(1-z\zeta^*)^{-1}\big(\xi(z),\xi(\zeta)\big)=$$
$$=-\operatorname{Re}\alpha(2\pi)^{-1}\int_0^{2\pi}(e^{i\vartheta}-z)^{-1}(e^{-i\vartheta}-\zeta^*)^{-1}d\vartheta\big(\xi(z),\xi(\zeta)\big).$$

If $z_1,z_2,\dots,z_k\in\mathfrak{D}$, then the $k\times k$ matrix

$$\Big(\int_0^{2\pi}(e^{i\vartheta}-z_\nu)^{-1}(e^{-i\vartheta}-z_\mu^*)^{-1}d\vartheta\big(\xi(z_\nu),\xi(z_\mu)\big)\Big)_{\nu,\mu=1,2,\dots,k}$$

has k positive eigenvalues. This follows from the fact that for arbitrary $\alpha_1,\alpha_2,\dots,\alpha_k\in\mathfrak{C}$, not all equal to zero, we have

$$\int_0^{2\pi}\Big\|\sum_{\nu=1}^{k}\frac{\xi(z_\nu)\alpha_\nu}{e^{i\vartheta}-z_\nu}\Big\|^2 d\vartheta>0.$$

If we choose $k>\varkappa$, from (1.2) we get a contradiction to the assumption $F\in\mathbf{C}_\varkappa^{n\times n}$.

Thus $\det(F(z)+\alpha I)\not\equiv 0$. Hence the meromorphic function $\det(F(z)+\alpha I)$ can vanish only on a set σ_α of isolated points of $\mathfrak{D}$. For $z, \zeta \notin \sigma_\alpha$ it follows that $I-\theta(\zeta)^*\theta(z)=2(\operatorname{Re}\alpha)(F(\zeta)^*+\alpha^*I)^{-1}(F(z)+F(\zeta)^*)(F(z)+\alpha I)^{-1}$. Therefore the kernel S_θ has $\varkappa$ negative squares on σ_α.

2. Let $\Pi_\varkappa$ be a $\pi_\varkappa$-space with indefinite scalar product $[.,.]$.[1]) A bounded linear operator T in $\Pi_\varkappa$ is called *contractive* if $\mathfrak{D}(T)=\Pi_\varkappa$ and $[Tx, Tx]\leqq[x, x]$ $(x\in\Pi_\varkappa)$, *isometric* if $[Tx, Tx]=[x, x]$ $(x\in\mathfrak{D}(T))$, and *unitary* if it is isometric and $\mathfrak{D}(T)==\mathfrak{R}(T)=\Pi_\varkappa$. An isometric operator T with $\mathfrak{D}(T)=\Pi_\varkappa$ or $\mathfrak{R}(T)=\Pi_\varkappa$ is called *maximal isometric.*

Proposition 1.2. *A contractive operator T in a $\pi_\varkappa$-space $\Pi_\varkappa$ has a $\varkappa$-dimensional nonpositive invariant subspace $\mathscr{L}$ such that $|\sigma(T|\mathscr{L})|\geqq 1$. If $\mathscr{L}$ is not uniquely determined, the points of $\sigma(T|\mathscr{L})$ and their algebraic multiplicities do not depend on the choice of $\mathscr{L}$.*

We shall write $\sigma_0(T):=\sigma(T|\mathscr{L})$ if T and $\mathscr{L}$ are as in Proposition 1.2. For $\lambda\in\sigma_0(T)$ the algebraic multiplicity of λ with respect to $T|\mathscr{L}$ will be called the index of λ with respect to T and denoted by $\varkappa_\lambda(T)$. Evidently, it is the dimension of the intersection $\mathscr{L}\cap\mathscr{S}_\lambda(T)$, where $\mathscr{S}_\lambda(T):=\{x\colon (T-\lambda I)^k x=0$ for some $k=1, 2, \ldots\}$. If $\mathfrak{U}\subset\{z\colon |z|\geqq 1\}$, the index $\varkappa_{\mathfrak{U}}(T)$ of $\mathfrak{U}$ is defined by

$$\varkappa_{\mathfrak{U}}(T):=\sum_{\lambda\in\sigma_0(T)\cap\mathfrak{U}}\varkappa_\lambda(T).$$

The first statement of Proposition 1.2, and the second statement for points $\lambda\in\sigma(T|\mathscr{L})$, $|\lambda|>1$, follow from [9, Theorem 11.2]. For a unitary operator T the second statement was completely proved in [10]; this result is also an immediate consequence of the spectral theorem [11]. In the following only these conclusions of Proposition 1.2 will be used.

However, for the sake of completeness, we prove the second statement for an abirtrary contractive operator T in $\Pi_\varkappa$. To this end, observe first that T has a unitary dilation $\tilde{U}$ in some larger $\pi_\varkappa$-space $\tilde{\Pi}_\varkappa\supset\Pi_\varkappa$, that is

$$T^n x = \tilde{P}\tilde{U}^n x \quad (x\in\Pi_\varkappa,\ n=0, 1, 2, \ldots), \tag{1.3}$$

where $\tilde{P}$ denotes the π-orthogonal projector of $\tilde{\Pi}_\varkappa$ onto $\Pi_\varkappa$ (see [12]), A relation between certain invariant subspaces of T and $\tilde{U}$ is established by the following lemma.

Lemma 1.3. *If T and $\tilde{U}$ are as above and $\mathscr{L}_0$ is a nonpositive subspace of $\Pi_\varkappa$ such that $T\mathscr{L}_0\subset\mathscr{L}_0$ and $|\sigma(T|\mathscr{L}_0)|=1$, then $\tilde{U}x=Tx$ $(x\in\mathscr{L}_0)$.*

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[1]) Here we use the notation of [8]. For the properties of $\pi_\varkappa$-spaces and their bounded linear operators see [9].

Proof. The operator T has the property

$$[Tx, Ty] = [x, y] \quad (x, y \in \mathscr{L}_0). \tag{1.4}$$

Indeed, consider $V := (T|\mathscr{L}_0)^{-1}$. Then $|\sigma(V)| = 1$. On the other hand, if $\mathscr{L}_0$ is equipped with the nonnegative scalar product $-[x, y]$ $(x, y \in \mathscr{L}_0)$, then V induces a contraction $\hat{V}$ in the factor space $\hat{\mathscr{L}} := \mathscr{L}_0 / \mathscr{L}_{00}$, where $\mathscr{L}_{00} := \{x \in \mathscr{L}_0 \colon [x, x] = 0\}$. Since $\sigma(\hat{V}) \subset \sigma(V)$, we have $|\sigma(\hat{V})| = 1$, and by a well known result on contractive operators in a unitary space, $\hat{V}$ is unitary. Therefore $[Vx, Vy] = [x, y]$ $(x, y \in \mathscr{L}_0)$ and (1.4) follows. Using (1.4) and (1.3), for $x \in \mathscr{L}_0$ we find

$$[x, x] = [Tx, Tx] = [\tilde{P}\tilde{U}x, \tilde{P}\tilde{U}x] \leqq [\tilde{U}x, \tilde{U}x] = [x, x];$$

hence $Tx = \tilde{P}\tilde{U}x = \tilde{U}x$.

Now we continue the proof of Proposition 1.2. The Lemma 1.3 implies that every $\lambda \in \sigma(T|\mathscr{L})$, $|\lambda| = 1$ belongs to $\sigma_0(\tilde{U})$. As the subspace $\mathscr{L}_0$ of Lemma 1.3 can always be extended to a $\varkappa$-dimensional nonpositive invariant subspace of U (see [16, Theorem VIII. 2.1]) we have for these λ

$$\varkappa_\lambda(T|\mathscr{L}) \leqq \varkappa_\lambda(\tilde{U}), \tag{1.5}$$

where $\varkappa_\lambda(T|\mathscr{L})$ denotes the dimension of $\mathscr{L} \cap \mathscr{S}_\lambda$. The same inequality (1.5) holds if $\lambda \in \sigma(T|\mathscr{L})$, $|\lambda| > 1$. Indeed, (1.3) implies that

$$(T - zI)^{-1} = \tilde{P}(\tilde{U} - z\tilde{T})^{-1} \quad \big(|z| > 1,\ z \notin \sigma(T) \cap \sigma(\tilde{U})\big),$$

and it follows that the dimension of the Riesz projector corresponding to λ and T is not greater than the dimension of the Riesz projector corresponding to λ and $\tilde{U}$. Now (1.5) yields

$$\varkappa = \sum_{\lambda \in \sigma(T|\mathscr{L})} \varkappa_\lambda(T|\mathscr{L}) \leqq \sum_{\lambda \in \sigma_0(\tilde{U})} \varkappa_\lambda(\tilde{U}) = \varkappa,$$

that is, in (1.5) the sign $=$ must hold. But the right hand side of (1.5) is independent of $\mathscr{L}$, and the statement follows.

The following proposition can be proved in the same way as Satz 1.2 in [1].

Proposition 1.4. *Let (T_n) be a sequence of contractive operators in $\Pi_\varkappa$, $\|T_n - T_0\| \to 0$ $(n \to \infty)$, and $\lambda_0 \in \sigma_0(T_0)$. Then for each sufficiently small neighbourhood $\mathfrak{U}$ of λ_0 there exists an $n(\mathfrak{U}) > 0$ such that for $n \geqq n(\mathfrak{U})$ we have $\varkappa_{\mathfrak{U}}(T_n) = \varkappa_{\lambda_0}(T_0)$.*

Because of the relation

$$\sum_{\lambda \in \sigma_0(T_0)} \varkappa_\lambda(T_0) = \sum_{\lambda \in \sigma_0(T_n)} \varkappa_\lambda(T_n) = \varkappa,$$

under the conditions of Proposition 1.9 the points of $\sigma_0(T_0)$ are the only "accumulation points" of $\sigma_0(T_n)$, $n = 1, 2, \ldots$.

3. A close connection between functions $F \in \mathbf{C}_\varkappa^{n \times n}$ and isometric operators in a $\pi_\varkappa$-space $\Pi_\varkappa$ is given by the following proposition ([2, Satz 2.2]):

a) *Let V be a maximal isometric $(\mathfrak{R}(V)=\Pi_\varkappa)$ operator in a $\pi_\varkappa$-space $\Pi_\varkappa$, S a hermitian $n\times n$ matrix and Γ a linear mapping from $\mathfrak{C}^n$ into $\Pi_\varkappa$. Then the function F:*

$$(1.6)\qquad F(z)=iS+\Gamma^*(V+zI)(V-zI)^{-1}\Gamma \quad (z^{-1}\notin\sigma(V^{-1}),\ |z|<1)$$

belongs to the class $\mathbf{C}_{\varkappa'}^{n\times n}$ for some $\varkappa'$, $0\leqq\varkappa'\leqq\varkappa$. If the operators V and Γ are closely i-connected then $\varkappa'=\varkappa$.

b) *If $F\in\mathbf{C}_\varkappa^{n\times n}$ and $0\in\mathfrak{D}_F$, then there exist a $\pi_\varkappa$-space $\Pi_\varkappa$, a maximal isometric $(\mathfrak{R}(V)=\Pi_\varkappa)$ operator V in $\Pi_\varkappa$ and a linear mapping Γ from $\mathfrak{C}^n$ into $\Pi_\varkappa$, closely i-connected with V, so that the representation* (1.6) *holds with* $S=\operatorname{Im} F(0)$.

We remind the reader that an operator Γ from $\mathfrak{C}^n$ into $\Pi_\varkappa$ is said to be *closely i-connected* with the maximal isometric $(\mathfrak{R}(V)=\Pi_\varkappa)$ operator V in $\Pi_\varkappa$ if $\Pi_\varkappa$ is the closed linear span of all elements $(V-zI)^{-1}\Gamma\xi$, $\xi\in\mathfrak{C}^n$, $z\in\varrho(V)$, $|z|<1$. Here, of course, $(V-zI)^{-1}$ is always to be understood as $V^{-1}(I-zV^{-1})^{-1}$ with the isometric operator $V^{-1}=V^+$ defined on all of $\Pi_\varkappa$, $z^{-1}\in\varrho(V^{-1})$.

The function $F\in\mathbf{C}_\varkappa^{n\times n}$, $0\in\mathfrak{D}_F$, admits also a representation (1.6) with a unitary operator V in $\Pi_\varkappa$. Consider this operator V, and let $\mathscr{L}$ be a $\varkappa$-dimensional nonpositive invariant subspace of V such that $|\sigma(V|\mathscr{L})|\geqq 1$. Denote the characteristic polynomial of $V|\mathscr{L}$, which does not depend on the choice of $\mathscr{L}$, by p and put $g(z)=p^*(z^{-1})p(z)$. Then we have $[g(V)x,x]\geqq 0$ $(x\in\Pi_\varkappa)$ and it follows that

$$\operatorname{Re}\Gamma^* g(V)(V+zI)(V-zI)^{-1}\Gamma\geqq 0\quad (z\in\mathfrak{D}).$$

Hence there exists a nondecreasing bounded $(n\times n)$-matrix function Σ on $[0,2\pi)$, such that

$$(1.7)\qquad \Gamma^* g(V)(V+zI)(V-zI)^{-1}\Gamma=\int_0^{2\pi}(e^{i\vartheta}+z)(e^{i\vartheta}-z)^{-1}\,d\Sigma(\vartheta).$$

Introducing the $(n\times n)$-matrix function G:

$$G(z):=\Gamma^*\big(g(z)I-g(V)\big)(V+zI)(V-zI)^{-1}\Gamma$$

we get from (1.6) and (1.7)

$$(1.8)\qquad F(z)=iS+\frac{1}{g(z)}\int_0^{2\pi}(e^{i\vartheta}+z)(e^{i\vartheta}-z)^{-1}\,d\Sigma(\vartheta)+\frac{1}{g(z)}G(z)\quad (z\in\mathfrak{D}).$$

As a consequence of b) we prove the following

Proposition 1.5. *The function $\theta\in\mathbf{S}_\varkappa^{n\times n}$, $0\in\mathfrak{D}_\theta$, admits the representation*

$$(1.9)\qquad \theta(z)=U_{22}+zU_{21}(I-zT)^{-1}U_{12}\quad (z\in\mathfrak{D}_\theta),$$

where T is a contractive operator in a space $\Pi_\varkappa$, which has no eigenvalues on the unit circle, and U_{12}, U_{21}, U_{22} are such mappings that the matrix

$$U = \begin{pmatrix} T & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \tag{1.10}$$

defines an isometric operator in the space $\Pi_\varkappa \oplus \mathfrak{C}^n$. The space $\Pi_\varkappa$ and the operator U can be chosen so that

$$\Pi_\varkappa = \text{c.l.s.}\ \{(I-zT)^{-1}U_{12}\xi:\ \xi\in\mathfrak{C}^n,\ z^{-1}\in\varrho(T)\};$$

then they are uniquely determined up to unitary equivalence.

Here, if $u, v\in\Pi_\varkappa$ and $\xi, \eta\in\mathfrak{C}^n$, the scalar product of $\{u, \xi\}$, $\{v, \eta\}\in\Pi_\varkappa\oplus\mathfrak{C}^n$ is defined by

$$[\{u, \xi\},\ \{v, \eta\}] = [u, v]+(\xi, \eta).$$

The operator U_{12} maps $\mathfrak{C}^n$ into $\Pi_\varkappa$, U_{21} maps $\Pi_\varkappa$ into $\mathfrak{C}^n$, and U_{22} maps $\mathfrak{C}^n$ into itself.

Proof. We may suppose that $\det(I-\theta(0))\neq 0$. Indeed, if this relation does not hold we consider θ_γ: $\theta_\gamma(z):=\gamma\theta(z)$ instead of θ for some γ: $|\gamma|=1$, $\det(I-\theta_\gamma(0))\neq 0$. Having found the representation of θ_γ with some operator U_γ, the representation of θ follows with an operator U, which is obtained from U_γ by multiplication of the second row by γ^{-1}.

Consider for $\alpha\in\mathfrak{C}$, $\operatorname{Re}\alpha>0$, the function F:

$$F(z):=\big(\alpha^*I+\alpha\theta(z)\big)\big(I-\theta(z)\big)^{-1}. \tag{1.11}$$

Then

$$F(z)+F(\zeta)^* = 2(\operatorname{Re}\alpha)\big(I-\theta(\zeta)^*\big)^{-1}\big(I-\theta(\zeta)^*\theta(z)\big)\big(I-\theta(z)\big)^{-1},$$

and it follows that $F\in\mathfrak{C}_\varkappa^{n\times n}$. From the relations (1.11), $F(0)=iS+\Gamma^*\Gamma$ and (1.6) we find

$$\begin{aligned}
\theta(z) &= I-2\operatorname{Re}\alpha\big(F(z)+\alpha I\big)^{-1} = I-2\operatorname{Re}\alpha\big(F(0)+\alpha I+2z\Gamma^*V^{-1}(I-zV^{-1})^{-1}\Gamma\big)^{-1} = \\
&= I-2(\operatorname{Re}\alpha)\big(F(0)+\alpha I\big)^{-1}+4(\operatorname{Re}\alpha)z\big(F(0)+\alpha I\big)^{-1}\Gamma^*V^{-1}\times \\
&\qquad\times\big(I-zV^{-1}+2z\Gamma(F(0)+\alpha I)^{-1}\Gamma^*V^{-1}\big)^{-1}\Gamma\big(F(0)+\alpha I\big)^{-1} = \\
&= \big(F(0)-\alpha I\big)\big(F(0)+\alpha I\big)^{-1}+ \\
&\qquad+4(\operatorname{Re}\alpha)z\big(F(0)+\alpha I\big)^{-1}\Gamma^*V^{-1}(I-zW_\alpha V^{-1})\Gamma\big(F(0)+\alpha I\big)^{-1}
\end{aligned}$$

with $W_\alpha:=I-2\Gamma\big(F(0)+\alpha I\big)^{-1}\Gamma^*$. Setting

$$T:=W_\alpha V^{-1},\quad U_{12}:=2\sqrt{\operatorname{Re}\alpha}\big(F(0)+\alpha I\big)^{-1},$$

$$U_{21}:=2\sqrt{\operatorname{Re}\alpha}\big(F(0)+\alpha I\big)^{-1}\Gamma^*V^{-1},\quad U_{22}:=\big(F(0)-\alpha^*I\big)\big(F(0)+\alpha I\big)^{-1},$$

and using the relation $\Gamma^*\Gamma=2^{-1}(F(0)+F(0)^*)$, it is not hard to verify that the matrix U satisfies $U^*U=I$.

The operator T is contractive in $\Pi_\varkappa$. Indeed, we have for $u\in\Pi_\varkappa$, $v:=V^{-1}u$:

$$\begin{aligned}[Tu,Tu] &= [v,v]-2\left[\Gamma(F(0)+\alpha I)^{-1}\Gamma^* v, v\right]-2\left[\Gamma(F(0)^*+\alpha^* I)^{-1}\Gamma^* v, v\right]+\\ &\quad+4\left(\Gamma^*\Gamma(F(0)+\alpha I)^{-1}\Gamma^* v, (F(0)+\alpha I)^{-1}\Gamma^* v\right)=\\ &= [v,v]-4\,\mathrm{Re}\,\alpha\left\|(F(0)+\alpha I)^{-1}\Gamma^* v\right\|^2 \leqq [u,u].\end{aligned}\tag{1.12}$$

Assume that $Tu_0=\lambda_0 u_0$, $|\lambda_0|=1$. Then, by (1.12), $\Gamma^* V^{-1}u_0=0$ and $W_\alpha V^{-1}u_0=V^{-1}u_0=\lambda_0 u_0$. Hence $\Gamma^* u_0=0$, $(V^{-1})^* u_0=Vu_0=\lambda_0^{-1}u_0$, and for arbitrary $\xi\in\mathbf{C}^n$, $z^{-1}\in\varrho(V^{-1})$, $|z|<1$, we get

$$[V^{-1}(I-zV^{-1})^{-1}\Gamma\xi, u_0] = (\xi, \Gamma^* u_0)(\lambda_0-z)^{-1} = 0.$$

As Γ and V are closely i-connected, this implies $u_0=0$. The proof of the uniqueness of U is left to the reader.

Remark. The function $F\in\mathbf{C}_\varkappa^{n\times n}$ in the proof of Proposition 1.5 admits also a representation (1.6) with a unitary operator V in a $\pi_\varkappa$-space $\Pi_\varkappa$. This implies a representation (1.9) of the function θ, where the operator (1.10) is unitary in the space $\Pi_\varkappa\oplus\mathfrak{C}^n$. Then θ is the characteristic function of the operator T^*, see (1.10) (the case $n=1$ was considered in [7]).

We mention that Proposition 1.5 is an immediate generalization of [7, Satz 6.5]. It can be reversed and generalized to functions θ with values in $[\mathfrak{H}]$, the Banach algebra of all bounded linear operators mapping the Hilbert space $\mathfrak{H}$ into itself.

4. In [2, Satz 3.2] it was shown that a function $\theta\in\mathbf{S}_\varkappa^{n\times n}$ admits also the representation

$$\theta(z) = B_0(z)^{-1}\theta_0(z) \quad (z\in\mathfrak{D}_\theta)\tag{1.13}$$

with a Blaschke—Potapov product B_0,

$$B_0(z)=U_0\overset{\curvearrowleft}{\prod_{j=1}^{l}} B_j(z),\quad B_j(z)=\overset{\curvearrowleft}{\prod_{k=1}^{k_j}}\left(\frac{z-\alpha_j}{1-z\alpha_j^*}P_{jk}+Q_{jk}\right),\tag{1.14}$$

and a function $\theta_0\in\mathbf{S}_0^{n\times n}$. Here $\alpha_j\in\mathfrak{D}$, $\alpha_j\neq\alpha_{j'}$, for $j\neq j'$; P_{jk} and Q_{jk} are idempotent hermitian matrices with $P_{jk}+Q_{jk}=I$ for $k=1,2,\ldots,k_j$ and $j=1,2,\ldots,l$; U_0 is a unitary matrix and $\theta_0\in\mathbf{S}_0^{n\times n}$.

The Blaschke—Potapov product B_0 is called *regular* if

$$P_{j1}\geqq P_{j2}\geqq\ldots\geqq P_{jk_j},\quad j=1,2,\ldots,l.$$

The representation (1.13) is called *regular* if B_0 is regular and

$$\mathfrak{R}(P_{j1}Y_{j+1}(\alpha_j))=\mathfrak{R}(P_{j1}),\quad j=1,2,\ldots,l\tag{1.15}$$

holds; here

$$(1.16)\quad Y_j(z) := \left(\overset{\curvearrowright}{\prod_{\nu=j}^{l}} B_\nu(z)^{-1}\right) U_0^{-1}\theta_0(z), \quad j = 1, 2, \dots, l, \quad Y_{l+1}(z) := U_0^{-1}\theta_0(z).$$

The *order* of the Blaschke—Potapov product B_0 in (1.14) is defined as

$$\sum_{j=1}^{l} \sum_{k=1}^{k_j} \dim P_{jk};$$

according to [2, Satz 3.2] it is equal to $\varkappa$, if the representation (1.13) is regular.

§ 2. Zeros and poles in $\mathfrak{D}$

1. The multiplicity of zeros and poles of a meromorphic matrix or operator function was defined e.g. in [5]. Here we use the following characterization of the pole multiplicity (see [1, Lemma 4.1]): If $A(z)$ is a meromorphic function whose values are bounded linear operators in a Banach space $\mathfrak{B}$ and which has a pole α with Laurent expansion

$$(2.1)\qquad A(z) = (z-\alpha)^{-k}A_{-k} + \dots + (z-\alpha)^{-1}A_{-1} + A_0 + \dots$$

for z near α, $z \neq \alpha$, then the *pole multiplicity* of α with respect to $A(z)$ is the dimension of the range of the operator

$$\mathfrak{A} := \begin{pmatrix} A_{-k} & 0 & \dots 0 & 0 \\ A_{-k+1} & A_{-k} & \dots 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ A_{-2} & A_{-3} & \dots A_{-k} & 0 \\ A_{-1} & A_{-2} & \dots A_{-k+1} & A_{-k} \end{pmatrix}$$

in the space $\mathfrak{B}^k$. The matrix A will be called *associated to the singular part* of the expansion (2.1).

In the following we need two simple properties of the pole multiplicity, which are easy consequences of the characterization given above.

a) If $A(z)$ is as above, Γ_1 is a bounded linear mapping from a Banach space $\mathfrak{B}_1$ into $\mathfrak{B}$, and Γ_2 is a bounded linear mapping from $\mathfrak{B}$ into $\mathfrak{B}_1$, then the pole multiplicity of α with respect to $\Gamma_2\, A(z)\Gamma_1$ is not greater than the pole multiplicity of α with respect to $A(z)$.

b) If α is an isolated eigenvalue of the operator T in $\mathfrak{B}$ and a pole of the resolvent of T, then its pole multiplicity with respect to this resolvent is equal to the algebraic multiplicity of the eigenvalue α.

Lemma 2.1. *Let $A(z)$ be a meromorphic $(n\times n)$-matrix function with a pole α of multiplicity $\varkappa$ and Laurent expansion* (2.1), *and let $Y(z)$ be an $(n\times n)$-matrix func-*

tion, holomorphic at $z=\alpha$. *If there exists a subspace* $\mathscr{L}\subset\mathfrak{R}(Y(\alpha))$ *such that*

(2.2) $$A_{-j}\mathscr{L}\subset\mathscr{L},\quad A_{-j}\mathscr{L}^{\perp}=\{0\},\quad j=1,2,\ldots,k,$$

then $A(z)Y(z)$ *has at* $z=\alpha$ *a pole of multiplicity* $\varkappa$.

Proof. The singular part of the Laurent expansion of $A(z)Y(z)$ at $z=\alpha$ has the associated matrix $\mathfrak{A}\mathfrak{Y}$, where $\mathfrak{Y}=(Y_{ij})_{i,j=1,2,\ldots,k}$, $Y_{ij}:=\frac{1}{(i-j)!}Y^{(i-j)}(\alpha)$ if $i\geqq j$, $Y_{ij}:=0$ if $i<j$, $i,j=1,2,\ldots,k$. Put $\mathfrak{Y}_0:=(P_0Y_{ij})_{i,j=1,2,\ldots,k}$, where P_0 is the orthogonal projector onto $\mathscr{L}$. According to (2.2), the range of $\mathfrak{A}\mathfrak{Y}$ coincides with the range of $\mathfrak{A}\mathfrak{Y}_0$, and the range of $\mathfrak{Y}_0$ is $\mathscr{L}^k=\mathscr{L}\dotplus\mathscr{L}\dotplus\ldots\dotplus\mathscr{L}$. On the other hand, the full range of $\mathfrak{A}$ is obtained if $\mathfrak{A}$ is applied to $\mathscr{L}^k$. The lemma is proved.

2. Consider now a function $\theta\in\mathbf{S}_{\varkappa}^{n\times n}$. If $\alpha\in\mathfrak{D}$ is a pole of θ, we denote its multiplicity by $\pi(\alpha)$. For some j, $1\leqq j\leqq l$, α coincides with α_j in a regular representation (1.13). We denote by $\varkappa(\alpha)$ the order of the corresponding factor B_j of the Blaschke—Potapov product in (1.13), that is

$$\varkappa(\alpha)=\sum_{k=1}^{k_j}\dim P_{jk}.$$

According to [2, § 3.4], $\varkappa(\alpha)$ coincides with the number of negative squares of the kernel S_{B_j}, and the number of negative squares of S_{Y_j} is $\varkappa(\alpha)$ plus the number of negative squares of $S_{Y_{j+1}}$, where Y_j is given by (1.16) and the representation (1.13) is again supposed to be regular.

If $0\in\mathfrak{D}_\theta$, then we denote by $\nu(\alpha)$ the dimension of the algebraic eigenspace, corresponding to α^{-1}, of a contractive operator T in $\Pi_\varkappa$ in a representation (1.9) of θ. This notation is correct because of [9, Theorem 11.2] and the following theorem.

Theorem 2.2. *If* $\theta\in\mathbf{S}_{\varkappa}^{n\times n}$ *and* $\alpha\in\mathfrak{D}$ *is a pole of* θ, *then* $\pi(\alpha)=\varkappa(\alpha)$. *If, additionally,* $0\in\mathfrak{D}_\theta$, *then* $\pi(\alpha)=\varkappa(\alpha)=\nu(\alpha)$.

Proof. First we show that the multiplicity of the pole α_j of B_j^{-1} in (1.14) is equal to $\sum_{k=1}^{k_j}\dim P_{jk}$. As the pole multiplicity is invariant under a fractional linear transformation of the independent variable, we may here suppose $\alpha_j=0$. Instead of P_{jk} we shall briefly write P_k, $k=1,2,\ldots,k_j$. Then the matrix associated with the singular part of the expansion of B_j^{-1} at $z=0$ is

$$\begin{pmatrix} P_{k_j} & 0 & \ldots & 0 & 0\\ P_{k_j-1}-P_{k_j} & P_{k_j} & \ldots & 0 & 0\\ \vdots & \vdots & & \vdots & \vdots\\ P_2-P_3 & P_3-P_4 & \ldots & P_{k_j} & 0\\ P_1-P_2 & P_2-P_3 & \ldots & P_{k_j-1}-P_{k_j} & P_{k_j}\end{pmatrix}.$$

Evidently, its range is $\mathfrak{R}(P_{k_j})+\mathfrak{R}(P_{k_j-1})+\ldots+\mathfrak{R}(P_1)$; therefore the dimension of this range is $\sum_{k=1}^{k_j}\dim P_k$. Thus the pole multiplicity of α_j coincides with the order of B_j.

Furthermore, we have

$$\theta(z) = B_1(z)^{-1}B_2(z)^{-1}\ldots B_{j-1}(z)^{-1}B_j(z)^{-1}Y_{j+1}(z).$$

From (1.15) it follows that Lemma 2.1 can be applied to $A=B_j^{-1}$, $Y=Y_{j+1}$ and $\mathscr{L}=\mathfrak{R}(P_{j1})$. Hence $B_j(z)^{-1}\,Y_{j+1}(z)$ has at $z=\alpha_j$ a pole of multiplicity $\sum_{k=1}^{k_j}\dim P_{jk}$. Finally

$$B_1(z)^{-1}B_2(z)^{-1}\ldots B_{j-1}(z)^{-1}$$

is holomorphic and boundedly invertible at $z=\alpha_j$. Therefore the pole multiplicity of $\theta(z)$ at $z=\alpha_j$ is $\sum_{k=1}^{k_j}\dim P_{jk}$, that is $\pi(\alpha_j)=\varkappa(\alpha_j)$.

To prove the second statement, consider a representation (1.9) of θ. According to the statements a) and b) in § 2.1, we have $\varkappa(\alpha_j)\leqq\nu(\alpha_j)$. On the other hand, the spectrum of T outside the unit disc consists of eigenvalues of total multiplicity $\varkappa$ (Propositions 1.2 and 1.5). Hence

$$\varkappa = \sum_{j=1}^{l}\varkappa(\alpha_j) \leqq \sum_{j=1}^{l}\nu(\alpha_j) = \varkappa,$$

and $\varkappa(\alpha_j)=\nu(\alpha_j)$, $j=1, 2, \ldots, l$, follows. The theorem is proved.

We mention that for a fractional linear transformation $z\to\zeta(z):=\dfrac{z-\beta}{1-z\bar\beta}$, $|\beta|<1$, $\beta\in\mathfrak{D}_\theta$ of $\mathfrak{D}$ onto $\mathfrak{D}$ the function θ_1: $\theta_1(\zeta):=\theta(z)$ always has the property $0\in\mathfrak{D}_{\theta_1}$. Also, it is easy to check that $\theta\in S_\varkappa^{n\times n}$ implies $\theta_1\in S_\varkappa^{n\times n}$.

Corollary 1. *$\theta\in\mathrm{S}_\varkappa^{n\times n}$ has poles in $\mathfrak{D}$ of total multiplicity $\varkappa$.*

Let now $F\in\mathrm{C}_\varkappa^{n\times n}$ be given. Choose α, $\mathrm{Re}\,\alpha>0$. Then by Proposition 1.1 the function θ:

$$\theta(z) = I-2\,\mathrm{Re}\,\alpha\big(F(z)+\alpha I\big)^{-1}$$

belongs to $\mathrm{S}_\varkappa^{n\times n}$, and (see [5]) the poles of θ coincide, including multiplicities, with the zeros of $F(z)+\alpha I$.

Corollary 2. *If $F\in\mathrm{C}_\varkappa^{n\times n}$ and* $\mathrm{Re}\,\alpha>0$, *then the function $F(z)+\alpha I$ has in $\mathfrak{D}$ zeros of total multiplicity $\varkappa$.*

The corresponding conclusion for a function $Q\in\mathrm{N}_\varkappa^{n\times n}$ reads as follows.

Corollary 3. *If* $Q\in\mathbf{N}_\varkappa^{n\times n}$ *and* $\operatorname{Im}\beta>0$, *then the function* $Q(z)+\beta I$ *has in* $\mathfrak{C}_+$ *zeros of total multiplicity* $\varkappa$.

As an application, consider the function Q:

$$Q(z)=Q_0(z)+\sum_{j=1}^{l}\sum_{k=1}^{k_j}\left((z-\alpha_j)^k B_{jk}+(z-\alpha_j^*)^k B_{jk}^*\right),$$

where $Q_0\in\mathbf{N}_0^{n\times n}$, B_{jk} are arbitrary $(n\times n)$-matrices and $\alpha_j\in\mathfrak{C}_+$, $k=1,2,\dots,k_j$; $j=1,2,\dots,l$. It follows as in [1, Satz 4.5] that $Q\in\mathbf{N}_\varkappa^{n\times n}$, where $\varkappa=\sum_{j=1}^{l}\varkappa_j$,

$$\varkappa_j=\dim\begin{pmatrix}B_{jk_j} & 0 & \dots & 0\\ \vdots & \vdots & & \vdots\\ B_{j2} & B_{j3} & & 0\\ B_{j1} & B_{j2} & \dots & B_{jk_j}\end{pmatrix}.$$

Hence Corollary 3 implies that for each β, $\operatorname{Im}\beta>0$, the function $Q(z)+\beta I$ has zeros of total multiplicity $\varkappa$ in $\mathfrak{C}_+$.

§ 3. Generalized zeros and poles of negative type on the boundary

1. Definition. Let $F\in\mathbf{C}_\varkappa^{n\times n}$. The point $z_0\in\partial\mathfrak{D}$ is called a *generalized pole (or zero) of negative type and multiplicity* $\pi(z_0)$ *for* F, if for each sufficiently small neighbourhood $\mathfrak{U}$ of z_0 there exists an $n(\mathfrak{U})>0$ such that for $\alpha>n(\mathfrak{U})$ (or $0<\alpha<n(\mathfrak{U})$, resp.) the function $F(z)+\alpha I$ has zeros of total multiplicity $\pi(z_0)$ in $\mathfrak{U}\cap\mathfrak{D}$.

To explain this definition e.g. in the case of a generalized pole, let us take a scalar function F. Instead of F we consider its continuation $\tilde{F}$ to $\{z\colon |z|\neq 1\}$ (see § 1.1) and assume that it has been continued analytically also to arcs of the unit circle $|z|=1$ if possible, that is if the boundary values of $\tilde{F}$ at the points of this arc exist and are purely imaginary. Suppose this continuation $\tilde{F}$ has a pole at $z_0\in\partial\mathfrak{D}$.

If $\varkappa=0$, that is $F\in\mathbf{C}_0$[1]), then $\operatorname{Re}F(z)\geqq 0$ for all $z\in\mathfrak{D}_F$. What is more, for each ϑ, $0<\vartheta<\pi/2$, there exists a ϑ_1, $0<\vartheta_1<\pi/2$, such that the relations $z_n\in\mathfrak{D}$, $-\vartheta<\arg(z_n-z_0)<\vartheta$ and $z_n\to z_0$ imply that $F(z_n)$ tends to infinity and $-\vartheta_1<\arg F(z_n)<\vartheta_1$.

On the other hand, if $\varkappa>0$, there may be poles z_0 on $\partial\mathfrak{D}$ with the property that there exists a sequence $(z_n)\subset\mathfrak{D}$, $z_n\to z_0$, such that $F(z_n)$ tends to infinity along the negative real half-axis. Moreover, it turns out that there may be a finite number of points z_0 on $\partial\mathfrak{D}$ which are no poles but which also do have the property $F(z_n)\to-\infty$ for some sequence $(z_n)\subset\mathfrak{D}$, $z_n\to z_0$. These two kinds of points z_0 are the generalized poles

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[1]) We write $\mathbf{C}_\varkappa$ etc. instead of $\mathbf{C}_\varkappa^{1\times 1}$.

of negative type. We mention already here that, for each point $\hat{z}\in\partial\mathfrak{D}$ which is not a generalized pole of negative type, there exists a neighbourhood $\hat{\mathfrak{U}}$ of $\hat{z}$ and a $\hat{\gamma}>0$ such that $\operatorname{Re} F(z)\geqq-\hat{\gamma}$ for all $z\in\hat{\mathfrak{U}}\cap\mathfrak{D}$ (see the Corollary subsequent to Theorem 3.5).

We show that the poles in $\mathfrak{D}$ of $F\in\mathbf{C}_\varkappa^{n\times n}$ have the same property as generalized poles on $\partial\mathfrak{D}$.

Proposition 3.1. *Let $F\in\mathbf{C}_\varkappa^{n\times n}$. If $z_0\in\mathfrak{D}$ is a pole of multiplicity $\pi(z_0)$ of F, then for each sufficiently small neighbourhood $\mathfrak{U}$ of z_0 there exists an $n(\mathfrak{U})>0$ such that $\alpha>n(\mathfrak{U})$ implies that the function $F(z)+\alpha I$ has zeros of total multiplicity $\pi(z_0)$ in $\mathfrak{U}$.*

Proof. For all $\alpha>0$, the point z_0 is also a pole of multiplicity $\pi(z_0)$ of $F(z)+\alpha I$. We choose a disc $\mathfrak{C}_0\subset\mathfrak{D}$ with centre z_0 such that z_0 is the only pole of F in $\mathfrak{C}_0$. Then F is holomorphic on $\mathfrak{C}_0\setminus\{z_0\}$ and we consider, for sufficiently large $\alpha>0$, the logarithmic residuum (see [5])

$$\frac{1}{2\pi i}\operatorname{trace}\int_{\partial\mathfrak{C}_0}F'(z)\big(F(z)+\alpha I\big)^{-1}dz=\frac{1}{2\pi i\alpha}\operatorname{trace}\int_{\partial\mathfrak{C}_0}F'(z)\big(\alpha^{-1}F(z)+I\big)^{-1}dz.$$

If α is large, this value is zero; hence for these α the total multiplicity of the zeros of $F(z)+\alpha I$ in $\mathfrak{C}_0$ is equal to $\pi(z_0)$.

For the zeros of $F\in\mathbf{C}_\varkappa^{n\times n}$ another simple application of the logarithmic residuum theorem gives the following result, the proof of which is left to the reader.

Proposition 3.2. *Let $F\in\mathbf{C}_\varkappa^{n\times n}$, $\det F(z)\not\equiv 0$. If z_0 is a zero of multiplicity $\mu(z_0)$ of F, then for each sufficiently small neighbourhood $\mathfrak{U}$ of z_0 there exists an $n(\mathfrak{U})>0$ such that $0<\alpha<n(\mathfrak{U})$ implies that the function $F(z)+\alpha I$ has zeros of total multiplicity $\mu(z_0)$ in $\mathfrak{U}$.*

Proposition 3.3. *Let $F\in\mathbf{C}_\varkappa^{n\times n}$, $\det F(z)\not\equiv 0$. Then the following statements are true:*

a) $F^{-1}\in\mathbf{C}_\varkappa^{n\times n}$ $\big(F^{-1}(z):=F(z)^{-1}\big)$;

b) *the zeros (poles) of F in $\mathfrak{D}$ coincide, multiplicities counted, with the poles (zeros) of F^{-1} in $\mathfrak{D}$;*

c) *the generalized zeros (poles) of F of negative type on $\partial\mathfrak{D}$ coincide, multiplicities counted, with the generalized poles (zeros) of F^{-1} of negative type.*

The proof of a) follows immediately from the definition of the class $\mathbf{C}_\varkappa^{n\times n}$, while b) is a general property of zeros and poles of matrix functions. To prove c), consider e.g. a generalized zero $z_0\in\partial\mathfrak{D}$ of F of negative type and choose a neighbourhood $\mathfrak{U}$ of z_0 such that $\mathfrak{U}\cap\mathfrak{D}$ does not contain any zero or pole of F. Then the statement follows easily from the identity

$$\alpha F(z)\big(F(z)^{-1}+\alpha^{-1}I\big)=F(z)+\alpha I\quad(z\in\mathfrak{U}\cap\mathfrak{D}).$$

2. Proposition 3.1 and Corollary 2 of Theorem 2.2 imply that the total multiplicity of poles in $\mathfrak{D}$ and generalized poles of negative type on $\partial\mathfrak{D}$ of a function $F\in \mathbf{C}_\varkappa^{n\times n}$ is at most $\varkappa$. We shall show that this multiplicity is exactly $\varkappa$ (Theorem 3.5). To this end, we consider the operator V of (1.6) which is maximal isometric in $\Pi_\varkappa$.

Proposition 3.4. *Let* $F\in\mathbf{C}_\varkappa^{n\times n}$, $0\in\mathfrak{D}_F$. *The point* $z_0\in\overline{\mathfrak{D}}$ *is a pole in* $\mathfrak{D}$, *or a generalized pole of negative type on* $\partial\mathfrak{D}$, *of F if and only if* z_0^{-1} *belongs to* $\sigma_0(V^{-1})$; *in this case* $\pi(z_0)=\varkappa_{z_0^{-1}}(V^{-1})$.

Proof. Let z_0 be, say, a generalized pole of negative type and multiplicity $\pi(z_0)$ of F. Then for each sufficiently small neighbourhood $\mathfrak{U}$ of z_0 there exists an $n(\mathfrak{U})>0$ such that for $\alpha>n(\mathfrak{U})$ the function $F(z)+\alpha I$ has zeros of total multiplicity $\pi(z_0)$ in $\mathfrak{U}\cap\mathfrak{D}$.

The function θ given by (1.1) and its contractive operator T in representation (1.9) will now be denoted by θ_α and T_α, respectively. Then, by (1.1), θ_α has poles of total multiplicity $\pi(z_0)$ in $\mathfrak{U}\cap\mathfrak{D}$. Theorem 2.2 implies that $T_\alpha|\mathscr{L}_\alpha$ has eigenvalues of total multiplicity $\pi(z_0)$ in $(\mathfrak{D}\cap\mathfrak{U})^{-1}$, where $\mathscr{L}_\alpha$ denotes a $\varkappa$-dimensional nonpositive invariant subspace of T_α with $|\sigma(T_\alpha|\mathscr{L}_\alpha)|\geqq 1$. If $\alpha\uparrow\infty$ then $\|T_\alpha-V^{-1}\|\to 0$ (see the proof of Proposition 1.5) and Proposition 1.4 implies that z_0^{-1} is an eigenvalue of algebraic multiplicity $\pi(z_0)$ of $V^{-1}|\mathscr{L}$. This reasoning can be reversed, and the statement follows.

We can now state the main result of this section.

Theorem 3.5. *Let* $F\in\mathbf{C}_\varkappa^{n\times n}$. *Then F has poles in* $\mathfrak{D}$ *and generalized poles of negative type on* $\partial\mathfrak{D}$ *of total multiplicity* $\varkappa$. *If, moreover,* $\det F(z)\not\equiv 0$, *then F has zeros in* $\mathfrak{D}$ *and generalized zeros of negative type on* $\partial\mathfrak{D}$ *of total multiplicity* $\varkappa$.

This follows immediately from Propositions 3.4 and 3.3 if we only observe that the condition $0\in\mathfrak{D}_F$ can always be fulfilled at the expense of a fractional linear transformation of $\mathfrak{D}$ onto itself.

By Proposition 3.4 and the definition of g appearing in the representation (1.8), the generalized poles of negative type of F are the zeros on $\partial\mathfrak{D}$ of the function g. Suppose the point $\hat{z}\in\partial\mathfrak{D}$ is not a generalized pole of negative type of $F\in\mathbf{C}_\varkappa^{n\times n}$, and choose an open arc $\hat{\Delta}\subset\partial\mathfrak{D}$ which contains $\hat{z}$ and has a positive distance from all generalized poles of negative type of F. Consider the decomposition

$$\Pi_\varkappa=\hat{\Pi}\oplus\Pi'_\varkappa,\quad \hat{\Pi}:=E(\hat{\Delta})\Pi_\varkappa,$$

where $\Pi_\varkappa$ is the space that plays a role in the representation (1.6) of F by a unitary operator V, and E denotes the spectral function of V (see [11]). Let the corresponding decomposition of V be $V=\hat{V}\oplus V'$. Then

$$F(z)=\hat{F}(z)+F'(z),$$

where $\hat{F}(z) := \Gamma^* E(\hat{\Delta})(\hat{V}+z\hat{I})(\hat{V}-z\hat{I})^{-1}E(\hat{\Delta})\Gamma$. As $\hat{\Pi}$ is a Hilbert space (see [11]), we have $\hat{F} \in \mathbf{C}_\varkappa^{n\times n}$. Moreover, $\widetilde{F'}$ (see the beginning of § 3) is holomorphic on $\hat{\Delta}$. This implies the following

Corollary. *Let $F \in \mathbf{C}_\varkappa^{n\times n}$. Then for each point $\hat{z} \in \partial\mathfrak{D}$ which is not a generalized pole of negative type of F, there exists a neighbourhood $\hat{\mathfrak{U}}$ of $\hat{z}$ and a number $\hat{\gamma}$ such that* $\operatorname{Re} F(z) \geqq -\hat{\gamma} I$ *for all* $z \in \hat{\mathfrak{U}} \cap \mathfrak{D}$.

§ 4. A generalization of Rouché's theorem

1. We denote by $\mathscr{D}^{n\times n}$ the set of all $(n\times n)$-matrix functions F which are defined and holomorphic in $\mathfrak{D}$ and admit a representation $F = y^{-1} Y$ with a bounded outer function y and a bounded holomorphic $(n\times n)$-matrix function Y in $\mathfrak{D}$ (equivalent definitions are given e.g. in [13]). Then the function

$$\det F(z) = y(z)^{-n} \det Y(z)$$

belongs to the class $\mathscr{D}(=\mathscr{D}^{1\times 1})$, hence it has, almost everywhere on $\partial\mathfrak{D}$, finite nontangential limits which are, almost everywhere, different from zero. Therefore the nontangential boundary values $F(\zeta)$ of F, which exist almost everywhere on $\partial\mathfrak{D}$, have an inverse $F(\zeta)^{-1}$ almost everywhere.

The function $F_0 \in \mathscr{D}^{n\times n}$ is called *outer* if $\det F_0(z)$ is an outer function. In this case we have $\det F_0(z) \neq 0$ $(z \in \mathfrak{D})$, hence $F_0(z)^{-1}$ exists for all $z \in \mathfrak{D}$ and the function F_0^{-1} belongs again to $\mathscr{D}^{n\times n}$.

The function $F \in \mathscr{D}^{n\times n}$ is said to have an *inner factor of order* $\varkappa$ if it admits a representation

$$F(z) = U_0 \left(\overset{\curvearrowleft}{\prod_{j=1}^{l}} B_j(z) \right) F_0(z), \tag{4.1}$$

where $F_0 \in \mathscr{D}^{n\times n}$ is an outer function and $U_0 \overset{\curvearrowleft}{\prod_{j=1}^{l}} B_j(z)$ is a regular Blaschke—Potapov product of order $\varkappa$ (see § 1).

Lemma 4.1. *Let f be a complex function which is holomorphic in $\mathfrak{D}$ and has no zeros there, and denote by* $\operatorname{Arg} f$ *a continuous branch of the function* $\arg f$. *If* $\gamma := \sup \{|\operatorname{Arg} f(z)| : z \in \mathfrak{D}\} < \infty$, *then f is an outer function.*

Proof. Choose an integer n such that $n > \frac{2\gamma}{\pi}$. Then the function f_1: $f_1(z) := (f(z))^{1/n} = |f(z)|^{1/n} \exp\left(\frac{i}{n} \operatorname{Arg} f(z)\right)$ has the property $\operatorname{Re} f_1(z) > 0$ $(z \in \mathfrak{D})$. By [14, p. 51, Exercise 1], f_1 is an outer function; thus f is an outer function.

2. Now we prove the following generalization of Rouché's theorem.

Theorem 4.2 *Suppose* $F, G \in \mathscr{D}^{n\times n}$, $\det(F(z)-G(z)) \not\equiv 0$ *in* $\mathfrak{D}$ *and*

$$\|G(\zeta)F(\zeta)^{-1}\| \leqq 1 \quad \textit{a.e. on } \partial\mathfrak{D}. \tag{4.2}$$

If F *has an inner factor of order* $\varkappa_F(<\infty)$, *then* $F-G$ *has an inner factor of order* $\varkappa_{F-G} \leqq \varkappa_F$. *If, additionally,* $F(F-G)^{-1}|_{\partial\mathfrak{D}} \in L_1^{n\times n}(\partial\mathfrak{D})$,[1]) *then* $\varkappa_{F-G} = \varkappa_F$.

Remark. From the proof it will follow that the difference $\varkappa_F - \varkappa_{F-G}$ is the total multiplicity of generalized poles of negative type on $\partial\mathfrak{D}$ of the function $(F+G)(F-G)^{-1}(=-I+2F(F-G)^{-1})$, which belongs to $\mathbf{C}_{\varkappa'}^{n\times n}$ for some $\varkappa' \leqq \varkappa_F$.

Proof of Theorem 3. We write the representation (4.1) of F in the form $F=BF_0$. Then $F-G=(B-GF_0^{-1})F_0$, $GF_0^{-1}\in\mathscr{D}^{n\times n}$, $F(F-G)^{-1}=B(B-GF_0^{-1})^{-1}$ and

$$\|G(\zeta)F(\zeta)^{-1}\| = \|G(\zeta)F_0(\zeta)^{-1}B(\zeta)^{-1}\| \leqq 1 \quad \text{a.e. on} \quad \partial\mathfrak{D}.$$

As F_0 is outer, the order of the inner factor of $F-G$ coincides with the order of the inner factor of $B-GF_0^{-1}$. Therefore, in the proof of the theorem we may suppose that $F=B$.

The matrix $B(\zeta)$, $|\zeta|=1$, is unitary, hence (4.2) implies $\|G(\zeta)\| \leqq 1$ a.e. on $\partial\mathfrak{D}$. Applying [13, Lemma 1.1] it follows that $\|G(z)\| \leqq 1$ for all $z\in\mathfrak{D}$. This is equivvalent (see [2, Lemma 3.1]) to $G\in\mathbf{S}_0^{n\times n}$ and $G^*\in\mathbf{S}_0^{n\times n}$, where G^* is the $(n\times n)$-matrix function $G^*(z):=G(z^*)^*$ $(z\in\mathfrak{D})$.

Consider now the function $B^{*-1}G^*$. According to [2, Lemma 3.5] it belongs to some class $\mathbf{S}_{\varkappa'}^{n\times n}$, where $\varkappa' \leqq \varkappa_F$. Then the same is true for GB^{-1} [2, Folgerung 3.3], and both functions have poles in $\mathfrak{D}$ of total multiplicity $\varkappa'$ (Corollary 1 of Theorem 2.2).

The condition $\det(B(z)-G(z)) \not\equiv 0$ implies that $(I-GB^{-1})^{-1}$ exists. Moreover, it is easy to check that the function C:

$$C(z) := (I+G(z)B(z)^{-1})(I-G(z)B(z)^{-1})^{-1} = -I+2(I-G(z)B(z)^{-1})^{-1} \tag{4.3}$$

belongs to $\mathbf{C}_{\varkappa'}^{n\times n}$. According to Theorem 3.5 it has poles of total multiplicity $\varkappa''(\leqq\varkappa')$ in $\mathfrak{D}$, and the difference $\varkappa'-\varkappa''$ is the total multiplicity of its generalized poles of negative type on $\partial\mathfrak{D}$. In view of (4.3), the function $I-G(z)B(z)^{-1}$ has zeros of total multiplicity $\varkappa''$ in $\mathfrak{D}$. By [5, (1.3)], for a meromorphic $(n\times n)$-matrix function the difference of the total zero and total pole multiplicities in $\mathfrak{D}$ coincides with

[1]) $L_1^{n\times n}(\partial\mathfrak{D})$ denotes the class of $(n\times n)$-matrix functions defined a.e. on $\partial\mathfrak{D}$ with entries in $L_1(\partial\mathfrak{D})$.

the corresponding difference for its determinant function. Therefore, using the relation

$$\det(B(z)-G(z)) = \det(I-G(z)B(z)^{-1})\det B(z)$$

we find that

$$\varkappa''' = \varkappa_F + \varkappa'' - \varkappa',$$

where $\varkappa'''$ denotes the total zero multiplicity of $B-G$ in $\mathfrak{D}$. Observing that $\varkappa'' \leqq \varkappa'$, the inequality $\varkappa''' \leqq \varkappa_F$ follows.

Therefore, the difference $B-G$ admits a regular representation $B-G=B_0H$ with a Blaschke—Potapov product B_0 of order $\varkappa'''$ and an $(n\times n)$-matrix function H which is holomorphic in $\mathfrak{D}$ and does not have any zeros there (cf. the proof of [2, Satz 3.2]). The first statement of the theorem follows if we show that H is an outer function. But this is true if and only if $\det H(z)^{-1}$ is an outer function.

We have $H=2B_0^{-1}(I+C)^{-1}B$ (see (4.3)). According to (1.8), C admits an integral representation

$$C(z) = iS + \frac{1}{g(z)} \int_0^{2\pi} \frac{e^{i\vartheta}+z}{e^{i\vartheta}-z}\, d\Sigma(\vartheta) + \frac{1}{g(z)} D(z), \tag{4.4}$$

where S, Σ, D and g have the properties mentioned in § 1.3. It follows that

$$H(z)^{-1} = \frac{1}{2} B(z)^{-1}((I+iS)g(z)+D(z)+C_0(z))B_0(z)g(z)^{-1},$$

where C_0 denotes the integral in (4.4). As $\operatorname{Re} C_0(z) \geqq 0$ $(z\in\mathfrak{D})$ and g, D are polynomials of z and z^{-1}, for each entry of the matrix $H(z)^{-1}$ the argument is bounded on $\mathfrak{D}$. Then the same is true for $\det H(z)^{-1}$. Hence Lemma 4.1 implies that $\det H(z)^{-1}$ is an outer function.

Let now $B(\zeta)(B(\zeta)-G(\zeta))^{-1}\in L_1^{n\times n}(\partial\mathfrak{D})$. Then according to (4.3) $C\in L_1^{n\times n}(\partial\mathfrak{D})$, and it remains to show that C does not have generalized poles of negative type on $\partial\mathfrak{D}$.

The function $C\in\mathbf{C}_{\varkappa'}^{n\times n}$ can be written as the sum of a rational function $C_1\in\mathbf{C}_{\varkappa_1}^{n\times n}$ with poles in $\mathfrak{D}$ and a function $C_2\in\mathbf{C}_{\varkappa_2}^{n\times n}$, which is holomorphic in $\mathfrak{D}$, $\varkappa'=\varkappa_1+\varkappa_2$. The function $C_2(\zeta)$, $\zeta\in\partial\mathfrak{D}$, also belongs to $L_1^{n\times n}(\partial\mathfrak{D})$, and it is sufficient to show that $\varkappa_2=0$. This will be accomplished if we prove the following two statements[1]):

a) $C_2\in H_1^{n\times n}$;

b) If for some $\varkappa<\infty$ we have $H_1^{n\times n}\cap\mathbf{C}_\varkappa^{n\times n}\neq\varnothing$, then $\varkappa=0$.

To prove a) we first observe that $E\in\mathbf{C}_\varkappa^{n\times n}$ implies $E\in H_\delta^{n\times n}$ for $\delta<(1+2\varkappa)^{-1}$. Indeed, as

$$\|E(z)\|^\delta \leqq (\sqrt{n}\max_{i,j}|e_{ij}(z)|)^\delta,$$

[1]) For the definition of the Hardy classes $H_\delta^{n\times n}$ see e.g. [13].

it is sufficient to show that the entries e_{ij} of E belong to H_δ. According to (1.8), every e_{ij} is of the form $g_1(z)^{-1}h(z)$, where $h\in H_\delta$ for all $\delta<1$ (see [15, II. 4.5]) and g_1 is a polynomial of degree $\leqq 2\varkappa$ with zeros on $\partial\mathfrak{D}$. If $\delta<(1+2\varkappa)^{-1}$, we choose $\delta_1<(2\varkappa)^{-1}$ and $\delta_2<1$ so that $\delta=\delta_1\delta_2(\delta_1+\delta_2)^{-1}$. Setting $p=\delta_2^{-1}(\delta_1+\delta_2)$ and $q=\delta_1^{-1}(\delta_1+\delta_2)$ we obtain

$$\int_0^{2\pi}|g_1(re^{i\vartheta})^{-1}h(re^{i\vartheta})|^\delta d\vartheta \leqq \Big(\int_0^{2\pi}|g_1(re^{i\vartheta})|^{-\delta p}d\vartheta\Big)^{1/p}\Big(\int_0^{2\pi}|h(re^{i\vartheta})|^{\delta q}d\vartheta\Big)^{1/q} =$$

$$= \Big(\int_0^{2\pi}|g_1(re^{i\vartheta})|^{-\delta_1}d\vartheta\Big)^{1/p}\Big(\int_0^{2\pi}|h(re^{i\vartheta})|^{\delta_2}d\vartheta\Big)^{1/q} \leqq K<\infty$$

for all $0<r<1$. Thus $E\in H_\delta^{n\times n}$. In particular, $C_2\in H_\delta^{n\times n}$. As $C_2(\zeta)\in L_1^{n\times n}(\partial\mathfrak{D})$, by a theorem of V. I. Smirnov (cf. [15, II. 6]) we have $C_2\in H_1^{n\times n}$.

To prove b), we use the representation

$$E(z) = i\,\mathrm{Im}\,E(0)+\frac{1}{2\pi}\int_0^{2\pi}(e^{i\vartheta}+z)(e^{i\vartheta}-z)^{-1}\,\mathrm{Re}\,E(e^{i\vartheta})\,d\vartheta \quad (z\in\mathfrak{D}),$$

which holds for arbitrary functions $E\in H_1^{n\times n}$, and the representation (1.8):

$$E(z) = iS+\frac{1}{g(z)}\int_0^{2\pi}(e^{i\vartheta}+z)(e^{i\vartheta}-z)^{-1}\,d\Sigma(\vartheta)+\frac{1}{g(z)}G(z),$$

valid for $E\in C_\varkappa^{n\times n}$, $0\in\mathfrak{D}_E$. Making the right-hand sides equal, multiplying by $g(z)$ and using Stieltjes—Livšic inversion formula it follows that

$$\int_{\vartheta_1}^{\vartheta_2} g(e^{i\vartheta})\,\mathrm{Re}\,E(e^{i\vartheta})\,d\vartheta = \int_{\vartheta_1}^{\vartheta_2} d\Sigma(\vartheta)\geqq 0$$

whenever $0\leqq\vartheta_1<\vartheta_2\leqq 2\pi$. Therefore $\mathrm{Re}\,E(e^{i\vartheta})\geqq 0$ almost everywhere on $[0, 2\pi]$. Hence $\mathrm{Re}\,E(z)>0$ $(z\in\mathfrak{D})$, and $\varkappa=0$.

The theorem is proved.

§ 5. Further examples

1. In this section we consider two examples of functions of the class $\mathbf{N}_\varkappa^{n\times n}$. By the connection between the classes $\mathbf{N}_\varkappa^{n\times n}$ and $\mathbf{C}_\varkappa^{n\times n}$ mentioned in § 1.1, the notions of generalized zeros and poles of negative type carry over to functions $Q\in\mathbf{N}_\varkappa^{n\times n}$ in the following way. Let φ be a linear fractional mapping from $\mathfrak{D}$ onto $\mathbb{C}_+$. The point $t_0\in R_1\cup\{\infty\}$, $t_0=\varphi(\zeta_0)$ $(|\zeta_0|=1)$, is said to be a *generalized pole (zero) of negative type and multiplicity* $\pi(t_0)$ *of* Q if ζ_0 is a generalized pole (or zero, resp.) of negative type and multiplicity $\pi(t_0)$ of $F=iQ\circ\varphi$, or equivalently, if for each

sufficiently small neighbourhood $\mathfrak{U}$ of t_0 (we admit the case $t_0=\infty$) in the closed complex plane there exists an $n(\mathfrak{U})>0$ such that for $\alpha>n(\mathfrak{U})$ (or $0<\alpha<n(\mathfrak{U})$, resp.) the function $Q(z)+i\alpha I$ has zeros of total multiplicity $\pi(t_0)$ in $\mathfrak{U}\cap\mathbb{C}_+$.

Let $Q_0\in\mathbf{N}_0^{n\times n}$. Then Q_0 has a representation

$$Q_0(z) = A_0+zB_0+\int_{-\infty}^{\infty}\left((t-z)^{-1}-t(1+z^2)^{-1}\right)d\Sigma(t)$$

with hermitian $(n\times n)$-matrices A_0, B_0; $B_0\geqq 0$, and a nondecreasing $(n\times n)$-matrix function Σ on R_1, $\int_{-\infty}^{\infty}(1+t^2)^{-1}d\Sigma(t)<\infty$. Now let B_1 be a hermitian $(n\times n)$-matrix and let us consider the function Q_1:

(5.1) $$Q_1(z):=Q_0(z)-zB_1.$$

Then

$$N_Q(z,\zeta) = \int_{-\infty}^{\infty}(t-z)^{-1}(t-\zeta^*)^{-1}d\Sigma(t)+B_0-B_1,$$

and, considering $N_{Q_1}(z,z)$ for $|z|$ sufficiently large, it follows that $Q_1\in\mathbf{N}_\varkappa^{n\times n}$, where $\varkappa$ denotes the number of negative eigenvalues of the matrix B_0-B_1. It is easy to see that Q_1 has a generalized pole of negative type and multiplicity $\varkappa$ at ∞. Moreover, Theorem 3.5 implies:

Proposition 5.1. *If* $\det Q_1(z)\not\equiv 0$, *then the function* Q_1 *in* (5.1) *has zeros in* $\mathbb{C}_+$ *and generalized zeros of negative type in* $R_1\cup\{\infty\}$ *of total multiplicity* $\varkappa$, *where* $\varkappa$ *denotes the number of negative eigenvalues of the matrix* B_0-B_1.

In special cases this result can be given a more explicit formulation. Here we consider the case where $n=1$ and

(5.2) $$Q_1(z) = \int_{-\infty}^{\infty}(t-z)^{-1}d\sigma(t)+\alpha-z$$

with α a real number and σ a nondecreasing function on R_1 such that

$$\int_{-\infty}^{\infty}(1+|t|)^{-1}d\sigma(t)<\infty;$$

without loss of generality, the coefficient of z has been chosen -1.

Proposition 5.2. *The function* Q_1 *in* (5.2) *has either exactly one zero in* $\mathbb{C}_+$ *or one generalized zero of negative type in* $R_1\cup\{\infty\}$. *This (possibly, generalized)* *zero* z_α *is* $\neq\infty$ *and of multiplicity* 1. *It can be characterized among the points of* $\bar{C}_+$ *by the following two properties:*

a) $$\int_{-\infty}^{\infty} |t-z_\alpha|^{-2}\, d\sigma(t) \leqq 1,$$

b) $$\int_{-\infty}^{\infty} (t-z_\alpha)^{-1}\, d\sigma(t)+\alpha-z_\alpha = 0.$$

Proof. The first statement including the claim about the multiplicity follows from Proposition 5.1.

Next we show that a zero or generalized zero $z_\alpha\in\bar{\mathfrak{C}}_+$ of Q_1 has the properties a) and b). If z_α is a zero $(z_\alpha\in\mathfrak{C}_+)$, this is obvious if we observe that

$$0 = \operatorname{Im} Q_1(z_\alpha) = \operatorname{Im} z_\alpha \Big(\int_{-\infty}^{\infty} \frac{d\sigma(t)}{|t-z_\alpha|^2} - 1 \Big).$$

Let now $z_\alpha\in R_1$. Then there exists a sequence $(z_n)\subset\mathfrak{C}_+$, $z_n\to z_\alpha$, $\operatorname{Im} z_n\downarrow 0$ $(n\to\infty)$ such that $Q_1(z_n)\to 0$, $\operatorname{Im} Q_1(z_n)<0$. It follows that $(\operatorname{Im} z_n)\Big(\int_{-\infty}^{\infty} |t-z_n|^{-2}\, d\sigma(t)-1\Big)<0$, or

$$\int_{-\infty}^{\infty} |t-z_n|^{-2}\, d\sigma(t) < 1, \quad n=1,2,\ldots.$$

Applying Fatou's lemma we get $\int_{-\infty}^{\infty} (t-z_\alpha)^{-2} d\sigma(t)\leqq 1$, and Lebesgue's theorem gives

$$0 = \lim_{n\to\infty} Q_1(z_n) = \int_{-\infty}^{\infty} (t-z_\alpha)^{-1}\, d\sigma(t)+\alpha-z_\alpha.$$

It remains to show that a) and b) have at most one solution z_α in $\bar{\mathfrak{C}}_+$. To this end we introduce the π_1-space $\Pi_1 := \mathfrak{C}\oplus L_2(\sigma)$ of all pairs $\{\xi, x\}$, $\xi\in\mathfrak{C}$, $x\in L_2(\sigma)$ with indefinite scalar product

$$[\{\xi, x\}, \{\eta, y\}] = -\xi\eta^* + \int_{-\infty}^{\infty} x(t)y(t)^*\, d\sigma(t) \quad (\xi,\eta\in\mathfrak{C};\ x,y\in L_2(\sigma)).$$

It is easy to check that the operator A:

$$A\{\xi, x\} := \Big\{\alpha\xi - \int_{-\infty}^{\infty} x(t)\, d\sigma(t),\ tx(t)+\xi\Big\}, \tag{5.3}$$

which is defined for every $\{\xi, x\}\in\Pi_1$ such that the function $t\to tx(t)+\xi$ belongs to $L_2(\sigma)$, is selfadjoint in Π_1. In order to find its eigenvalues λ we have to solve the equation

$$(A-\lambda I)\{\xi, x\}=0. \tag{5.4}$$

From (5.3) and (5.4) it follows that $x(t)=-\xi(t-\lambda)^{-1}(\sigma-a.e)$. In particular, $\int_{-\infty}^{\infty}|t-\lambda|^{-2}\,d\sigma(t)<\infty$. Moreover, the first component in (5.4) gives

$$\alpha-\lambda+\int_{-\infty}^{\infty}(t-\lambda)^{-1}\,d\sigma(t)=0. \tag{5.5}$$

Conversely, it is easy to see that any solution λ of (5.5) with $\int_{-\infty}^{\infty}|t-\lambda|^{-2}d\sigma(t)<\infty$ is an eigenvalue of A with corresponding eigenelement $\{\xi, -\xi(t-\lambda)^{-1}\}$ $(\xi\neq 0)$.

Since A is a selfadjoint operator in Π_1, it has exactly one eigenvalue $\lambda_0\in\overline{\mathfrak{C}}_+$ such that the corresponding eigenvector is nonpositive:

$$-1+\int_{-\infty}^{\infty}|t-\lambda_0|^{-2}\,d\sigma(t)\leqq 0.$$

Therefore, $z_\alpha=\lambda_0$ is the only solution of the system a)—b) in $\overline{\mathfrak{C}}_+$. The proposition is proved.

Remark 1. The zeros of the function Q_1 are the fixed points of the function Q_0: $Q_0(z):=\int_{-\infty}^{\infty}(t-z)^{-1}d\sigma(t)+\alpha$, and by a fractional linear transformation of $\mathfrak{C}_+$ onto $\mathfrak{D}$ the equation $Q_0(z)=z$ is transformed into $G_0(\zeta)=\zeta$, where G_0 maps $\mathfrak{D}$ holomorphically into itself $(G_0\in S_0)$. However, the usual fixed point argument does not seem to be applicable in this case, as the boundary values of G_0 on $\partial\mathfrak{D}$ are, in general, discontinuous.

Remark 2. Suppose that the function σ in (5.2) satisfies the additional condition

$$\int_{-\infty}^{\infty}(t-x)^{-2}d\sigma(t)=\infty \quad \text{for all } x\in\mathfrak{E}_\sigma,$$

where $\mathfrak{E}_\sigma$ denotes the set of all points of increase of σ. Then for every real α the function Q_1 in (5.2) has one and only one zero $z_\alpha\in\overline{\mathfrak{C}}_+\setminus\mathfrak{E}_\sigma$ and $\int_{-\infty}^{\infty}|t-z_\alpha|^{-2}d\sigma(t)\leqq 1$.

Remark 3. Besides the zero z_α, the function Q_1 can have an arbitrary number $(\leqq\infty)$ of real zeros which do not satisfy condition a).

To see this, we suppose that the function σ in (5.2) is constant on some interval (a, b) which is *special* in the sense that

$$\lim_{x\downarrow a}\int_{-\infty}^{a}(t-x)^{-1}\,d\sigma(t)=-\infty,\quad \lim_{x\uparrow b}\int_{b}^{\infty}(t-x)^{-1}\,d\sigma(t)=\infty.$$

Then Q_1 is holomorphic in (a, b) and $Q_1(a+0)=-\infty$, $Q_1(b-0)=\infty$. Therefore it has at least one zero in (a, b), more exactly, it has an odd number of zeros in (a, b), counted with multiplicities. Denote these zeros by $x_1 \leqq x_2 \leqq \ldots \leqq x_{2k+1}$. It is easy to see that $Q_1'(x_{2j}) \leqq 0$, $j=1, 2, \ldots, k$, that is $x_{2j}=z_\alpha$, $j=1, 2, \ldots, k$. Hence the function Q_1 has in (a, b) either one simple zero, either zeros of total multiplicity three; in the second case the zero x_2 coincides with z_α.

Consequently, if σ has $N\,(\leqq\infty)$ special intervals (a_j, b_j), $j=1, 2, \ldots, N$, then the corresponding function Q_1 has in no more than one of these intervals zeros of total multiplicity three, in each of the remaining intervals it has exactly one (simple) zero. The case $N=\infty$ occurs, for example, if Q_1 is a meromorphic function with an infinite number of poles.

We mention that any simple selfadjoint operator $\tilde{A}$ in a π_1-space $\tilde{\Pi}_1$ is unitarily equivalent to the operator A appearing in the proof of Proposition 5.2. Here $\tilde{A}$ is called simple if there exists an $e \in \mathfrak{D}(\tilde{A})$, $[e, e]<0$, such that

$$\tilde{\Pi}_1 = \text{c.l.s.}\,\{(\tilde{A}-\zeta\tilde{I})^{-1}e : \pm\zeta\in\mathbb{C}_+\cap\varrho(\tilde{A})\}.$$

Indeed, suppose $[e, e]=-1$ and consider the decomposition

$$\Pi_1 = \mathcal{L}_0 \dotplus \mathcal{L}_1,\ \mathcal{L}_0 = \text{l.s.}\,\{e\},\ \mathcal{L}_1 = \mathcal{L}_0^{[\perp]}. \tag{5.6}$$

Then $\mathcal{L}_1$ is a Hilbert space with scalar product $[.,.]$, and $\mathcal{L}_0$ can be identified with $\mathbb{C}$ by writing $e=\{1, 0\}$ with respect to the decomposition (5.6).

If $\tilde{A}e=\{\alpha, h\}$, $\alpha\in\mathbb{C}$, $h\in\mathcal{L}_1$, then the matrix representation of $\tilde{A}$ is

$$\tilde{A} = \begin{pmatrix} \alpha & -[.,h] \\ h & A_{11} \end{pmatrix} \tag{5.7}$$

with some selfadjoint operator A_{11} in the Hilbert space $\mathcal{L}_1$. Now an easy calculation gives

$$(\tilde{A}-\zeta\tilde{I})^{-1}e = \xi\{1, -(A_{11}-\zeta I)^{-1}h\},\quad \xi = (\alpha-\zeta+[(A_{11}-\zeta I)^{-1}h, h])^{-1}.$$

It follows that $\mathcal{L}_1=\text{c.l.s.}\,\{(A_{11}-\zeta I)^{-1}h : \pm\zeta\in\mathbb{C}_+\}$. Hence A_{11} is unitarily equivalent to the operator of multiplication by the independent variable in the space $L_2(\sigma)$, $\sigma(t):=[E_t h, h]$, where E_t is the spectral function of A_{11} and h corresponds to the function $h(t)\equiv 1$ belonging to $L_2(\sigma)$. With this realization of $\mathcal{L}_1$ and A_{11}, the matrix in (5.7) defines the operator A in (5.3).

This model of an arbitrary simple selfadjoint operator in a π_1-space (or, more generally, in a $\pi_\varkappa$-space) and the characterization of its eigenvalues by conditions a) and b) were first given in [4; III, § 6].

2. In this section we consider an $(n\times n)$-matrix function $Q_0\in \mathbf{N}_0^{n\times n}$ of the form

$$Q_0(z) = -\sum_{j=1}^{l}\frac{B_j}{t_j+z}+\int_0^\infty (t-z)^{-1}\,d\Sigma(t),$$

where B_j are nonnegative hermitian matrices, $0<t_1<t_2<\ldots<t_l$, and Σ is a nondecreasing $(n\times n)$-matrix function on $[0,\infty)$ with the properties $\Sigma(0+)=\Sigma(0)=0$, $\int_0^\infty (1+t)^{-1}d\Sigma(t)<\infty$. Then the function Q: $Q(z)=zQ_0(z^2)$ has the representation

$$Q(z)=\frac{1}{2}\sum_{j=1}^{l} B_j\{(i\sqrt{t_j}-z)^{-1}-(i\sqrt{t_j}+z)^{-1}\}+\int_{-\infty}^{\infty}(z-s)^{-1}\,d\tilde{\Sigma}(s),$$

$$\tilde{\Sigma}(s):=\begin{cases} 2^{-1}\Sigma(s^2) & s\geqq 0,\\ -2^{-1}\Sigma(s^2) & s\leqq 0.\end{cases}$$

According to the example at the end of § 2.2 we have $Q\in \mathbf{N}_\varkappa^{n\times n}$, where $\sum_{j=1}^{l}\dim B_j=\varkappa$ Evidently, Q is antisymmetric with respect to the imaginary axis: $Q(-z^*)= -Q(z)^*$.

Proposition 5.3. *The function $Q(z)+iI$ has zeros of total multiplicity $\varkappa$ in $\mathbf{C}_+$. These zeros z_j, $j=1,2,\ldots,m\ (\leqq\varkappa)$, are on the imaginary axis and $0<|z_j|<t_1^2$.*

Proof. The first statement follows immediately from Corollary 3 of Theorem 2.2. To prove the second statement, we first consider the case $n=1$. To find the solutions of the equation $Q_0(z^2)=-iz^{-1}$, $\operatorname{Im} z>0$, we put $z=is$. Then it takes the form $Q_0(-s^2)=-s^{-1}$, and a simple consideration of the graphs of $Q_0(-s^2)$ and $-s^{-1}$ shows that this equation has $\varkappa$ zeros in $(0,\infty)$ and that these zeros are smaller than t_1. By the first statement of the proposition, these $\varkappa$ zeros give the only zeros of $Q(z)+iI$ in $\mathbf{C}_+$.

Let now n be arbitrary and consider a zero $z_0\in\mathbf{C}_+$ of $Q(z)+iI$. If z_0 is outside the interval $(0, it_1^2)$ of the imaginary axis, then Q is holomorphic at z_0. Hence there exists a vector $\xi\neq 0$ such that

$$z_0(Q_0(z_0^2)\xi,\xi)+i(\xi,\xi)=0.$$

But we have shown (case $n=1$) that this is impossible.

As an application of Proposition 5.3 we consider the Schrödinger equation

$$\frac{d^2\psi(r)}{dr^2}-V(r)\psi(r)+k^2\psi(r)=0,\quad \psi(0)=0, \tag{5.8}$$

with a short range potential V: $V(r)=0$ if $r>a$ for some $a<\infty$, $V\in L_1$. To find the nonreal resonances k of the problem (5.8) we observe that for $r>a$ the solution ψ of (5.8) is $\psi(r;k^2)=Ce^{ikr}$ and, considering $r=a$, we get $\psi'(a;k^2)\psi(a;k^2)^{-1}=ik$. This equation can be written as $k\psi(a;k^2)\psi'(a;k^2)^{-1}+i=0$. But $\psi(a;z)\psi'(a;z)^{-1}$ is a function of class $\mathbf{N}_0$. Indeed, $\psi(r;z)$ satisfies the equation

$$\psi''(r;z)-V(r)\psi(r;z)+z\psi(r;z)=0, \quad \psi(0;z)=0,$$

and it follows that

$$\psi(r;z)\psi'(r;z)^{-1}-\psi'(r;z^*)^{-1}\psi(r;z^*)=$$

$$=(z-z^*)\psi'(r;z^*)^{-1}\int_0^r|\psi(s;z^*)|^2ds\cdot\psi'(r;z)^{-1}.$$

Evidently, the number of negative poles of $\psi(a;z)\,\psi'(a;z)^{-1}$ is equal to the number of negative zeros of $\psi'(a;z)$, that is the number of negative eigenvalues λ of the boundary problem

$$\psi''(r)-V(r)\psi(r)+\lambda\psi(r)=0, \quad \psi(0)=0, \quad \psi'(a)=0. \tag{5.9}$$

Proposition 5.3 now implies the following statement:

The number of nonreal resonances of (5.8) *in* $\mathfrak{C}_+$ *is equal to the number of negative eigenvalues* λ *of the boundary problem* (5.9).

Without going into details we mention that Proposition 5.3 can be used to prove a similar statement in the case of a vector equation (5.8).

Note. We use this opportunity to mention that in our paper [7] the statement of Satz 3.4 is incorrect. To make it correct, in formula (3.10) one has to replace ϱ_0 (R_0) by $\hat{\varrho}_0$ ($\hat{R}_0$, resp.) and to define

$$\varrho_0:=\hat{\varrho}_0-1,\ R_0(z):=\hat{R}_0(z)-z(1-z^2)^{\varrho_0}\int_{\mathfrak{U}_0}\prod_{j=1}^{r}\frac{(1+t^2)^{\varrho_j}}{(t-\alpha_j)^{2\varrho_j}}\,d\sigma(t).$$

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