Acta Sci. Math., 43 (1981), 45—54

On the convergence of solutions of functional differential equations

T. KRISZTIN

1. Introduction

The application of Ljapunov functions and functionals has proved to be useful in the study of the stability of solutions of functional differential equations. Such investigations were initiated by N. N. KRASOVSKIĬ [9] and B. S. RAZUMIKHIN [10]. The Ljapunov functions and functionals are usable for studying other properties, too. For instance, S. R. BERNFELD and J. R. HADDOCK ([1], [2], [4]) examined the existence of the limit of solutions as $t\to\infty$ by the aid of Ljapunov functions. But their method was not applicable when the right-hand side of the equation is the sum of an ordinary and a functional part of the same order. But such equations have occurred in the applications, for example in the investigation of biological populations [3]. In this case the problem was solved for certain autonomous and periodic equations only [5], [6]. In this paper we give a sufficient condition for the existence of the limit of solutions in case of non-periodic equations. Our main result guarantees the existence of the limit of a Ljapunov function along the solutions as $t\to\infty$. We present several applications in which we show that the solutions or their norm tend to a constant as $t\to\infty$. Among these, we study a stability example of N. N. KRASOVSKIĬ proving that his assumptions imply the existence of the limit of solutions in addition to the stability of the zero solution.

The main theorems are valid results for functional differential equations in any Banach space X. But they also yield new results for the special case $X=R$ (Section 4).

Received May 9, 1980.

2. Notations and definitions

Let R be the set of real numbers and R^+ the set of nonnegative real numbers. Let X be a Banach space with norm $|\cdot|$ and let $C=C([-r, 0], X)$ denote the space of continuous functions which map the interval $[-r, 0]$ into X, where $r>0$. For $\varphi\in C$ define $\|\varphi\|=\max_{-r\leqq s\leqq 0}|\varphi(s)|$. If $x\colon [t_0-r, t_0+A)\to X$ is a continuous function ($t_0\in R^+$, $0<A\leqq\infty$), then for $t\in[t_0, t_0+A)$ the function $x_t\in C$ is defined by $x_t(s)=x(t+s)$, $-r\leqq s\leqq 0$.

We consider the *nonlinear, non-autonomous* functional differential equation

$$\dot{x}(t) = F(t, x_t), \tag{2.1}$$

where $F\colon R^+\times C_F\to X$, $C_F\subset C$.

Let $t_0\in R^+$ and $\varphi_0\in C_F$ be given. A function $x(\cdot)=x(t_0, \varphi_0)(\cdot)$ is said to be a solution of (2.1) (with the initial function φ_0 at t_0) if there exists a number A ($0<A\leqq\infty$) such that $x(\cdot)$ is defined and continuous on $[t_0-r, t_0+A)$, absolutely continuous on the bounded intervals of $[t_0, t_0+A)$, $x_{t_0}=\varphi_0$, $x_t\in C_F$ for $t\in[t_0, t_0+A)$ and $\dot{x}(t)=F(t, x_t)$ almost everywhere on $[t_0, t_0+A)$. In this paper we suppose $A=\infty$, i.e. the solutions of (2.1) exist for $t\geqq t_0$ (see, for example, [7], [8]).

By a *Ljapunov function* we mean a continuous function $V\colon [-r, \infty)\times X\to R$. The upper *right-hand derivative* $D^+_{(2.1)}V$ of a Ljapunov function V with respect to system (2.1) is defined by

$$D^+_{(2.1)}V(t, \varphi) = \overline{\lim_{h\to 0+}}\, \frac{1}{h}\left[V\big(t+h, \varphi(0)+hF(t, \varphi)\big)-V\big(t, \varphi(0)\big)\right] \quad \big((t, \varphi)\in R^+\times C_F\big).$$

If V is a Ljapunov function and $(t, \varphi)\in R^+\times C_F$, then let

$$\overline{V}(t, \varphi) = \sup_{-r\leqq s\leqq 0} V\big(t+s, \varphi(s)\big), \quad \underline{V}(t, \varphi) = \inf_{-r\leqq s\leqq 0} V\big(t+s, \varphi(s)\big).$$

Finally, for a Ljapunov function V and given numbers $0<\eta\leqq\varepsilon$ define

$$\bar{S}(V, \eta, \varepsilon) = \big\{(t, \varphi)\in R^+\times C_F\colon V\big(t, \varphi(0)\big)\geqq \varepsilon,\ \overline{V}(t, \varphi)\leqq 2\varepsilon,\ \overline{V}(t, \varphi)-V\big(t, \varphi(0)\big) < \eta\big\},$$

$$\underline{S}(V, \eta, \varepsilon) = \big\{(t, \varphi)\in R^+\times C_F\colon V\big(t, \varphi(0)\big)\leqq -\varepsilon,$$
$$\underline{V}(t, \varphi)\geqq -2\varepsilon,\ V\big(t, \varphi(0)\big)-\underline{V}(t, \varphi) < \eta\big\}.$$

3. The main result

The main result guarantees the existence of the limit of a Ljapunov function along the solutions of (2.1) as $t\to\infty$.

Theorem 3.1. *Suppose that for a nonnegative Ljapunov function V there exists a functional $W\colon R^+\times C_F\to R$ with the following property: for every $\varepsilon>0$ there exist $\eta=\eta(\varepsilon)>0$ and $\xi=\xi(\varepsilon)>0$ such that*

(i) *if* $(t, \varphi)\in \bar{S}(V, \eta, \varepsilon)$, *then*

$$W(t, \varphi) \leqq \xi[\bar{V}(t, \varphi)-V(t, \varphi(0))], \tag{3.1}$$

(ii) *if* $x(\cdot)$ *is a solution of* (2.1) *and* $(t, x_t)\in \bar{S}(V, \eta, \varepsilon)$ *for every* $t\in[t_1, t_2]$ $(t_0\leqq t_1\leqq t_2)$, *then*

$$V(t_2, x(t_2))-V(t_1, x(t_1)) \leqq \int_{t_1}^{t_2} W(t, x_t)\, dt. \tag{3.2}$$

Then for each solution $x(\cdot)$ *of* (2.1) $\lim_{t\to\infty} V(t, x(t))$ *exists.*

We first prove the following lemma.

Lemma. *If the conditions of Theorem* 3.1 *are satisfied, then for each solution* $x(\cdot)$ *of* (2.1) *the function* $\bar{V}(\cdot, x_.)$ *is non-increasing.*

Proof. Assume that (2.1) has a solution $x(\cdot)$ such that $\bar{V}(\cdot, x_.)$ is not a non-increasing function. Then there exists a $t_1\geqq t_0$ and in any right-hand side neighbourhood of t_1 there exists a t such that $\bar{V}(t, x_t)>\bar{V}(t_1, x_{t_1})=V(t_1, x(t_1))>0$. Let ε be chosen such that $0<\varepsilon<V(t_1, x(t_1))<2\varepsilon$ and choose $\eta=\eta(\varepsilon)$, $\xi=\xi(\varepsilon)$ according to assumptions of the lemma. Obviously there exist t_2, t_3 such that $t_3>t_2\geqq t_1$, $t_3-t_2<\frac{1}{\xi}$, $V(t_2, x(t_2))=V(t_1, x(t_1))<V(t_3, x(t_3))\leqq 2\varepsilon$, $V(t_3, x(t_3))-V(t_2, x(t_2))<\eta$ and if $t\in[t_2, t_3]$, then $V(t_2, x(t_2))\leqq V(t, x(t))$ and $\bar{V}(t, x_t)\leqq V(t_3, x(t_3))$. For such t_2, t_3 we have $(t, x_t)\in \bar{S}(V, \eta, \varepsilon)$ provided $t\in[t_2, t_3]$. Also

$$\bar{V}(t,x_t)-V(t,x(t)) \leqq V(t_3,x(t_3))-V(t_2,x(t_2)) \quad (t\in[t_2, t_3]). \tag{3.3}$$

It follows from (3.1), (3.2) and (3.3) that

$$V(t_3, x(t_3))-V(t_2, x(t_2)) \leqq \int_{t_2}^{t_3} W(t, x_t)dt \leqq$$
$$\leqq \int_{t_2}^{t_3} \xi[\bar{V}(t, x_t)-V(t, x(t))]\, dt \leqq \int_{t_2}^{t_3} \xi[V(t_3, x(t_3))-V(t_2, x(t_2))]\, dt =$$
$$= (t_3-t_2)\xi[V(t_3, x(t_3))-V(t_2, x(t_2))].$$

Hence $t_3-t_2\geqq\frac{1}{\xi}$. This is a contradiction. The lemma is proved.

Proof of Theorem 3.1. Suppose that (2.1) has a solution $x(\cdot)$ such that the limit $\lim_{t\to\infty} V(t, x(t))$ does not exist. Then $\lim_{t\to\infty} \bar{V}(t, x_t)=\alpha>0$ (this limit exists by Lemma). Let ε be chosen so that $0<\varepsilon<\alpha<2\varepsilon$ and choose $\eta=\eta(\varepsilon)$ and $\xi=\xi(\varepsilon)$ according to the assumptions of the theorem. Then we can find a constant β

$\left(0<\beta<\min\left\{\frac{\eta}{3}, \frac{\alpha-\varepsilon}{3}\right\}\right)$ and numbers t_1, t_2 such that

$$t_2 > t_1 \geqq t_0, \quad t_2 - t_1 \leqq r, \tag{3.4}$$

$$V(t_2, x(t_2)) - V(t_1, x(t_1)) = \beta, \tag{3.5}$$

$$\bar{V}(t_1, x_{t_1}) - \alpha < \frac{\eta}{3}, \quad |\alpha - V(t_2, x(t_2))| < \frac{\eta}{3}, \tag{3.6}$$

$$0 \leqq V(t_2, x(t_2)) - V(t, x(t)) \leqq \beta \quad (t \in [t_1, t_2]),$$

$$V(t_1, x(t_1)) \geqq \varepsilon, \quad \bar{V}(t_1, x_{t_1}) \leqq 2\varepsilon, \tag{3.7}$$

$$\bar{V}(t_1, x_{t_1}) - V(t_2, x(t_2)) \leqq \gamma, \tag{3.8}$$

where $\gamma>0$ and

$$\frac{1}{\xi} \sum_{k=1}^{\left[\frac{\beta}{\gamma}\right]} \frac{1}{k+1} > r. \tag{3.9}$$

From (3.8) and the monotonicity of $\bar{V}(\cdot, x.)$ it follows that

$$\bar{V}(t, x_t) - V(t_2, x(t_2)) \leqq \gamma \quad (t \in [t_1, t_2]). \tag{3.10}$$

Since $\beta<\frac{\eta}{3}$, by (3.6) we have

$$\bar{V}(t, x_t) - V(t, x(t)) \leqq \bar{V}(t, x_t) - \alpha + |\alpha - V(t_2, x(t_2))| + V(t_2, x(t_2)) - V(t, x(t)) < \tag{3.11}$$

$$< \frac{\eta}{3} + \frac{\eta}{3} + \frac{\eta}{3} = \eta \quad (t \in [t_1, t_2]).$$

From (3.5), (3.6), (3.7) and (3.11) we obtain $(t, x_t) \in \bar{S}(V, \eta, \varepsilon)$ for $t \in [t_1, t_2]$. Thus, (3.1) and (3.2) hold for (t, x_t) as $t \in [t_1, t_2]$. Let $\tau_k \in [t_1, t_2]$ be the greatest number for which

$$V(t_2, x(t_2)) - V(\tau_k, x(\tau_k)) = k\gamma \quad \left(k = 1, 2, \ldots, \left[\frac{\beta}{\gamma}\right]\right). \tag{3.12}$$

From (3.10) and the choice of τ_k it follows that

$$\bar{V}(t, x_t) - V(t, x(t)) \leqq (k+1)\gamma \quad \left(t \in [\tau_k, t_2];\ k = 1, 2, \ldots, \left[\frac{\beta}{\gamma}\right]\right). \tag{3.13}$$

By (3.1), (3.2), (3.12) and (3.13) we have

$$\gamma = V(\tau_{k-1}, x(\tau_{k-1})) - V(\tau_k, x(\tau_k)) \leqq \int_{\tau_k}^{\tau_{k-1}} W(t, x_t)\, dt \leqq$$

$$\leqq \int_{\tau_k}^{\tau_{k-1}} \xi[\bar{V}(t, x_t) - V(t, x(t))]\, dt \leqq \int_{\tau_k}^{\tau_{k-1}} \xi(k+1)\gamma\, dt =$$

$$= (\tau_{k-1} - \tau_k)\xi(k+1)\gamma \quad \left(k = 1, 2, \ldots, \left[\frac{\beta}{\gamma}\right];\ \tau_0 = t_2\right).$$

Hence $\tau_{k-1}-\tau_k \geqq \frac{1}{\xi}\frac{1}{k+1}$. Thus, by (3.9)

$$t_2-t_1 \geqq t_2-\tau_{\left[\frac{\beta}{\gamma}\right]} = (t_2-\tau_1)+(\tau_1-\tau_2)+\ldots+\left(\tau_{\left[\frac{\beta}{\gamma}\right]-1}-\tau_{\left[\frac{\beta}{\gamma}\right]}\right) \geqq \frac{1}{\xi}\sum_{k=1}^{\left[\frac{\beta}{\gamma}\right]}\frac{1}{k+1} > r,$$

which contradicts (3.4). This completes the proof.

Remark 3.1. If the Ljapunov function V in Theorem 3.1 is locally Lipschitzian and $W=D^+_{(2.1)}V$, then for each solution $x(\cdot)$ of (2.1) the assumption (3.2) is satisfied and even

$$V(t_2, x(t_2))-V(t_1, x(t_1)) = \int_{t_1}^{t_2} D^+_{(2.1)}V(t, x_t)\,dt \quad (t_0 \leqq t_1 \leqq t_2).$$

This can be shown as follows. If $\dot{x}(t)=F(t, x_t)$, then $x(t+h)=x(t)+hF(t, x_t)+ +o(h)$ $(h\to 0+)$. From the Lipschitz condition for V we obtain

$$V(t+h, x(t+h))-V(t, x(t)) \leqq V(t+h, x(t)+hF(t, x_t))+L|o(h)|-V(t, x(t))$$
$$(h \to 0+),$$

where L is the Lipschitz constant for V on a neighbourhood of $(t, x(t))$. Hence $D^+V(t, x(t)) \leqq D^+_{(2.1)}V(t, x_t)$, where $D^+V(t, x(t))$ is the upper right-hand derivative of V along the solution $x(t)$ of (2.1), that is

$$D^+V(t, x(t)) = \overline{\lim_{h\to 0+}}\frac{1}{h}\left[V(t+h, x(t+h))-V(t, x(t))\right].$$

Likewise we can prove $D^+V(t, x(t)) \geqq D^+_{(2.1)}V(t, x_t)$ and we obtain

(3.14) $$D^+V(t, x(t)) = D^+_{(2.1)}V(t, x_t).$$

(3.14) was proved by T. YOSHIZAWA [11] for ordinary differential equations in the case $X=R^m$. Since V is locally Lipschitzian, $V(\cdot, x(\cdot))$ is absolutely continuous on every bounded interval of $[t_0, \infty)$ and thus

(3.15) $$V(t_2, x(t_2))-V(t_1, x(t_1)) = \int_{t_1}^{t_2} D^+V(t, x(t))\,dt \quad (t_0 \leqq t_1 \leqq t_2).$$

From (3.14) and (3.15) it follows that our statement holds.

Corollary 3.1. *If for every* $\varepsilon>0$ *there exists an* $\eta=\eta(\varepsilon)>0$ *such that* $(t, \varphi)\in \bar{S}(|\varphi(0)|, \eta, \varepsilon)$ *implies* $D^+_{(2.1)}|\varphi| \leqq 0$, *then for each solution* $x(\cdot)$ *of* (2.1) $\lim_{t\to\infty}|x(t)|$ *exists.*

Proof. We apply Theorem 3.1. Let $V(t, x)=|x|$ and $W(t, \varphi)=D^+_{(2.1)}|\varphi|$. Since the condition of Corollary 3.1 is stronger than condition (i) of Theorem 3.1 and the function V is locally Lipschitzian, from Remark 3.1 it is obvious that the limit exists.

Corollary 3.1 is due to J. R. HADDOCK [4].

In the next corollary of Theorem 3.1 we do not assume that the Ljapunov function is nonnegative.

Corollary 3.2. *Suppose that for a Ljapunov function V there exist functionals W_1, W_2: $R^+ \times C_F \to R$ with the following property: for every $\varepsilon > 0$ there exist $\eta = \eta(\varepsilon) > 0$ and $\xi = \xi(\varepsilon) > 0$ such that*

(i) $(t, \varphi) \in \bar{S}(V, \eta, \varepsilon)$ *implies* $W_1(t, \varphi) \leqq \xi[\bar{V}(t, \varphi) - V(t, \varphi(0))]$,

(ii) $(t, \varphi) \in \underline{S}(V, \eta, \varepsilon)$ *implies* $W_2(t, \varphi) \leqq \xi[V(t, \varphi(0)) - \underline{V}(t, \varphi)]$,

(iii) *if* $(t, x_t) \in \bar{S}(V, \eta, \varepsilon)$ $(t \in [t_1, t_2],\ t_0 \leqq t_1 \leqq t_2)$, *then*

$$V(t_2, x(t_2)) - V(t_1, x(t_1)) \leqq \int_{t_1}^{t_2} W_1(t, x_t)\, dt,$$

(iv) *if* $(t, x_t) \in \underline{S}(V, \eta, \varepsilon)$ $(t \in [t_1, t_2],\ t_0 \leqq t_1 \leqq t_2)$, *then*

$$V(t_1, x(t_1)) - V(t_2, x(t_2)) \leqq \int_{t_1}^{t_2} W_2(t, x_t)\, dt,$$

where $x(\cdot)$ is a solution of (2.1).

Then for each solution $x(\cdot)$ of (2.1) $\lim_{t\to\infty} V(t, x(t))$ *exists.*

Proof. Let $V_1(t, x) = \max\{V(t, x), 0\}$, $V_2(t, x) = -\min\{V(t, x), 0\}$. From conditions (i), (ii), (iii), (iv) of Corollary 3.2 it follows that V_1, W_1 and V_2, W_2 satisfy conditions (i), (ii) of Theorem 3.1. This implies that for every solution $x(\cdot)$ of (2.1) the limits $\lim_{t\to\infty} V_1(t, x(t))$ and $\lim_{t\to\infty} V_2(t, x(t))$ exist. Thus the corollary is proved.

4. Applications and examples

I. Consider the equation

$$\dot{x}(t) = f(t, x(t)) + g(t, x_t), \tag{4.1}$$

where f: $R^+ \times X_f \to X$, $X_f \subset X$, g: $R^+ \times C_g \to X$, $C_g \subset C$. (4.1) is the special case of the equation (2.1), when

$$F(t, \varphi) = f(t, \varphi(0)) + g(t, \varphi).$$

Theorem 4.1. *Suppose that for a nonnegative, locally Lipschitzian Ljapunov function V there exist functions α, p: $R^+ \to R^+$ with the following properties:*

(i) *$\alpha(t)$ is bounded for* $t \geqq t_0$,

(ii) *the function p is locally Lipschitzian on* $(0, \infty)$,

(iii) $V(t, x+y) \leqq V(t, x) + V(t, y)$ *for all* $(t, x), (t, y) \in R^+ \times X$,

(iv) $\overline{\lim\limits_{h \to 0+}} \frac{1}{h} [V(t+h, x+hf(t, x)) - V(t, x)] \leqq -\alpha(t) p(V(t, x))$

for all $(t, x) \in R^+ \times X_f$,

(v) *for every* $\varepsilon > 0$ *there exists an* $\eta = \eta(\varepsilon) > 0$ *such that* $(t, \varphi) \in \bar{S}(V, \eta, \varepsilon)$ *implies* $\overline{\lim\limits_{h \to 0+}} \frac{1}{h} V(t+h, hg(t, \varphi)) \leqq \alpha(t) p(\bar{V}(t, \varphi))$.

Then for each solution $x(\cdot)$ *of* (4.1) $\lim\limits_{t \to \infty} V(t, x(t))$ *exists.*

Proof. We apply Theorem 3.1. Let $W = D^+_{(4.1)} V$. By Remark 3.1 it is sufficient to prove that condition (i) of Theorem 3.1 is satisfied. Let $\varepsilon > 0$ be given and choose $\eta = \eta(\varepsilon)$ according to assumption (v). If $(t, \varphi) \in \bar{S}(V, \eta, \varepsilon)$, then from conditions (i)—(v) we obtain

$$D^+_{(4.1)} V(t, \varphi) = \overline{\lim_{h \to 0+}} \frac{1}{h} [V(t+h, \varphi(0) + hf(t, \varphi(0)) + hg(t, \varphi)) - V(t, \varphi(0))] \leqq$$

$$\leqq \overline{\lim_{h \to 0+}} \frac{1}{h} [V(t+h, \varphi(0) + hf(t, \varphi(0))) - V(t, \varphi(0))] + \overline{\lim_{h \to 0+}} \frac{1}{h} V(t+h, hg(t, \varphi)) \leqq$$

$$\leqq \alpha(t) [p(\bar{V}(t, \varphi)) - p(V(t, \varphi(0)))] \leqq KL [\bar{V}(t, \varphi) - V(t, \varphi(0))] =$$

$$= \xi [\bar{V}(t, \varphi) - V(t, \varphi(0))],$$

where L is the Lipschitz constant of p on $[\varepsilon, 2\varepsilon]$ and K is an upper bound for α on $[t_0, \infty)$. This completes the proof.

II. We now apply Theorem 4.1 to obtain a result for equation (4.1) in the case $X = R$.

Theorem 4.2. *Let* $X = R$. *If* $f(t, 0) \equiv 0$, $xf(t, x) \leqq -a(t) x^2$ *for all* $(t, x) \in R^+ \times X_f$, $|g(t, \varphi)| \leqq a(t) \|\varphi\|$ *for all* $(t, \varphi) \in R^+ \times C_g$ *and* $a(t)$ *is bounded for* $t \geqq t_0$, *then for each solution* $x(\cdot)$ *of* (4.1) $\lim\limits_{t \to \infty} x(t)$ *exists.*

Proof. In Theorem 4.1 let $V(t, x) = |x|$, $\alpha(t) = a(t)$, $p(u) \equiv 1$, $\eta(\varepsilon) = \varepsilon$. Thus, V is a Lipschitzian function and conditions (i), (ii), (iii) and in the case $x = 0$ condition (iv) in Theorem 4.1 are obviously satisfied. We have

$$\overline{\lim_{h \to 0+}} \frac{1}{h} (|x + hf(t, x)| - |x|) \leqq \overline{\lim_{h \to 0+}} \frac{1}{h} |x| \left(1 + h \frac{f(t, x)}{x} - 1\right) \leqq -a(t) |x|,$$

if $x \neq 0$ and

$$\overline{\lim_{h \to 0+}} \frac{1}{h} |hg(t, \varphi)| = |g(t, \varphi)| \leqq a(t) \|\varphi\|,$$

that assures also conditions (iv), (v) in Theorem 4.1 to be satisfied. This completes the proof.

Example 4.1. Let us consider the scalar equation

$$\dot{x}(t) = -ax(t)+b(t)x(t-\tau(t)), \tag{4.2}$$

where $a>0$, $b(t)$ and $\tau(t)$ are continuous for $t\geqq t_0$, $|b(t)|\leqq a$, $0\leqq\tau(t)\leqq r$.

For equation (4.2) in this case N. N. KRASOVSKIĬ [9] proved that the zero solution is uniformly stable. Applying Theorem 4.2 we obtain that $x(t)$ tends to a constant as $t\to\infty$, where $x(\cdot)$ is a solution of (4.2).

III. Let us consider the following special form of equation (4.1):

$$\dot{x}(t) = -a(t)x(t)+\sum_{k=1}^{n} b_k(t)x(t-\tau_k(t)), \tag{4.3}$$

where $a, b_k, \tau_k\colon R^+\to R$ are continuous functions and $0\leqq\tau_k(t)\leqq r$ $(k=1, 2, \ldots, n)$.

Theorem 4.3. *Let $k\colon [-r,\infty)\to(0,\infty)$ be a continuous and locally Lipschitzian function. If there exists a $K\in R^+$ such that*

$$k(t)\sum_{k=1}^{n}\frac{|b_k(t)|}{k(t-\tau_k(t))} \leqq a(t)-\frac{D^+k(t)}{k(t)} \leqq K \quad (t\in R^+), \tag{4.4}$$

then for each solution $x(\cdot)$ of (4.3) $\lim_{t\to\infty}|k(t)x(t)|$ exists.

Proof. Apply Theorem 4.1 setting $V(t, x)=|k(t)x|$, $\alpha(t)=a(t)-\frac{D^+k(t)}{k(t)}$, $p(u)\equiv 1$, $\eta(\varepsilon)=\varepsilon$. It is clear that conditions (i), (ii), (iii) in Theorem 4.1 are satisfied. Using (4.4) we can check conditions (iv), (v) in Theorem 4.1 as follows

$$\overline{\lim_{h\to 0+}}\frac{1}{h}\big(|k(t+h)(x-ha(t)x)|-|k(t)x|\big) = |k(t)x|\overline{\lim_{h\to 0+}}\frac{1}{h}\frac{k(t+h)(1-ha(t))-k(t)}{k(t)} =$$

$$= |k(t)x|\left(\frac{D^+k(t)}{k(t)}-a(t)\right) = -\alpha(t)V(t, x),$$

$$\overline{\lim_{h\to 0+}}\frac{1}{h}\left|k(t+h)h\sum_{k=1}^{n} b_k(t)x(t-\tau_k(t))\right| = k(t)\left|\sum_{k=1}^{n} b_k(t)x(t-\tau_k(t))\right| \leqq$$

$$\leqq k(t)\sum_{k=1}^{n}\frac{|b_k(t)|}{k(t-\tau_k(t))}\,\bar{V}(t, x_t) \leqq \left(a(t)-\frac{D^+k(t)}{k(t)}\right)\bar{V}(t, x_t) = \alpha(t)\,\bar{V}(t, x_t).$$

This completes the proof.

Remark 4.1. If for equation (4.3) the inequality $\sum_{k=1}^{n}|b_k(t)|\leq a(t)\leq K$ holds, then $k(t)\equiv 1$ satisfies (4.4) and by Theorem 4.3 $\lim_{t\to\infty}|x(t)|$ exists. But Theorem 4.3 can be used even if this inequality does not hold, as the following example shows.

Example 4.2. Let us consider the equation

$$\dot{x}(t) = -a(t)x(t)+b(t)x(t-\tau(t)), \tag{4.5}$$

where $a(t)$, $b(t)$ and $\tau(t)$ are continuous for $t\geq t_0$, $0\leq\tau(t)\leq r$ and there exists a $K\in R^+$ such that $a(t)\leq|b(t)|\leq K$ for $t\geq t_0$. Let $k(t)=\exp\left(\int_0^t (a(s)-|b(s)|)ds\right)$. We have

$$\frac{D^+k(t)}{k(t)} = a(t)-|b(t)|,$$

$$\exp\left(\int_0^t (a(s)-|b(s)|)\,ds\right)\frac{|b(t)|}{\exp\left(\int_0^{t-\tau(t)} (a(s)-|b(s)|)ds\right)} \leq |b(t)| \leq K.$$

Thus, from Theorem 4.3 it follows that for each solution $x(\cdot)$ of (4.5)

$$\lim_{t\to\infty}\left|x(t)\exp\left(\int_0^t (a(s)-|b(s)|)\,ds\right)\right|$$

exists.

IV. Let us consider the equation

$$\dot{x}(t) = -h(x(t))+h(x(t-\tau(t))), \tag{4.6}$$

where $\tau(t)$ is continuous for $t\geq t_0$, $0\leq\tau(t)\leq r$ and $h(s)$ is continuous for $s\in R$.

Theorem 4.4. *If the function h is increasing and locally Lipschitzian on $(-\infty, 0)$ and $(0, \infty)$, then for each solution $x(\cdot)$ of (4.6) $\lim_{t\to\infty} x(t)$ exists.*

Proof. Apply Corollary 3.2 setting $V(t, x)=x$, $W_1(t, \varphi)=-W_2(t, \varphi)=$ $=D^+_{(4.6)}V(t, \varphi)$. Let $\varepsilon>0$ be given, $\eta(\varepsilon)=\varepsilon$ and $\xi(\varepsilon)=\max\{L_1, L_2\}$, where L_1, L_2 are the Lipschitz constants of h on $[\varepsilon, 2\varepsilon]$, $[-2\varepsilon, -\varepsilon]$, respectively. Since $W_1(t, x_t)=-W_2(t, x_t)=\dot{x}(t)$ it is obvious that conditions (iii), (iv) in Corollary 3.2 are satisfied. If $(t, x_t)\in\bar{S}(x(t), \eta, \varepsilon)$ then

$$\dot{x}(t) = -h(x(t))+h(x(t-\tau(t))) \leq -h(x(t))+h(\bar{x}_t) \leq \xi(\bar{x}_t-x(t)).$$

If $(t, x_t)\in\underline{S}(x(t), \eta, \varepsilon)$, then

$$-\dot{x}(t) = h(x(t))-h(x(t-\tau(t))) \leq h(x(t))-h(\underline{x}_t) \leq \xi(x(t)-\underline{x}_t).$$

Thus conditions (i), (ii) in Corollary 3.2 are satisfied and the theorem is proved.

Remark 4.2. Applying Theorem 4.4 to case $h(u)=u^{1/3}$, $\tau(t)\equiv r$ we get a new proof for the following conjecture of S. R. BERNFELD and J. R. HADDOCK [1], which was solved by C. JEHU [5]: each solution of the scalar equation $\dot{x}(t)=-x^{1/3}(t)+ +x^{1/3}(t-r)$ tends to a constant as $t\to\infty$.

Acknowledgement. The author wishes to thank Professors L. Hatvani, L. Pintér and Dr. J. Terjéki for many discussions.

References

[1] S. R. BERNFELD, J. R. HADDOCK, A variation of Razumikhin's method for retarded functional differential equations, *Nonlinear Systems and Applications, An International Conference* (V. Lakshmikantham ed.), Academic Press (New York, 1977), 561—566.

[2] S. R. BERNFELD, J. R. HADDOCK, Liapunov—Razumikhin functions and convergence of solutions of scalar functional differential equations, *Applicable Anal.* (to appear).

[3] K. L. COOKE, J. A. YORKE, Some equations modelling growth process and gonorrhea epidemics, *Math. Biosci.*, **16** (1973), 75—101.

[4] J. R. HADDOCK, Asymptotic behavior of solutions of nonlinear functional differential equations in Banach space, *Trans. Amer. Math. Soc.*, **231** (1977), 83—92.

[5] C. JEHU, Comportement asymptotique des solutions de l'equation $\dot{x}(t)=-f(t, x(t))+ +f(t, x(t-1))+h(t)$, *Ann. Soc. Sci. Bruxelles, Sér. I*, **92** (1978), 263—269.

[6] J. L. KAPLAN, M. SORG, J. A. YORKE, Solutions of $\dot{x}(t)=f(x(t), x(t-L))$ have limits when f is an order relation, *Nonlinear Anal.*, **3** (1979), 53—58.

[7] G. E. LADAS, V. LAKSHMIKANTHAM, *Differential Equations in Abstract Spaces*, Academic Press (New York, 1972).

[8] V. LAKSHMIKANTHAM, A. R. MITCHELL, R. W. MITCHELL, On the existence of solutions of differential equations of retarded type in a Banach space, *Ann. Polon. Math.*, **35** (1978), 253—260.

[9] Н. Н. Красовский, *Некоторые задачи теории устойчивости движения*, Физматгиз (Москва, 1959).

[10] Б. С. Разумихин, Об устойчивости систем с запаздыванием, *Прикл. мат. мех.*, **20** (1956), 500—512.

[11] T. YOSHIZAWA, *Stability Theory by Liapunov's Second Method*, Math. Society of Japan (Tokyo, 1966).

BOLYAI INSTITUTE
UNIVERSITY SZEGED
ARADI VÉRTANÚK TERE 1
H—6720 SZEGED, HUNGARY