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Compact and Hilbert—Schmidt composition operators on weighted sequence spaces

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1. Preliminaries. If $(X, \mathcal{S}, \lambda)$ is a σ-finite measure space, then every non-singular measurable transformation T from X into itself induces the composition transformation C_T from $L^p(\lambda)$ into the linear space of all complex valued functions on X defined by $C_T f = f \circ T$ for every $f \in L^p(\lambda)$. If C_T turns out to be a continuous linear transformation from $L^p(\lambda)$ into itself, then we designate it as a composition operator on $L^p(\lambda)$.

If $w = \{w_n\}$ is a sequence of strictly positive real numbers, then we define the measure λ on the measurable space $(N, \mathcal{P}(N))$ as

$$\lambda(E) = \sum_{n \in E} w_n \quad \text{for every} \quad E \in \mathcal{P}(N),$$

the power set of the set N of positive integers. Thus $(N, \mathcal{P}(N), \lambda)$ becomes a σ-finite measure space. The L^p-space of this measure space is known as a weighted sequence space and w is called the sequence of weights. We denote this weighted sequence space by l_w^p. It is a well established fact that l_w^p (more generally $L^p(\lambda)$) is a Banach space. If $p=2$, then l_w^p is a Hilbert space under pointwise addition and scalar multiplication with the inner product defined as

$$\langle f, g \rangle = \int_N f \bar{g}\, d\lambda = \sum_{n=1}^{\infty} w_n f(n) \bar{g}(n)$$

for every f and g in l_w^2. It is also interesting to note that the space l_w^2 is a functional Hilbert space. By $B(l_w^2)$ we denote the Banach algebra of all bounded operators on l_w^2.

The main purpose of this note is to characterise compact, finite rank and Hilbert—Schmidt composition operators on l_w^2.

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2. Compact composition operators. If $(X, \mathscr{S}, \lambda)$ is a non-atomic measure space, then it has been shown in [6] that no composition operator on $L^2(\lambda)$ is compact. If the sequence w of weights is a constant sequence, then it can be easily established that l_w^2 does not admit any compact composition operator. Thus in particular no composition operator on l^2 is compact, though it is an L^2-space of an atomic measure space. But if the sequence w is a non-constant suitably chosen sequence, then there are compact composition operators on l_w^2. This fact makes this study a little interesting. Before the characterisation of compact composition operators on l_w^2 we shall need the following easy lemma.

Lemma 2.1. *Every weakly convergent sequence in l_w^2 is pointwise convergent.*

Proof. Let $\{f_n\}$ be a sequence in l_w^2 converging to zero weakly. Then, since $\{w_j f_n(j)\} = \{\langle f_n, e_j\rangle\}$ converges to zero, where $e_j(i) = \delta_{ij}$ (the Kronecker delta), it follows that $\{f_n\}$ is pointwise convergent.

Remark. The above lemma is true in any functional Hilbert space.

Let $T\colon N \to N$ be a mapping and let $\varepsilon > 0$. Then the set M_ε is defined as

$$M_\varepsilon = \{n\colon n \in N \text{ and } \lambda T^{-1}(\{n\}) > \varepsilon\lambda(\{n\})\}.$$

The following theorem characterises compact composition operators on l_w^2 in terms of the cardinality of M_ε.

Theorem 2.2. *Let $C_T \in B(l_w^2)$. Then C_T is compact if and only if M_ε, for every $\varepsilon > 0$, contains finitely many elements.*

Proof. Let $\varepsilon > 0$ be given and let $\{f_n\}$ be a sequence in l_w^2 converging weakly to zero. Suppose M_ε contains k elements. Then, since $\lambda T^{-1}(\{n\}) \leqq \varepsilon\lambda(\{n\})$ for every $n \in N \setminus M_\varepsilon$ and $\lambda T^{-1}(\{n\}) \leqq M\lambda(\{n\})$ for every $n \in N$ and for some finite $M > 0$ [7],

$$\|C_T f_n\|^2 = \int_N |f_n|^2 \, d\lambda T^{-1} = \int_{M_\varepsilon} |f_n|^2 \, d\lambda T^{-1} + \int_{N/M_\varepsilon} |f_n|^2 d\lambda T^{-1} =$$

$$\leqq M \cdot k |f_n(m_r)|^2 \lambda(\{m_s\}) + \varepsilon\|f_n\|^2,$$

where $|f_n(m_r)| = \max\{|f_n(m_t)|\colon m_t \in M_\varepsilon\}$ and $\lambda(\{m_s\}) = \max\{\lambda(\{m_t\})\colon m_t \in M_\varepsilon\}$. Since by the above lemma $\{f_n\}$ converges to zero pointwise, we can find $m \in N$ such that for every $n > m$,

$$\|C_T f_n\|^2 \leqq \varepsilon_1 M k \lambda(\{m_s\}) + \varepsilon\|f_n\|^2.$$

Since every weakly convergent sequence is norm bounded [1, p. 145] and ε_1 and ε are arbitrary, we conclude that the sequence $\{\|C_T f_n\|\}$ converges to zero. Hence C_T is compact.

Conversely, suppose M_ε contains infinitely many elements for some $\varepsilon>0$. Let M_ε^e be the closure of span $\{e_n: n\in M_\varepsilon\}$. Then for $f\in M_\varepsilon^e$,

$$\|C_T f\|^2 = \int_N |f|^2 \, d\lambda T^{-1} > \int_{M_\varepsilon} |f|^2 \, d\lambda = \varepsilon \|f\|^2.$$

Thus C_T is bounded away from zero on M_ε^e. This shows that the range of $C_T|M_\varepsilon^e$, the restriction of C_T to the subspace M_ε^e, is a closed infinite dimensional subspace contained in the range of C_T. Hence by problem 141 of [3] C_T is not compact.

Corollary 1. *Let $C_T\in B(l_w^2)$. Then C_T is compact if and only if $\lambda(T^{-1}(\{n\}))/\lambda(\{n\})$ tends to zero as n tends to ∞.*

Corollary 2. *No composition operator on l^2 is compact.*

Proof. If $C_T\in B(l^2)$, then the range of T contains infinitely many elements [8]. Hence $M_\varepsilon=T(N)$ whenever $\varepsilon<1$. Thus C_T is not compact.

Let a be a strictly positive real number and let $w=\{w_n\}$ be the sequence defined as $w_n=a^n$ for $n\in N$. Then the corresponding l_w^2 is denoted by l_a^2. In the light of the following two theorems it is comparatively easier to locate compact composition operators on l_a^2.

Theorem 2.3. *Let $C_T\in B(l_a^2)$, where $0<a<1$. Then C_T is compact if and only if the sequence $\{n-T(n)\}$ tends to ∞ as n tends to ∞.*

Proof. Suppose the sequence $\{n-T(n)\}$ tends to ∞ as n tends to ∞. Let m be in the range of T and let $T^{-1}(\{m\})=\{m_1, m_2, m_3, \dots\}$ be the arrangement of $T^{-1}(\{m\})$ in the ascending order. Then

$$\frac{\lambda(T^{-1}(\{m\}))}{\lambda(\{m\})} = \sum_i a^{m_i-m} < \sum_{i=0}^{\infty} a^{m_1-m+i} = \frac{a^{m_1-m}}{1-a} = \frac{a^{m_1-T(m_1)}}{1-a}.$$

Since $a<1$, we can conclude from the hypothesis that $\lim\limits_{m\to\infty} \dfrac{\lambda(T^{-1}(\{m\}))}{\lambda(\{m\})}=0$. Hence by the Corollary 1 of Theorem 2.2 C_T is compact.

Conversely, suppose the sequence $\{n-T(n)\}$ does not tend to ∞ as n tends to ∞. By Theorem 1 of [7] the sequence is bounded from below. Hence the sequence $\{n-T(n)\}$ has bounded subsequences. Let $\{n_k-T(n_k)\}$ be a bounded subsequence with a bound M. Then

$$\frac{\lambda(T^{-1}(\{T(n_k)\}))}{\lambda(\{T(n_k)\})} > a^{n_k-T(n_k)} > a^M > 0.$$

Hence again by the Corollary 1 of Theorem 2.2 C_T is not compact. This completes the proof of the theorem.

Example. Let $T: N \to N$ be defined as $T(m)=n/3$ if $n-2 \leqq m \leqq n$, where n is a multiple of 3. Then C_T is a composition operator on l_a^2, $0<a<1$. Since $\lambda T^{-1}(\{n\})/\lambda(\{n\})\left(=a^{2n}(1+a^{-1}+a^{-2})\right)$ tends to zero as n tends to ∞, we can conclude that C_T is compact.

Theorem 2.4. *Let $C_T \in B(l_a^2)$, where $a>1$. Then C_T is compact if and only if $\{T(n)-n\}$ tends to ∞ as n tends to ∞.*

Proof. The proof is dual to the proof of Theorem 2.3.

Example. Let $T: N \to N$ be the mapping defined by $T(m)=n^2$ if $n-2 \leqq \leqq m \leqq n$, where n is a multiple of 3. Then, since $\lambda T^{-1}(\{n\})/\lambda(\{n\})=0$ for $n \in N \setminus T(N)$ and $(a^{-2}+a^{-1}+1)/a^{\sqrt{n}(\sqrt{n}-1)}$ for $n \in T(N)$, C_T is a compact composition operator.

We now give several sufficient conditions for non-compactness of composition operators on l_a^2.

Theorem 2.5. *Let $T: N \to N$ be an injection and $C_T \in B(l_a^2)$, where $0<a<1$. Then C_T is not compact.*

Proof. Suppose C_T is compact. We infer from Theorem 2.3 that $\{n-T(n)\} \to \infty$. Therefore there exists a number $n_0 \in N$ such that for every $n>n_0$, we have $T(n)<n$. Let $n_1'=\max\{T(i)|1 \leqq i \leqq n_0\}$, $n_1=\max\{n_1', n_0\}$, $N_1=\{1, 2, \ldots, n_1\}$ and $N_2=N_1 \cup \cup \{n_1+1\}$. Since $T(n)<n$ for $n>n_0$ and T is injective, $T(N_1)=N_1=T(N_2)$ which contradicts the injectivity of T.

The following is an example of a function T which is not an injection, but it induces a compact composition operator.

Example. Let $E_n=\{2^{n-1}(2k-1)|k \in N\}$. Then $\bigcup_n E_n=N$. Let $T: N \to N$ be defined as $T(m)=n$ for every $m \in E_n$. Then C_T is a composition operator on l_a^2, $0<a<1$. Since

$$\lambda T^{-1}(\{n\})/\lambda(\{n\}) = a^{2^{n-1}}/a^n(1-a^{2^n}),$$

C_T is compact.

Theorem 2.6. *Let $T: N \to N$ be a surjection and $C_T \in B(l_a^2)$, where $a>1$. Then C_T is not a compact composition operator.*

Proof. Suppose C_T is compact. We infer from Theorem 2.4 that $\{T(n)-n\}$ tends to ∞. Therefore there exists a number $n_0 \in N$ such that for every $n>n_0$ we have $T(n)>n$. Let $n_1'=\max\{T(i)|1 \leqq i \leqq n_0\}$, $n_1=\max\{n_1'+1, n_0+1\}$ and $N_1= =\{1, 2, \ldots, n_1\}$. Then $T(N \setminus N_1) \subset N \setminus N_1$ and so $T(N) \cap N_1 = T(N_1) \cap N_1$. Since $T(n_1)>n_1$, we get $\operatorname{Card}\left(T(N) \cap N_1\right)=\operatorname{Card}\left(T(N_1) \cap N_1\right)<\operatorname{Card} N_1$. This means T is not surjective. This proves the Theorem.

The following is an example of a compact composition operator induced by a non-surjective mapping.

Example. Let $T\colon N\to N$ be defined as $T(n)=2n$. Then C_T is a composition operator on l_a^2, $a>1$. Since

$$\lambda T^{-1}(\{n\})/\lambda(\{n\}) = \begin{cases} 0, & \text{if } n \text{ is odd} \\ 1/a^{n/2}, & \text{if } n \text{ is even,} \end{cases}$$

C_T is compact.

The following Theorem characterises finite rank composition operators on l_w^2, where $\Sigma w_i<\infty$.

Theorem 2.7. *Let $C_T\in B(l_w^2)$, where $\Sigma w_i<\infty$. Then C_T is a finite rank operator if and only if the range of T contains finitely many elements.*

Proof. Since the range of C_T is dense in $l_w^2(N, T^{-1}(\mathcal{P}(N)), \lambda)$, $\lambda(E)=\sum_{n\in E} w_n$ [10, Lemma 2.4], the proof follows trivially.

3. Hilbert—Schmidt composition operators

Definition. A bounded linear operator A on an infinite dimensional separable Hilbert space H is said to be a Hilbert—Schmidt operator if there exists an orthonormal basis $\{e_n\colon n\in N\}$ in H such that $\Sigma\|Ae_n\|^2<\infty$. It is well known that the definition is independent of the choice of the orthonormal basis.

Let $T\colon N\to N$ be a mapping and let $y=\{y(m)\}$ be the sequence defined by $y(m)=\|K_{T(m)}\|$ for every $m\in N$, where K_m is the kernel function for l_w^2 defined by $K_m=e_m/w_m$. Then we prove the following Theorem.

Theorem 3.1. *Let $C_T\in B(l_w^2)$. Then C_T is a Hilbert—Schmidt operator if and only if $y\in l_w^2$.*

Proof. Since the family $\{f_n\}$ defined by $f_n=e_n/\sqrt{w_n}$ forms an orthonormal basis for l_w^2, C_T is a Hilbert—Schmidt operator if and only if

$$\sum_n \|C_T f_n\|^2 = \sum_n\sum_m w_m \left|\frac{e_n(T(m))}{\sqrt{w_n}}\right|^2 = \sum_n \sum_{m\in T^{-1}(\{n\})} w_m\cdot\frac{1}{w_n} = \sum_n \sum_{m\in T^{-1}(\{n\})} w_n\cdot w_m \frac{1}{w_n^2} =$$

$$= \sum_n\sum_m w_n\cdot w_m \left|\frac{e_{T(m)}(n)}{w_{T(m)}}\right|^2 = \sum_m w_m \sum_n w_n \left|\frac{e_{T(m)}(n)}{w_{T(m)}}\right|^2 = \sum_m w_m \|K_{T(m)}\|^2 = \|y\|^2<\infty.$$

This finishes the proof of the Theorem.

Example. Let the sequence $\{w_n\}$ of weights be the sequence $\{n\}$ and let $T\colon N\to N$ be defined as $T(n)=n^3$. Then C_T is a composition operator on l_w^2. Since

$$\|y\|^2 = \sum_n w_n/w_{T(n)} = \sum_n n/n^3 = \sum_n 1/n^2<\infty,$$

C_T is Hilbert—Schmidt.

The following exampale shows that the set of Hilbert—Schmidt composition operators is properly contained in the set of all compact composition operators on l_w^2.

Example. Let $\{w_n\}=\{n\}$ and let C_T be the composition operator on l_w^2 induced by the mapping $T(n)=n^2$. Then C_T is compact but is not Hilbert—Schmidt.

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