Acta Sci. Math., 43 (1981), 37—39

The boundedness of closed linear maps in C^*-algebras

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The domain of a closed *-derivation in a C^*-algebra has many properties. In particular, ŌTA [6] studied such domains by using Lorentz representation and obtained some interesting results on the boundedness of closed *-derivations. Especially, he showed that a closed *-derivation, which is bounded on the unitary group of the domain, is bounded.

Now in connection with strongly continuous one-parameter semi-groups of positive maps on C^*-algebras, we are interested in the boundedness of more general closed linear maps. One of the crucial points in [6] is that the domain of a closed *-derivation becomes a semi-simple Banach *-algebra under the graph norm. Although such fact is not valid in our general situation, we have some generalizations of results in [6] by virtue of a simple lemma on Banach algebras.

Let A and A_0 be respectively a unital C^*-algebra and a *-subalgebra of A which contains the identity e of A. The following lemma is elementary, but it is essential in what follows.

Lemma. *Suppose that there exists a closed linear map Φ of A_0 into a Banach space. Then A_0 is a semi-simple Banach algebra with an isometric involution under some norm $\|\cdot\|'$ which is equivalent to the graph norm $\|\cdot\|_\Phi=\|\cdot\|+\|\Phi(\cdot)\|$.*

Proof. Since $(A_0, \|\cdot\|_\Phi)$ is a Banach space, by the closed graph theorem, the product in A_0 is separately continuous with respect to $\|\cdot\|_\Phi$, and hence A_0 is a Banach algebra under some norm which is equivalent to $\|\cdot\|_\Phi$ (see [8, p. 5]). Since A_0 is semi-simple by the proof of [8, Theorem 4.4.10], JOHNSON's theorem [5] implies that the involution is continuous in $\|\cdot\|_\Phi$, and hence we have the desired norm $\|\cdot\|'$ by another equivalent renorming. The proof is complete.

By the above lemma and [8, Theorem 4.1.5], it follows that a *-subalgebra A_0, which is the domain of a closed linear map, has sufficiently many unitary elements, more precisely, every element of A_0 is a linear combination of unitary elements of A_0.

Received August 28, 1980.

An involutive Banach algebra is said to be C^*-equivalent if it is *-isomorphic to some C^*-algebra. B. Russo and H. A. Dye [9] showed that a linear map on a unital C^*-algebra, which is bounded on the unitary group, is bounded. This result and the above mentioned remark suggest the following:

Theorem 1. *Let Φ be a closed linear map of A_0 into a Banach space. If Φ is norm bounded on the unitary group of A_0, then A_0 is a C^*-algebra and Φ is bounded.*

Proof. Since the norm $\|\cdot\|'$ in the Lemma is equivalent to the graph norm $\|\cdot\|_\Phi$, there exists a constant $N>0$ such that $\|a\|'\leqq N\|a\|_\Phi$ for all $a\in A_0$. Then we have

$$\sup\{\|u\|': u \text{ is unitary in } A_0\}\leqq N\sup\{1+\|\Phi(u)\|: u \text{ is unitary in } A_0\}\leqq$$
$$\leqq N+N\sup\{\|\Phi(u)\|: u \text{ is unitary in } A_0\}<+\infty.$$

Hence from [7, Corollary 12] A_0 is C^*-equivalent, which implies that A_0 is a C^*-algebra. Hence by the closed graph theorem or by Corollary 1 in [9] Φ is bounded.

Theorem 1 implies that any closed *-homomorphism of A_0 into A is automatically bounded. Moreover, this assertion is true for a more general class of maps. More precisely, let Φ be a 2-positive map from A_0 into another C^*-algebra B, that is, for all pairs $\{x_1, x_2\}$ in A_0, the matrices $(\Phi(x_i^* x_j))$ are positive in the C^*-algebra of all 2×2 matrices over B. Then the Schwarz inequality $\Phi(a^*)\Phi(a)\leqq\|\Phi(e)\|\,\Phi(a^*a)$ $(a\in A_0)$ follows easily ([1], [4]), and hence Φ is bounded if it is closed.

It is natural to ask if every closed positive linear map Φ from A_0 into another C^*-algebra B is automatically bounded, where positivity of Φ means that $\Phi(a^*a)$ is positive in B for all $a\in A_0$. We have however no answer to this question.

Now let Φ be a completely positive linear map on A and put $L_\Phi(x)=\Phi(x)-\frac{1}{2}\{\Phi(e)x+x\Phi(e)\}$ for $x\in A$. Then the generator of a uniformly continuous semi-group of unital completely positive maps on A is essentially determined by two classes of operators, that is, *-derivations on A and maps of the form L_Φ for Φ ([2]). In this connection, the following corollary is interesting.

Corollary. *Suppose that A_0 is strongly dense in A. Let Φ be a completely positive map from A_0 into A. If L_Φ generates a strongly continuous semi-group of linear maps on A, then $A_0=A$, that is, Φ is everywhere defined.*

A linear map δ from A_0 into A is called a Jordan derivation if $\delta(h^2)=h\delta(h)+\delta(h)h$ for all $h=h^*$ in A_0. Then we have the following theorem, which is a generalization of Theorem 2.4 in [6].

Theorem 2. *Suppose that A_0 is strongly dense in A. Let δ be a closed Jordan derivation from A_0 into A. If A_0 is closed under the square root operation of positive* *elements $A_0 \cap A^+$ where A^+ denotes the positive part of A, then δ is everywhere defined and is bounded.*

Proof. Since the norm $\|\cdot\|'$ in the Lemma is equivalent to the graph norm $\|\cdot\|_\delta$, $\lim_{n\to\infty} \|x^n\|_\delta^{1/n}$ exists and is equal to $\lim_{n\to\infty} \|x^n\|'^{1/n}$ for $x \in A_0$. Hence, for $h = h^* \in A_0$ we have

$$\lim_{n\to\infty} \|h^n\|_\delta^{1/n} = \lim_{n\to\infty} (\|h\|^{2^n} + \|\delta(h^{2^n})\|)^{1/2^n} \leqq$$
$$\leqq \lim_{n\to\infty} \|h\| \{1 + (2^n \|\delta(h)\|)/\|h\|\}^{1/2^n} = \|h\|$$

because $\|\delta(h^{2^n})\| \leqq 2^n \|h\|^{2^n-1} \|\delta(h)\|$ $(n = 1, 2, 3, \ldots)$ where $\|\cdot\|$ is the norm of A. Hence

$$\lim_{n\to\infty} \|h^n\|'^{1/n} \leqq \|h\| = \inf \{\sum |\lambda_i| : h = \sum \lambda_i u_i, u_i\text{'s are unitaries in } A\} \leqq$$
$$\leqq \inf \{\sum |\lambda_i| : h = \sum \lambda_i u_i, u_i\text{'s are unitaries in } A_0\}$$

which implies that the semi-simple involutive Banach algebra A_0 is hermitian from [7, Corollary 5 and 9]. Denote the spectrum of an element x of A_0 in A (resp. A_0) by $\mathrm{sp}(x)$ (resp. $\mathrm{sp}_0(x)$). Now let h be a hermitian element of A_0. If $\mathrm{sp}_0(h) \geqq 0$, then $\mathrm{sp}(h) \geqq 0$, and hence there exists a hermitian element k in A_0 such that $k^4 = h$ from our assumption. Hence $\mathrm{sp}_0(k^2) = \{\lambda^2 : \lambda \in \mathrm{sp}_0(k)\} \geqq 0$ since A_0 is hermitian. Therefore, A_0 is C^*-equivalent from [3, Corollary], which implies that $A_0 = A$, and hence δ is bounded from the closed graph theorem. The proof is completed.

References

1] M. D. Choi, A Schwarz inequality for positive linear maps on C^*-algebras, *Ill. J. Math.*, **18** (1974), 565—574.

[2] E. Christensen and D. E. Evans, Cohomology of operator algebras and quantum dynamical semi-groups, *Preprint*, 1978.

[3] J. Cuntz, Locally C^*-equivalent algebras, *J. Functional Analysis*, **23** (1976), 95—106.

[4] D. E. Evans, Positive linear maps on operator algebras, *Comm. Math. Phys.*, **48** (1976), 15—22.

[5] B. E. Johnson, The uniqueness of the complete norm topology, *Bull. Amer. Math. Soc.*, **73** (1967), 537—539.

[6] S. Ōta, Certain operator algebras induced by *-derivations in C^*-algebras on an indefinite inner product space, *J. Functional Analysis*, **30** (1978), 238—244.

[7] T. W. Palmer, The Gelfand—Naimark pseudonorm on Banach *-algebras, *J. London Math. Soc.*, **3** (1971), 59—66.

[8] C. E. Rickart, *General theory of Banach algebras*, van Nostrand (Princeton, 1960).

[9] B. Russo and H. A. Dye, A note on unitary operators in operator algebras, *Duke Math. J.*, **33** (1966), 413—416.

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