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A Rees matrix semigroup over a semigroup and its maximal right quotient semigroup

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As with the Rees matrix semigroup over a group with zero, one can construct a semigroup $M^0(I, W, \Lambda; P)$ where W is a semigroup with zero [3]. Under certain conditions, we show that the maximal right quotient semigroup $Q(S)$ of a Rees matrix semigroup $S=M^0(W, n; P)$ is isomorphic to the endomorphism monoid of $S=M^0(Q(W), n; P)$ as a right S-system.

1. Preliminaries. Although much of the basic notations and definitions are given in this section, we assume the reader is familiar with the basic terminology and results on algebraic semigroups as presented in CLIFFORD and PRESTON [2]. Those wishing a more indepth view of S-systems and semigroups of quotients should read the survey article by WEINERT [6].

A *right S-system with zero M_S* is a semigroup S with zero, a set M, and a function $M\times S\to M$ with $(m, s)\to ms$ for which the following properties hold:

(i) $(ms)t=m(st)$ for $m\in M$ and $s, t\in S$;

(ii) M contains an element ϑ (necessarily unique) such that $\vartheta s=\vartheta$ for all $s\in S$;

(iii) for all $m\in M$, $m0=\vartheta$ where 0 is the zero of S.

An *S-subsystem* N of M_S is a subset N of M such that $NS\subseteqq N$; this will be denoted by $N_S\subseteqq M_S$. Let M_S and N_S be S-systems with $f\colon M_S\to N_S$ a mapping such that $f(ms)=f(m)s$ for all $m\in M$ and $s\in S$, then f is called an *S-homomorphism.* The set of all S-homomorphisms from M_S to N_S is denoted by $\mathrm{Hom}_S(M, N)$. Let $N_S\subseteqq M_S$, then N_S is *intersection large* (*$\cap$-large*) in M_S if for $\{\vartheta\}\neq X_S\subseteqq M_S$, $X\cap N\neq\{\vartheta\}$. We denote this by $N_S\subseteqq' M_S$. Note that this is equivalent to saying that for all $\vartheta\neq m\in M$ there exists $s\in S^1$ such that $\vartheta\neq ms\in N$. The *singular congruence* ψ_M on M_S is a right congruence such that $a\psi_M b$ if and only if $ax=bx$ for all x in an $\cap$-large right ideal of S. If every nonzero S-subsystem of M is $\cap$-large, then M_S

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is said to be *intersection uniform* ($\cap$-*uniform*). An S-subsystem N of M_S is *dense* in M_S if for each $m_1, m_2, m_3 \in M$ where $m_1 \neq m_2$ there exists an $s \in S^1$ such that $m_1 s \neq m_2 s$ and $m_3 s \in N$. It is easy to see that when N_S is dense in M_S then $N_S \subseteq' M_S$.

The construction of the maximal right quotient semigroup is due to McMorris [4] and will not be repeated here. You will recall that every $\cap$-large right ideal of S is dense if and only if $\psi_S = \iota_S$ the identity congruence. In view of this result, we define S as being *right nonsingular* if every $\cap$-large right ideal of S is dense.

2. The Maximal Right Quotient Semigroup. Let $S = M^0(I, G, \Lambda; P)$ be a Rees matrix semigroup over a group G. In studying these structures, Botero de Meza [1] showed that if each row of P has a nonzero entry and S is nonsingular then the maximal right quotient semigroup of S, denoted $Q(S)$, is isomorphic to $\mathrm{Hom}_S(S, S)$. In general, this is not the case for $M^0(I, W, \Lambda; P)$ where W is a semigroup. What restrictions must we place on $M^0(I, W, \Lambda; P)$ to obtain similar results?

First let us consider the fact that the only $\cap$-large right ideal of S is S itself. This is not true for $M^0(I, W, \Lambda; P)$ as the next theorem and example will illustrate.

Theorem 1. *Let $S = M^0(I, W, \Lambda; P)$ where W is a semigroup with 0 and 1, T is a unitary subgroup of W, and P has an entry from T in each row. If L is an $\cap$-large right ideal of W then $\mathscr{L} = \{(i, l, \lambda) | l \in L, i \in I, \lambda \in \Lambda\}$ is an $\cap$-large right ideal of S.*

Proof. Let $0 \neq (i, a, \lambda) \in S$. Since P has an entry from T in each row then there exists $p_{\lambda j} \in T$ for some $j \in I$. For $0 \neq a \in W$, there exists $x \in W$ such that $0 \neq ax \in L$ since $L_W \subseteq' W_W$. Hence we let $b = p_{\lambda j}^{-1} x$ and choose $\mu \in \Lambda$ then $(i, a, \lambda) * (j, b, \mu) = (i, ap_{\lambda j} b, \mu) = (i, ap_{\lambda j} p_{\lambda j}^{-1} x, \mu) = (i, ax, \mu) \in \mathscr{L}$. It is easy to see that $\mathscr{L}$ is a right ideal of S and hence an $\cap$-large right ideal of S.

Example 2. Let $W = \{0, e, 1\}$, the semilattice $0 < e < 1$ and consider $M^0(I, W, I; P)$ where $P = (p_{ij})$ and $p_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$. The right ideal $\{0, e\}$ is $\cap$-large in W and so by Theorem 1, $\mathscr{L} = \{(i, a, j) | a \in \{0, e\}\ i, j \in I\}$ is an $\cap$-large right ideal of $M^0(I, W, I; P)$; however, it is clear that $\mathscr{L} \neq M^0(I, W, I; P)$.

If $\mathscr{L}$ is an $\cap$-large right ideal of $M^0(I, W, \Lambda; P)$ what is its relation to the $\cap$-large right ideals of W, if any? If we let $I = \Lambda$, we are able to obtain some results to this question.

Theorem 3. *Let $S = M^0(I, W, I; P)$ where W is a semigroup with 0 and 1, T is a unitary subgroup of W, and P has an entry from T in each row. If $\mathscr{L}$ is an $\cap$-large right ideal of S, then for each $k \in I$, $M_k = \{s \in W | (k, s, k) \in \mathscr{L}\}$ is an $\cap$-large right ideal of W.*

Proof. Let $m \in M_k$ where $k \in I$ arbitrary but fixed, and let $s \in W$. Since each row of P has an entry from T there exists $p_{kj} \in T$ for some $j \in I$. Since $\mathscr{L}$ is a right ideal of S, then $(k, m, k)*(j, p_{kj}^{-1}s, k)\in\mathscr{L}$. But then $(k, ms, k)\in\mathscr{L}$ and so $ms\in M_k$. Thus M_k is a right ideal of W. To see that M_k is $\cap$-large, we let $0\neq z\in W$ and consider $(k, z, k)\in S$. Since $\mathscr{L}_S\subseteq' S_S$ then either $(k, z, k)\in\mathscr{L}$ or there exists $(n, s, m)\in S$ such that $0\neq(k, z, k)*(n, s, m)\in\mathscr{L}$. In the former case, $z\in M_k$ and there is nothing to prove. In the latter case, we have $0\neq(k, zp_{kn}s, m)\in\mathscr{L}$. Hence there exist $p_{mj}\in T$ and $0\neq(k, zp_{kn}s, m)*(j, 1, k)\in\mathscr{L}$ and so $0\neq zp_{kn}sp_{mj}\in M_k$. Thus M_k is $\cap$-large in W.

If we restrict our Rees matrix semigroup over a semigroup further to $|I|=n<\infty$, then $\bigcap_{k\in I} M_k$ is also an $\cap$-large right ideal of W and we develop the following results for $M^0(W, n; P)$.

Theorem 4. *Let $S=M^0(W, n; P)$ where W is a semigroup with 0 and 1, T is a unitary subgroup of W, and P has an entry from T in each row. If $\mathscr{L}$ is an $\cap$-large right ideal of S then $\mathscr{R}=\{(i, t, j)|t\in\bigcap_{k\in I} M_k,\ i, j\in I\}$ is an $\cap$-large right ideal of S contained in $\mathscr{L}$.*

Proof. Since $\bigcap_{k\in I} M_k$ is $\cap$-large in W then by Theorem 1, $\mathscr{R}$ is an $\cap$-large right ideal of S. Let $(i, t, j)\in\mathscr{R}$. Since $t\in\bigcap_{k\in I} M_k$ then $t\in M_i$ a right ideal of S and so $tp_{ih}^{-1}\in M_i$ for some $h\in I$ with $p_{ih}\in T$. Consequently, $(i, tp_{ih}^{-1}, i)\in\mathscr{L}$ and since $\mathscr{L}$ is a right ideal of S then $(i, tp_{ih}^{-1}, i)*(h, 1, j)\in\mathscr{L}$. But this says that $(i, t, j)\in\mathscr{L}$ since $(i, tp_{ih}^{-1}, i)*(h, 1, j)=(i, tp_{ih}^{-1}p_{ih}1, j)=(i, t, j)$.

Theorem 5. *Let $S=M^0(W, n; P)$ where W is a semigroup with 0 and 1, T is a unitary subgroup of W, and P has an entry from T in each row. If S is right nonsingular then W is right nonsingular.*

Proof. Let $k, j\in I$ such that $p_{kj}\in T$ and suppose $a\,\psi_W b$. Then since ψ_W is a right congruence $(ap_{kh})\psi_W(bp_{kh})$ for all $h\in I$. Hence there exists an $\cap$-large right ideal L of W such that for $x\in L$, $h\in I$ we have $ap_{kh}=bp_{kh}x$. By Theorem 1, L induces an $\cap$-large right ideal $\mathscr{L}=\{(i, x, j)|x\in L$ and $i, j\in I\}$ on S. Hence for (j, a, k), $(j, b, k)\in S$ and $(i, x, m)\in\mathscr{L}$ we have $(j, a, k)*(i, x, m)=(j, ap_{ki}x, m)$ and $(j, b, k)*(i, x, m)=(j, bp_{ki}x, m)$. Thus $(j, a, k)\psi_S(j, b, k)$ and since S is right nonsingular then $(j, a, k)=(j, b, k)$ and so $a=b$.

The converse to this result is in general false since if G is a group with zero adjoined and $S=M^0(I, G, \Lambda; P)$ is regular then S is right reductive if and only if no two rows of P are left proportional; that is, for any two rows μ and λ of P there does not exist $c\in G$ such that $p_{\mu i}=cp_{\lambda i}$ for all $i\in I$ [5, p. 156]. Hence if S is not right reductive then S is not right nonsingular.

McMorris [4] showed that a semigroup W with 0 and 1 can be embedded in $Q(W)$ by $\xi\colon W\to Q(W)$ defined by $x\to[\lambda_x]$ where $\lambda_x\in\mathrm{Hom}_W(W, W)$ defined by $t \mapsto xt$. The zero and identity of W are the zero and identity of $Q(W)$. If P is a sandwich matrix defined on W, we can define a new sandwich matrix on $Q(W)$ by allowing the entries of P to be operated on by ξ; that is, $p_{ij} \in P$ would become $[p_{pij}]_{ij}$. For convenience, we simply write p_{ij} and let $P = \xi(P)$. It is not difficult to see that $M^0(W, n; P)$ can be embedded into $M^0(Q(W), n; P)$.

We are interested in obtaining a characterization of $Q(S)$, the maximal right quotient semigroup of $S = M^0(W, n; P)$, where W is a semigroup with 0 and 1, T is a unitary subgroup of W and P has an entry from T in each row.

Theorem 6. *Let $S = M^0(W, n; P)$ where W is an $\cap$-uniform semigroup with 0 and 1, T is a unitary subgroup of W and P has an entry from T in each row. If $f\colon \mathscr{L} \to S$ is an S-homomorphism where $\mathscr{L}$ is an $\cap$-large right ideal of S then there exists an indexing function $i\colon I \to I$, and for each $h \in I$ a W-homomorphism $\hat{f}_h\colon \bigcap_{k \in I} M_k \to W$ such that $f|_{\mathscr{R}}((m, x, t)) = (i(m), \hat{f}_m(x), t)$ where $\mathscr{R}$ is defined in Theorem 4.*

Proof. Let $f\colon \mathscr{L} \to S$ be an S-homomorphism and $\mathscr{L}$ an $\cap$-large right ideal of S. By Theorem 3, for each $k \in I$, $M_k = \{s \in W \mid (k, s, k) \in \mathscr{L}\}$ is an $\cap$-large right ideal of W and so is $\bigcap_{k \in I} M_k$. By Theorem 4, we can construct $\mathscr{R} = \{(d, z, g) \mid z \in \bigcap_{k \in I} M_k$ and $d, g \in I\}$ an $\cap$-large right ideal of S contained in $\mathscr{L}$. Let ∂ be a fixed element of I. For $m \in I$ and $x \in \bigcap_{k \in I} M_k$ we have $f((m, x, \partial)) = f((m, x, \partial) * (j, p_{\partial j}^{-1}, \partial)) = f((m, x, \partial)) * (j, p_{\partial j}^{-1}, \partial)$ for some $j \in I$ with $p_{\partial j} \in T$, since f is an S-homomorphism. Thus $f((m, x, \partial)) = (i, y, \partial)$ for some $i \in I$ and $y \in W$. Now let $s \in I$ and $a, b \in \bigcap_{k \in I} M_k$ and suppose $f((s, a, \partial)) = (i, y, \partial)$ and $f((s, b, \partial)) = (h, z, \partial)$. Since W is $\cap$-uniform then $aW \cap bW \neq 0$ and so there exists $0 \neq x \in aW \cap bW$ such that $x = aw$ and $x = bu$ for some $w, u \in W$. Since (s, aw, ∂), $(s, bu, \partial) \in \mathscr{R}$ then $f((s, aw, \partial)) = (i, y, \partial) * (j, p_{\partial j}^{-1} w, \partial)$ and similarly $f((s, bu, \partial)) = (h, z, \partial) * (j, p_{\partial j}^{-1} u, \partial)$ for some $j \in I$. But $f((s, aw, \partial)) = f((s, bu, \partial))$ and so $(i, yw, \partial) = (h, zu, \partial)$. Hence $i = h$ and we can consider the first index as a function of s, denoted $i(s)$. Now for each $h \in I$, we define $\hat{f}_h\colon \bigcap_{k \in I} M_k \to W$ by $x \mapsto y$ where $f((h, x, \partial)) = (i(h), y, \partial)$. Each $\hat{f}_h$ is a W-homomorphism since for $s \in W$ and $x \in \bigcap_{k \in I} M_k$ we have

$$f((h, xs, \partial)) = f((h, xp_{\partial j} p_{\partial j}^{-1} s, \partial)) = f((h, x, \partial)) * (j, p_{\partial j}^{-1} s, \partial) =$$

$$= (i(h), y, \partial) * (j, p_{\partial j}^{-1} s, \partial) = (i(h), yp_{\partial j} p_{\partial j}^{-1} s, \partial) = (i(h), ys, \partial)$$

and so $\hat{f}_h(x)s = ys = \hat{f}_h(xs)$. The remainder of the theorem now follows from the fact that for $(m, x, t) \in \mathscr{R}$, $f((m, x, t)) = f((m, x, \partial) * (j, p_{\partial j}^{-1}, t)) = f((m, x, \partial)) * (j, p_{\partial j}^{-1}, t) = (i(m), \hat{f}_m(x), \partial) * (j, p_{\partial j}^{-1}, t) = (i(m), \hat{f}_m(x), t)$.

We now prove the main result of this paper.

Theorem 7. *Let* $S=M^0(W,n;P)$ *where* W *is an* $\cap$*-uniform semigroup with* 0 *and* 1, T *is a unitary subgroup of* W *and* P *has an entry from* T *in each row. If* S *is right nonsingular then* $Q(S)\approx \mathrm{Hom}_S(\hat{S},\hat{S})$ *where* $\hat{S}=M^0(Q(W),n;P)$.

Proof. Let $[f]\in Q(S)$ and define $\mu_{[f]}\colon \hat{S}\to\hat{S}$ by $(s,q,t)\to(i(s),q_sq,t)$ where $q_s=[\hat{f}_s]$ and $i(s)$ are defined in Theorem 6. We should note that by Theorem 5, S being right nonsingular implies that W is right nonsingular and so $[\hat{f}_s]\in Q(W)$. To see that $\mu_{[f]}$ is an $\hat{S}$-homomorphism we consider $\mu_{[f]}((s,q,t))*(r,g,u)=$ $=(i(s),q_sq,t)*(r,g,u)=(i(s),q_sqp_{tr}g,u)$ and

$$\mu_{[f]}((s,q,t)*(r,g,u))=\mu_{[f]}((s,qp_{tr}g,u))=(i(s),q_sqp_{tr}g,u).$$

Thus $\mu_{[f]}$ is an $\hat{S}$-homomorphism. We now define $\varphi\colon Q(S)\to\mathrm{Hom}_S(\hat{S},\hat{S})$ by $[f]\to\mu_{[f]}$. To see that φ is well defined suppose $[f]=[g]$. Then f and g agree on some dense right ideal $\mathscr{L}$ of S. We must show that $[\hat{f}_j]=[\hat{g}_j]$ for all $j\in I$. Since $\mathscr{L}$ is a dense right ideal of S then it is also $\cap$-large. By Theorem 3, for each $k\in I$, $M_k=\{s\in W|(k,s,k)\in\mathscr{L}\}$ is an $\cap$-large right ideal of W and so is $\bigcap_{k\in I} M_k$. By Theorem 4, $\mathscr{R}=\{(d,z,h)|z\in\bigcap_{k\in I} M_k \text{ and } d,h\in I\}$ is an $\cap$-large right ideal of S contained in $\mathscr{L}$ so $f|_{\mathscr{R}}=g|_{\mathscr{R}}$ agree. Thus for $j\in I$, $\hat{f}_j\colon \bigcap_{k\in I} M_k\to W$ and $\hat{g}_j\colon \bigcap_{k\in I} M_k\to W$ agree on their domains and by Theorem 5 $\bigcap_{k\in I} M_k$ is a dense right ideal of W so $[\hat{f}_j],[\hat{g}_j]\in Q(W)$ and $[\hat{f}_j]=[\hat{g}_j]$. We now show that φ is one-to-one by supposing that $\mu_{[f]}=\mu_{[g]}$. Since for each $k\in I$ $(k,1,\partial)\in\hat{S}$, $\mu_{[f]}((k,1,\partial))=(i(k),[\hat{f}_k],\partial)$ and $\mu_{[g]}((k,1,\partial))=(f(k),[\hat{g}_k],\partial)$ then $[\hat{f}_k]=[\hat{g}_k]$ and $i(k)=j(k)$ for all $k\in I$. Thus for each $k\in I$, $\hat{f}_k$ and $\hat{g}_k$ agree on some $\cap$-large right ideal of W call it L_k. Let $L=\bigcap_{k\in I} L_k$. Since L is an $\cap$-large right ideal of W then by Theorem 1 $\mathscr{L}=\{(d,z,h)|z\in L \text{ and } d,h\in I\}$ is an $\cap$-large right ideal of S. We claim that f and g agree on $\mathscr{L}$. Let $(d,z,h)\in\mathscr{L}$ then there exists $m\in I$ such that $p_{\partial m}\in T$ in P and so $f((d,z,h))=f((d,z,\partial))*(m,p_{\partial m}^{-1},h)$ and $g((d,z,h))=g((d,z,\partial))*(m,p_{\partial m}^{-1},h)$. But $f((d,z,\partial))=g((d,z,\partial))$ since $\hat{f}_d$ and $\hat{g}_d$ agree on $L_d\subseteq L$. Consequently, the claim is established and $[f]=[g]$ in $Q(S)$; furthermore φ is one-to-one. To show that φ is onto let $\sigma\in\mathrm{Hom}_S(\hat{S},\hat{S})$ and consider $\sigma^{-1}S=\{x\in\hat{S}|\sigma(x)\in S\}$. Since $S\subseteq'\hat{S}$ then $\sigma^{-1}S\subseteq'\hat{S}$ and $S\cap\sigma^{-1}S\subseteq'S$. Next define $\tau\colon S\cap\sigma^{-1}S\to S$ by $x\mapsto\sigma(x)$. Clearly, τ is an S-homomorphism so by Theorem 6, there exists $\mathscr{D}$ an $\cap$-large right ideal of S, an indexing function $i\colon I\to I$, and for each $h\in I$ a W-homomorphism $\hat{\tau}\colon\bigcap_{k\in I} D_k\to W$ such that $\tau|_{\mathscr{D}}((m,x,t))=(i(m),\hat{\tau}_m(x),t)$. Note that $\mathscr{D}$ is a dense right ideal of S since S is right nonsingular and that $\tau\mathscr{D}\subseteq S$. Let $\bar{\tau}=\tau|_{\mathscr{D}}$ and consider $[\bar{\tau}]\in Q(S)$. Since S_S is dense in $\hat{S}_S$ then $\mu_{[\bar{\tau}]}=\sigma$ and so φ is onto.

What remains to be shown is that φ is a semigroup homomorphism; that is, $\mu_{[fg]}=\mu_{[f]}\mu_{[g]}$ where the operation on the right is composition of functions. Let $(s, q, t)\in\hat{S}$ then $\mu_{[f]}\mu_{[g]}((s, q, t))=\mu_{[f]}(\mu_{[g]}((s, q, t)))=\mu_{[f]}((j(s), [\hat{g}_s]q, t))=(i(j(s)), [\hat{f}_{j(s)}][\hat{g}_s]q, t)=(i(j(s)), [\hat{f}_{j(s)}\hat{g}_s]q, t)=\mu_{[fg]}((s, q, t))$ since $fg((s, x, t))=f(g((s, x, t)))=f((j(s), \hat{g}_s(x), t))=(i(j(s)), \hat{f}_{j(s)}(\hat{g}_s(x)), t)=(i(j(s)), \hat{f}_{j(s)}\hat{g}_s(x), t)$.

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