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# The algebraic representation of semigroups and lattices; representing subsemigroups

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A monoid $S$ and a lattice $L$ are *jointly algebraic*, if there is a universal algebra $\mathfrak{A}=\langle A,\mathscr{P}\rangle$ such that $S\cong\mathrm{End}\,\mathfrak{A}$ and $L\cong\mathrm{Su}\,\mathfrak{A}$. The major result of this paper is that if either $S$ or $L$ are finite and if they are jointly algebraic, then every submonoid $T$ of $S$ is jointly algebraic with $L$. We prove a slightly stronger theorem.

## § 1. Introduction

We adopt the notation of [1] and [2]. If $M$ is a set of partial functions on the set $A$ then we will write sometimes $M^{\sim}$ for $\tilde{M}$ and we will use the following additional notation: $\Gamma$, $\Sigma$ denote systems of equations with coefficients from $M$. For $D\subset A$, $\bar{D}=\mathscr{C}(D;A,M)=\bigcap\limits_{D\subset\mathrm{Spt}\,\Sigma}$ ($\mathrm{Spt}\,\Sigma$ is the set of all points on which $\Sigma$ has a solution). We write simply $\bar{D}$ if $A$ and $M$ are understood. For $B\subset A$, $\mathscr{S}B=\mathscr{S}(B;A,M)=\bigcup\limits_{D\text{ finite},\,D\subset B}\bar{D}$. We write $\mathscr{S}B$ if $A$ and $M$ are understood. If $D\subset B$ and $D$ is finite we will henceforth write $D\subset_f B$.

## § 2. Concrete Results

Lemma 1. *If $\mathfrak{A}$ is any algebra on $A$ whose operations are all substitutive with $M$ and $\Sigma$ is a system of equations over $M$, then* $\mathrm{Spt}\,\Sigma$ *is a subalgebra of* $\mathfrak{A}$.

Proof. It is enough to prove that $\mathrm{Spt}\,\Sigma$ is a subalgebra of $\mathfrak{A}_M$, the algebra of all the operations substitutive over $M$. According to [1] Theorem 1 we have to show that $\mathrm{Spt}\,\Sigma=\mathscr{S}(\mathrm{Spt}\,\Sigma;A,M)$. Now if $D\subset\mathrm{Spt}\,\Sigma$, then $\bar{D}=\bigcap\limits_{D\subset\mathrm{Spt}\,\Gamma}\mathrm{Spt}\,\Gamma\subset\mathrm{Spt}\,\Sigma$ and

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therefore $\mathcal{S}(\mathrm{Spt}\,\Sigma; A, M) = \bigcup_{D \subset_f \mathrm{Spt}\,\Sigma} \subset \bigcup \mathrm{Spt}\,\Sigma = \mathrm{Spt}\,\Sigma$. By [1] Lemma 5, $\mathcal{S}$ is a closure operator and hence $\mathrm{Spt}\,\Sigma \subset \mathcal{S}(\mathrm{Spt}\,\Sigma; A, M)$. □

Lemma 2. *If* $D \subset_f A$ *then* $\bar{D}$ *is the subalgebra of* $\mathfrak{A}_M$ *generated by* $D$ *and* $\bar{D} = \mathrm{dom}\, g$ *for some partial identity function* $g \in \tilde{M}$.

Proof. Let $B$ be the subalgebra of $\mathfrak{A}_M$ generated by $D$. Then by Theorem 1 of [1] $B = \bigcup_{C \subset_f B} \bar{C}$, thus $\bar{D} \subset B$. But also $\bar{D} = \bigcap_{D \subseteqq \mathrm{Spt}\,\Sigma} \mathrm{Spt}\,\Sigma = \mathrm{Spt}\,\Gamma$ for some system $\Gamma$ by Lemma 2 of [1], and hence $\bar{D} \in \mathrm{Su}\,\mathfrak{A}_M$ by Lemma 1 above. Thus $\bar{D} = B$. Clearly $\bar{D} = \mathrm{dom}\, g$ for $g = \mathrm{id} \restriction \mathrm{Spt}\,\Gamma$*, and since $\mathrm{Spt}\,\Gamma \in \mathrm{Su}\,\mathfrak{A}_M$ we have $g \in \tilde{M}$ because every identity on a subalgebra of $\mathfrak{A}_M$ is a partial endomorphism. □

Lemma 3. *If* $D$ *is finite, then* $\mathcal{C}(D; A, M) = \mathcal{C}(D; A, \tilde{M})$.

Proof. Note $\mathfrak{A}_M = \mathfrak{A}_{\tilde{M}}$, hence the subalgebra generated by $D$ is the same in both algebras and the result follows from Lemma 2 above. □

Corollary 1. $\mathcal{S}(B; A, M) = \bigcup_{D \subset_f B} \mathcal{C}(D; A, M) = \bigcup_{D \subset_f B} \mathcal{C}(D; A, \tilde{M}) = \mathcal{S}(B; A, \tilde{M})$.

Definition 1. We will write the *ordered triple* $(A; S, L)$ for a representation of $S$ as a transformation monoid on $A$ and $L$ as an algebraic intersection structure on $A$ (i.e. $L$ is a set of subsets of $A$, which forms by intersection an algebraic lattice). Then $\mathrm{St}_1(A; S, L)$ and $\mathrm{St}_2(A; S, L)$ are abbreviations for the following statements:

$$\mathrm{St}_1(A; S, L):\ S \Rightarrow \overline{S \cup L},$$

(where if $M$ is a set of partial functions on $A$, $\bar{M}$ is the set of total functions in $\tilde{M}$) and

$$\mathrm{St}_2(A; S, L):\ B = \mathcal{S}(B; A, S \cup L) \Rightarrow B \in L.$$

If $(A; S, L)$ and $S = \mathrm{End}\,\mathfrak{A}$ and $L = \mathrm{Su}\,\mathfrak{A}$ for some algebra $\mathfrak{A} = \langle A, \mathcal{P} \rangle$ then we will say that $(A; S, L)$ is *algebraic*.

Remark. Then Theorem 3 of [1] reads (using also Theorem 4 of [2]): $(A; S, L)$ is algebraic if and only if $\mathrm{St}_1(A; S, L)$ and $\mathrm{St}_2(A; S, L)$.

Lemma 5. *If* $\mathrm{St}_2(A; S, L)$, *then* $(A; \overline{S \cup L}, L)$ *is algebraic.*

Proof. a) $\mathrm{St}_1(A, \overline{S \cup L}, L)$. Note that $\overline{S \cup L} \subset \overline{\overline{S \cup L} \cup L}$. Put on the other hand $\overline{\overline{S \cup L} \cup L} = A^A \cap [(A^A \cap (S \cup L)^\sim) \cup L]^\sim \subset A^A \cap [(S \cup L)^\sim \cup L]^\sim = A^A \cap (S \cup L)^\sim = \overline{S \cup L}$. This proves $\mathrm{St}_1(A; \overline{S \cup L}, L)$ which says: $\overline{S \cup L} = \overline{\overline{S \cup L} \cup L}$.

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* For a function $f$ and $A \subseteqq \mathrm{dom}\, f$, $f \restriction A$ denotes the restriction of $f$ to $A$.

b) $\mathrm{St}_2(A, \overline{S\cup L}, L)$. We have $(S\cup L)^{\sim}=(\overline{S\cup L}\cup L)^{\sim}$ because obviously $(S\cup L)^{\sim}\subset(\overline{S\cup L}\cup L)^{\sim}$ and $(\overline{S\cup L}\cup L)^{\sim}=[(A^A\cap(S\cup L)^{\sim})\cup L]^{\sim}\subset((S\cup L)^{\sim}\cup L)^{\sim}=$ $=(S\cup L)^{\sim}$. Therefore, by Corollary 1, $\mathscr{S}(B; A, \overline{S\cup L}\cup L)=\mathscr{S}(B; A(\overline{S\cup L}\cup L)^{\sim})=$ $=\mathscr{S}(B; A, (S\cup L)^{\sim})=\mathscr{S}(B; A, S\cup L)$. So, if $B=\mathscr{S}(B; A, \overline{S\cup L}\cup L)$, then $B=$ $=\mathscr{S}(B; A, S\cup L)$ and hence $B\in L$ because $\mathrm{St}_2(A; S, L)$ holds. □

## § 3. Representations

Definition 2. If $S$ is a monoid and $L$ an algebraic lattice, then the partial universal algebra $\langle A;f\rangle_{f\in S\cup L}$ is a *representation* of $S$ and $L$, if all of the operations in $S$ form a transformation monoid of $A$, with $(fg)(a)=f(g(a))$, $\mathrm{id}(a)=a$ and if all of the operations in $L$ are partial identities with range $p\cap$range $q=$range$(p\wedge q)$ and the 1 of the lattice is the identity transformation of $A$. Furthermore we require that a representation be faithful: for any two $f, g\in S, f\neq g$ there exists an $a\in A$ with $f(a)\neq g(a)$ and if for any two $p, q\in L, p\neq q$, range $p\neq$range $q$. We write simply $\langle A,f\rangle$ for $\langle A,f\rangle_{f\in S\cup L}$ when $S\cup L$ is understood. Note $(A, \{f; f\in S\},$ $\{f(A); f\in L\})$ iff $\langle A,f\rangle_{f\in S\cup L}$ is a representation.

We will adopt the notions of [3] for homomorphism, subalgebra, embedding of partial algebras and will also say that $\mathfrak{B}$ is an extension of $\mathfrak{A}$ if $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$.

Definition 3. If $\langle A;f\rangle_{f\in S\cup L}$ is a representation of $S$ and $L$ then we will write $\overline{S\cup L}^A$ to emphasize the function closure cited in $\mathrm{St}_1$ taken with respect to that representation of $S$ and $L$ on $A$.

Lemma 6. *Let $\psi: A\to B$ be a homomorphism from the representation $\langle A;f\rangle_{f\in S\cup L}$ into the representation $\langle B;f\rangle_{f\in S\cup L}$. If the system $\Sigma$ of equations with coefficients in $S\cup L$ has a solution $h$ at some $a\in A$, then $\Sigma$ has also a solution $\psi h$ at $\psi(a)\in B$.*

Proof. If $\alpha$ is an assignment which satisfies $\Sigma$ at $a$, then clearly $\psi\alpha$ is an assignment which satisfies $\Sigma$ at $\psi(a)$. □

Definition 4. Let $\langle A;f\rangle_{f\in S\cup L}$ be a representation of $S$ and $L$ and let $(A_i: i\in I)$ be a family of subalgebras of $A$ with $\bigcup_{i\in I} A_i=A$ and $(\varphi_i: i\in I)$ homomorphisms from $A$ onto $A_i$ which leave $A_i$ elementwise fixed. $((A_i, \varphi_i): i\in I)$ is called a *cover* of $\langle A;f\rangle_{f\in S\cup L}$.

Lemma 7. *If $((A_i, \varphi_i): i\in I)$ is a cover of $\langle A;f\rangle_{f\in S\cup L}$ and $h\in\overline{S\cup L}^A$, then $h=\bigcup_{i\in I} h_i$ with each $h_i\in\overline{S\cup L}^{A_i}$.*

Proof. We prove that $h\restriction A_i\in\overline{S\cup L}^{A_i}$. For each $a\in A_i$ there exists a system $\Sigma$ whose unique solution at $a$ is $h$. If $h(a)\notin A_i$, then by applying $\varphi_i$ and Lemma 6 we observe that $\Sigma$ has a solution at $\varphi_i(a)=a$ which is equal to $\varphi_i(h(a))\in A_i$. But then $\Sigma$ has two different solutions at $a$, a contradiction. So $h\restriction A_i\in A_i^{A_i}$. Because $h\in\overline{S\cup L}^A$ there exists for every finite subset $D$ of $A_i$ a system $\Sigma$ whose unique solution at $D$ is $h$. We will have proven that $h\restriction A_i\in\overline{S\cup L}^{A_i}$ if there exists an assignment $\alpha$ of $\Sigma$ at $D$ for $D\subset_f A_i$ with $\alpha(x)\in A_i$ for all the variables $x\in\Sigma$. If $\beta$ is any assignment of $\Sigma$ at $D$, then clearly $\varphi_i\beta=\alpha$ has the desired property. Thus $h=\bigcup_{i\in I} h_i$ for $h_i=h\restriction A_i$. □

Lemma 8. *If* $((A_i,\varphi_i)\colon i\in I)$ *is a cover of* $\langle A;f\rangle$ *and* $h\in\overline{S\cup L}^A$ *and* $x\in A_i\cap A_j$, *then* $h(x)\in A_i\cap A_j$.

Proof. According to Lemma 7, $h\restriction A_i\in A_i^{A_i}$ and $h\restriction A_j\in A_j^{A_j}$ which implies the assertion. □

## § 4. The Foliation of a Representation

Definition 5. If $\mathcal{R}=\langle A;f\rangle$ is a representation of $S$ and $L$, and $a\in A$, then $[a]$ is the subalgebra of $\mathcal{R}$ generated by $\{a\}$.

Definition 6. If $\mathcal{R}=\langle A;f\rangle$ is a representation of $S$ and $L$, then $\mathcal{F}(\mathcal{R})=$ $=\langle\mathcal{F}(A),f\rangle_{f\in S\cup L}$, the *foliation* of $\mathcal{R}$, is an extension of $\mathcal{R}$ which is constructed as follows: for each $x\in A$, $A_x^-=\{a_x;\, a\in A-[x]\}$ and $A_x=A_x^-\cup[x]$,

$$\mathcal{F}(A)=\bigcup_{x\in A} A_x;$$

with $y\in\mathcal{F}(A)$ and $f\in S$,

$$f(y)=\begin{cases} f(y) & \text{if } y\in A,\\ (f(a))_x & \text{if } y=a_x \text{ and } f(a)\in A-[x],\\ f(a) & \text{if } y=a_x \text{ and } f(a)\in[x];\end{cases}$$

with $y\in\mathcal{F}(A)$ and $p\in L$,

$$p(y)=\begin{cases} p(y) & \text{if } y\in A,\\ (p(a))_x & \text{if } y=a_x.\end{cases}$$

$(p(y)$ is either $y$ or is undefined$)$.

Lemma 9. $\mathcal{F}(\mathcal{R})$ *is a representation of* $S$ *and* $L$.

Proof. Let $f,g,h\in S$ with $(fg)=h$. We want to prove that for all $y\in\mathcal{F}(A)$, $f(g(y))=h(y)$. This is clearly true for $y\in A$, so let $y=a_x$ and assume first that $g(a)\notin[x]$ and $f(g(a))\notin[x]$, so $h(a)\notin[x]$ and then: $f(g(y))=f(g(a_x))=f((g(a))_x)=$ $=((fg)(a))_x=(h(a))_x=h(a_x)=h(y)$. If $g(a)\notin[x]$ but $f(g(a))\in[x]$, then $h(a)\in[x]$

and then: $f(g(y))=f((g(a))_x)=f(g(a))=(fg)(a)=h(a)=h(a_x)=h(y)$. If $g(a)\in[x]$, then $f(g(a))\in[x]$ and $h(a)\in[x]$ and then: $f(g(y))=f(g(a_x))=f(g(a))=(fg)(a)==h(a)=h(a_x)=h(y)$.

If id is the unit element in $S$ and $y\in\mathscr{F}(A)$, then if $y\in A$ clearly $\mathrm{id}(y)=y$ and if $y=a_x$ then $a\notin[x]$ and $\mathrm{id}(a)=a\notin[x]$ and hence $\mathrm{id}(a_x)=(\mathrm{id}(a))_x=a_x$.

Let $p, q$ be two elements in $L$, then $p(a_x)$ is defined and equal to $a_x$ if and only if $p(a)$ is defined. But (on $A$), range $p\cap$range $q$=range $(p\wedge q)$ is equivalent to the condition: $(\forall a\in A)\,[p(a)$ and $q(a)$ are defined iff $(p\wedge q)(a)$ is defined]. Therefore $p(a_x)$ and $q(a_x)$ are defined iff $(p\wedge q)(a_x)$ is defined. Furthermore we have shown that the identity map on $A$ extends to the identity map on $\mathscr{F}(A)$. Observe that $\mathscr{F}(\mathscr{R})$ is faithful iff $\mathscr{R}$ is faithful. □

Definition 7. If $\mathscr{F}(\mathscr{R})=\langle\mathscr{F}(A);f\rangle_{f\in S\cup L}$ is the foliation of $\mathscr{R}=\langle A;f\rangle_{f\in S\cup L}$, then the maps $\varphi$, $(\varphi_x;\, x\in A)$, $(\varepsilon_x;\, x\in A)$, $(\nu_x;\, x\in A)$ are defined as follows:

$$\nu_x\colon \mathscr{F}(A)\to A\cup A_x \text{ with } \nu_x(y)=\begin{cases} y & \text{if } y\in A\cup A_x,\\ a & \text{if } y=a_z,\ z\neq x;\end{cases}$$

$$\varepsilon_x\colon A\cup A_x\to A_x \text{ with } \varepsilon_x(y)=\begin{cases} y & \text{if } y\in A_x,\\ y_x & \text{if } y\in A-[x];\end{cases}$$

$$\varphi\colon \mathscr{F}(A)\to A \text{ with } \varphi(y)=\begin{cases} y & \text{if } y\in A,\\ a & \text{if } y=a_x;\end{cases}$$

$$\varphi_x\colon \mathscr{F}(A)\to A_x \text{ with } \varphi_x=\varepsilon_x\nu_x.$$

Lemma 10. *Each of the maps above is a homomorphism onto the indicated subalgebra of $\mathscr{F}(\mathscr{R})$.*

Proof. a) $\nu_x$. Because $A\cup A_x$ is a subalgebra, the restriction of $\nu_x$ to $A\cup A_x$ is a homomorphism. First let $a_z\in A_z$ and $f\in S$ with $f(a)\notin[z]$. Then $\nu_x(f(a_z))==\nu_x((f(a))_z)=f(a)=f(\nu_x(a_z))$. If $f(a)\in[z]$, then $\nu_x(f(a_z))=\nu_x(f(a))=f(a)=f(\nu_x(a_z))$. For $p\in L$, $p$ is defined at $a_z$ iff $p$ is defined at $a$ and $p(a_z)=a_z$ and $p(a)=a$, hence $p(\nu_x(a_z))=p(a)=a=\nu_x(a_z)=\nu_x(p(a_z))$.

b) $\varphi$. This proof is almost identical to the one for $\nu_x$.

c) $\varepsilon_x$. Because $A_x$ is a subalgebra of the representation $\langle A\cup A_x, f\rangle$ of $S$ and $L$, the restriction of $\varepsilon_x$ to $A_x$ is a homomorphism. So if $y\in A-[x]$ and $f\in S$, with $f(y)\notin[x]$, then $\varepsilon_x f(y)=f(y)_x=f(y_x)=f(\varepsilon_x(y))$; further if $f(y)\in[x]$, then $\varepsilon_x f(y)=f(y)=f(y_x)=f(\varepsilon_x(y))$. If $p\in L$ and $p$ is defined at $y\in A_x^-$, then $p$ is defined at $y_x$ and $p(y)=y, p(y_x)=y_x$ hence $p(\varepsilon_x(y))=p(y_x)=y_x=\varepsilon_x(y)=\varepsilon_x(p(y))$.

d) $\varphi_x$. $\varphi_x$ is a homomorphism as a product of two homomorphisms. □

Lemma 11. *The sets $(A_x\colon x\in A)$ together with $A$ and the maps $(\varphi_x\colon x\in A)$ and $\varphi$ form a cover of $\mathscr{F}(\mathscr{R})=\langle\mathscr{F}(A);f\rangle_{f\in S\cup L}$.*

Proof. Clearly $(\bigcup_{x\in A} A_x)\cup A=\mathscr{F}(A)$ and furthermore $A$ and each of the sets $A_x$ are subalgebras of $\mathscr{F}(\mathscr{R})$. By Lemma 10 the maps $\varphi$, and $(\varphi_x: x\in A)$ are homomorphisms which leave $A$ and $(A_x: x\in A)$ pointwise fixed as required. □

Lemma 12. $\bigcup_{x\in A} \varphi_x(\mathscr{C}(\varphi(D); A, S\cup L))=\mathscr{C}(D: \mathscr{F}(A), S\cup L)$, *for* $D\subseteqq\mathscr{F}(A)$.

Proof. If $y\in A$ then $\varphi(y)=y$ and if $y=a_x$, then $\varphi_x\varphi(y)=\varphi_x\varphi(a_x)=$ $=\varepsilon_x v_x\varphi(a_x)=a_x=y$ and hence we see by Lemma 6 that a system $\Sigma$ of equations has a solution at $y$ iff $\Sigma$ has a solution at $\varphi(y)$. Furthermore $\Sigma$ has a solution at $y\in A$ iff $\Sigma$ has a solution at $\varphi_x(y)$ for each $x\in A$, because again $\varphi\varphi_x(y)=y$. (For $y\in A$: $y\in[x]\Rightarrow\varphi\varphi_x(y)=y$, and $y\notin[x]\Rightarrow\varphi\varphi_x(y)=y$.) This means that $D\subset\operatorname{Spt}\Sigma$ iff $\varphi(D)\subset\operatorname{Spt}\Sigma$. In fact for $a\in A$, $a\in\operatorname{Spt}\Sigma$ iff $\forall x\in A, a\notin[x], a_x\in\operatorname{Spt}\Sigma$, thus $\bigcup_{x\in A}\varphi_x(A\cap\operatorname{Spt}\Sigma)=\operatorname{Spt}\Sigma$. Hence

$$\mathscr{C}(D, \mathscr{F}(A), S\cup L) = \bigcap_{D\subset\operatorname{Spt}\Sigma}\operatorname{Spt}\Sigma = \bigcap_{\varphi(D)\subset\operatorname{Spt}\Sigma}\operatorname{Spt}\Sigma =$$

$$= \bigcap_{\varphi(D)\subset\operatorname{Spt}\Sigma}\Big(\bigcup_{x\in A}\varphi_x(A\cap\operatorname{Spt}\Sigma)\Big) \supset \bigcup_{x\in A}\Big(\bigcap_{\varphi(D)\subset\operatorname{Spt}\Sigma}\varphi_x(A\cap\operatorname{Spt}\Sigma)\Big) \supset$$

$$\supset \bigcup_{x\in A}\varphi_x\Big(\bigcap_{\varphi(D)\subset\operatorname{Spt}\Sigma}(A\cap\operatorname{Spt}\Sigma)\Big) = \bigcup_{x\in A}\varphi_x\Big(\bigcap_{\varphi(D)\subset\operatorname{Spt}^*\Sigma}\operatorname{Spt}^*\Sigma\Big) =$$

$$= \bigcup_{x\in A}\varphi_x(\mathscr{C}(\varphi(D), A, S\cup L))$$

(cf. Lemma 6), where $\operatorname{Spt}^*\Sigma$ is the support in the original representation $\mathscr{R}=\langle A; f\rangle$.

On the other hand, because $\varphi_x=\varepsilon_x v_x$ is one-to-one on $A$, we get

$$\varphi_x(\mathscr{C}(\varphi(D); A, S\cup L)) = \varphi_x\Big(\bigcap_{\varphi(D)\subset\operatorname{Spt}^*\Sigma}\operatorname{Spt}^*\Sigma\Big) = \bigcap_{\varphi(D)\subset\operatorname{Spt}^*\Sigma}\varphi_x(\operatorname{Spt}^*\Sigma) =$$

$$= \bigcap_{\varphi(D)\subset\operatorname{Spt}\Sigma}\varphi_x(A\cap\operatorname{Spt}\Sigma) \subset \bigcap_{D\subset\operatorname{Spt}\Sigma}\operatorname{Spt}\Sigma = \mathscr{C}(D; \mathscr{F}(A), S\cup L).$$ □

Lemma 13. $\bigcup_{x\in A}\varphi_x(\mathscr{S}(\varphi(B); A, S\cup L))=\mathscr{S}(B; \mathscr{F}(A), S\cup L)$.

Proof. Observe that $\varphi(D)\subset_f\varphi(B)\Rightarrow\exists E\subset_f B$ such that $\varphi(E)=\varphi(D)$ hence

$$\bigcup_{x\in A}\varphi_x(\mathscr{S}(\varphi(B); A, S\cup L)) = \bigcup_{x\in A}\varphi_x\Big(\bigcup_{\varphi(D)\subset_f\varphi(B)}\mathscr{C}(\varphi(D); A, S\cup L)\Big) =$$

$$= \bigcup_{x\in A}\varphi_x\Big(\bigcup_{E\subset_f B}\mathscr{C}(\varphi(E); A, S\cup L)\Big) = \bigcup_{D\subset_f B}\Big(\bigcup_{x\in A}\varphi_x(\mathscr{C}(\varphi(D); A, S\cup L)\Big) =$$

$$= \bigcup_{D\subset_f B}\mathscr{C}(D; \mathscr{F}(A), S\cup L) = \mathscr{S}(B; \mathscr{F}(A), S\cup L).$$ □

Now by intersecting $A$ with each of the expressions in Lemma 13 we have:

Corollary 2. $\mathscr{S}(\varphi(B); A, S\cup L)=\mathscr{S}(B; \mathscr{F}(A), S\cup L)\cap A$.

Definition 8. If $\mathscr{R}=\langle A;f\rangle$ is a representation of $S$ and $L$, then we write $\mathrm{St}_2\,\mathscr{R}$ or $\mathrm{St}_2\,\langle A;f\rangle$ to mean $\mathrm{St}_2$ holds for the corresponding triple (see Definition 2): $\mathrm{St}_2(A, \{f; f\in S\}, \{f(A); f\in L\})$.

Lemma 14. *If $\mathscr{R}=\langle A;f\rangle$ is a representation of $S$ and $L$ with $\mathrm{St}_2\,\mathscr{R}$, and $\mathscr{F}(\mathscr{R})=\langle\mathscr{F}(A),f\rangle$ is the foliation of $\mathscr{R}$, then $\mathrm{St}_2\,\mathscr{F}(\mathscr{R})$.*

Proof. Let $\mathscr{S}(B;\mathscr{F}(A), S\cup L)=B$; then $\mathscr{S}(\varphi(B); A, S\cup L)=\varphi(B)$ (otherwise $A\cap B\subset\varphi(B)\subsetneqq\mathscr{S}(\varphi(B); A, S\cup L)\subset\mathscr{S}(B;\mathscr{F}(A), S\cup L)\cap A=B\cap A$). Hence there is $p\in L$ with $\varphi(B)=\text{range}\,p$ in $A$. Then:

$$B=\mathscr{S}(B;\mathscr{F}(A), S\cup L)=\bigcup_{x\in A}\varphi_x(\mathscr{S}(\varphi(B); A, S\cup L))=\bigcup_{x\in A}\varphi_x\varphi(B)=$$

$$=\bigcup_{x\in A}\varepsilon_x v_x\varphi(B)=\bigcup_{x\in A}\varepsilon_x\varphi(B)=\bigcup_{x\in A}\varepsilon_x(\text{range } p \text{ in } A)=\text{range } p \text{ in } \mathscr{F}(A).$$

Thus $\mathrm{St}_2\,\mathscr{F}(\mathscr{R})$ holds. □

Lemma 15. *If $h\in\overline{S\cup L}^{\mathscr{F}(A)}$, then $m=h\restriction A\in\overline{S\cup L}^{A}$, and for all $a_x\in\mathscr{F}(A)$, $h(a_x)=(ma)_x$ if $m(a)\notin[x]$ and $h(a_x)=m(a)$ otherwise.*

Proof. By Lemma 7 and Lemma 11 $h=m\cup(\bigcup_{x\in A}h_x)$ with $m\in\overline{S\cup L}^{A}$, and $h\restriction a_x=h_x\in\overline{S\cup L}^{A_x}$. Now to each $a_x\in\mathscr{F}(A)$ there is a system $\Sigma$, such that $h$ is the unique solution to $\Sigma$ on $\{a, a_x\}$. Thus $m$ is the unique solution to $\Sigma$ at $a$ and $h_x$ is the unique solution to $\Sigma$ at $a_x$. Note $m$ is a solution to $\Sigma$ at $a$ and $\varphi_x$ is a homomorphism, thus by Lemma 6, $\varphi_x m$ is a solution to $\Sigma$ at $a_x=\varphi_x(a)$. But $h$ is the unique solution to $\Sigma$ at $a_x$, thus $h(a_x)=\varphi_x(ma)=\varepsilon_x v_x(ma)=\varepsilon_x(ma)$. Hence if $ma\notin[x]$, $h(a_x)=(ma)_x$ and if $ma\in[x]$, $h(a_x)=ma$. □

Corollary 3. *If $h\in\overline{S\cup L}^{\mathscr{F}(A)}$ and if $(h\restriction A)\in S$ on $\mathscr{R}$ then $h\in S$ on $\mathscr{F}(\mathscr{R})$.*

Definition 9. If $\mathscr{R}=\langle A;f\rangle$ is a representation of $S$ and $L$ and $h\in A^A$, then we write $h$ is in the *one closure* of $S$ in $\mathscr{R}$ (or shortly $h\in\mathrm{oc}\,(S)_{\mathscr{R}}$ or $h\in\mathrm{oc}(S)$) if for each $a\in A$ there exists $f\in S$ with $h(a)=f(a)$. Local closure of $S$ is denoted by $\mathrm{l.c.}(S)$.

Lemma 16. *If $h\in\overline{S\cup L}^{\mathscr{F}(A)}$, then $m=(h\restriction A)$ is in the one closure of $S$ in $\mathscr{R}$.*

Proof. Assume there is $a\in A$ such that for all $f\in S$ $f(a)\neq m(a)=h(a)$. Then there exists a system $\Sigma$ of equations, whose unique solution at $a$ is $m(a)\notin[a]$. The unique solution of $\Sigma$ at $\varphi_a(a)=a$ is $\varphi_a(m(a))=(m(a))_a\neq m(a)$ which is a contradiction. □

Definition 10. The representation $\mathscr{R}=\langle A,f\rangle$ of $S$ and $L$ on $A$ is *algebraic,* if the corresponding triple $(A; S, L)$ is *algebraic*.

Definition 11. $\mathrm{oc}\,\mathscr{F}(\mathscr{R})=\langle\mathscr{F}(A);f\rangle_{f\in\mathrm{oc}(S)\cup L}$ where $\mathscr{R}=\langle A;f\rangle_{f\in S\cup L}$ a representation of $S$ and $L$ on $A$ and the action of the operations in $\mathrm{oc}\,(S)\cup L$ are as determined in $\overline{S\cup L}^{\mathscr{F}(A)}$.

Lemma 17. *If the representation* $(A;\,S,L)$ *has each compact* $t\in L$ *singleton generated, then* $(\mathscr{F}(A);\,S,L)$ *also has each compact* $t\in L$ *singleton generated.*

Proof. Observe that for all $a\in A$, we have for each $p\in L$ $\exists x[a_x\in p$ in $(\mathscr{F}(A);\,S,L)]$ iff $[a\in p$ on $(\mathscr{F}(A);\,S,L)]$ iff $\forall x[a_x\in p$ in $(\mathscr{F}(A);\,S,L)]$.

Lemma 18. *Let* $(B;\,S,L)$ *satisfy* $\mathrm{St}_2$. *Suppose* $a,b\in B$ *are such that for every* $p\in L$ $[a\in p\Rightarrow b\in p]$. *Then each system of equations* $\Sigma$ *over* $S\cup L$ *which has a solution at* $a$ *also has a solution at* $b$.

Proof. Let $\Sigma$ be a system of equations over $S\cup L$ which has a solution at $a$. $\mathrm{Spt}\,\Sigma$ denotes the set of all points in $B$ on which $\Sigma$ has a solution. Clearly $\mathrm{Spt}\,\Sigma=\bigcup_{D\subset_f \mathrm{Spt}\,\Sigma}\ \bigcap_{D\subseteq \mathrm{Spt}\,\Gamma}\mathrm{Spt}\,\Gamma=\mathscr{S}(\mathrm{Spt}\,\Sigma;\,B,\,S\cup L)$ hence by $\mathrm{St}_2\,(B;\,S,L)$, $\mathrm{Spt}\,\Sigma\in L$.

Hence $b\in\mathrm{Spt}\,\Sigma$ as required. □

Lemma 19. *Given* $(B;\,S,L)$ *which satisfies* $\mathrm{St}_2$ *and for which each compact* $t\in L$ *is singleton generated, if* $h\in\overline{S\cup L}^{B}$ *and* $h\in\mathrm{oc}\,(S)$ *on* $(B;\,S,L)$ *then* $h\in\mathrm{l.c.}(S)$ on $(B;\,S,L)$.

Proof. Fix $\{b_1,\dots,b_n\}\subset_f B$. Let $p\in L$ be generated by $\{b_1,\dots,b_n\}$; thus $p$ is compact, and there exists $b\in B$ which generates $p$ as well. Let $\Sigma$ be a system of equations with coefficients from $S\cup L$ such that $h$ is the unique solution on $\{b,b_1,b_2,\dots,b_n\}$. Since $h\in\mathrm{oc}\,(S)$ there is some $f\in S$ with $f(b)=h(b)$. Hence $h$ is also the unique solution on $\{b\}$ to the system $\Gamma=\Sigma\cup\{fx_0=x_1\}$. By Lemma 18 $\Gamma$ has also a solution on each $b_i$, $i=1,\dots,n$. But $\Gamma\supseteq\Sigma$ so the solution to $\Gamma$ on $\{b_1,\dots,b_n\}$ is $h$. On the other hand $(fx_0=x_1)\in\Gamma$ hence the solution to $\Gamma$ on $\{b_1,\dots,b_n\}$ is $f$. Thus $f(b_i)=h(b_i)$ for $i=1,\dots,n$, so $h\in\mathrm{l.c.}(S)$ as required. □

Lemma 20. *Let* $N$ *be a monoid and* $L$ *an algebraic lattice such that* $(A;N,L)$ *with* $\mathrm{St}_2\,(A;N,L)$, *then if* $S$ *is a submonoid of* $N$ *we have* $\mathrm{St}_2\,(A;\,S,L)$.

Proof. Clearly $\mathscr{C}(D;A,N\cup L)\subset\mathscr{C}(D;A,\,S\cup L)$ and hence for each $B\subset A$, $B\subset\mathscr{S}(B;A,N\cup L)\subset\mathscr{S}(B;A,\,S\cup L)$. So if $B=\mathscr{S}(B;A,\,S\cup L)$ we get $B==\mathscr{S}(B;A,N\cup L)$ and then $B\in L$ in $(A;N,L)$. □

Theorem 1. *If* $(A;N,L)$ *is algebraic and each compact* $t\in L$ *is singleton generated in that representation then for each submonoid* $S\subseteq N$ *we have* $(\mathscr{F}(A);\,\mathrm{l.c.}(S),L)$ *is algebraic, where* $\mathrm{l.c.}(S)$ *is the local closure of* $S$ *in the representation* $(\mathscr{F}(A);\,S,L)$.

Proof. Let $(A; N, L)$ satisfy the hypothesis of the theorem and let $S$ be a submonoid of $N$. By Lemma 20 $(A; S, L)$ satisfies $St_2$, and clearly each compact $t \in L$ is singleton generated in $(A; S, L)$ as well. By Lemmas 14 and 17 $(\mathscr{F}(A); S, L)$ also satisfies $St_2$ and each compact $t \in L$ is singleton generated in that representation. Furthermore by Lemma 5 $(\mathscr{F}(A); \overline{S \cup L}^{\mathscr{F}(A)}, L)$ is algebraic, and here again each compact $t \in L$ is singleton generated. We claim that $\overline{S \cup L}^{\mathscr{F}(A)} = \text{l.c.}(S)$, the local closure of $S$ in $(\mathscr{F}(A); S, L)$; this will establish the result of the Theorem. Evidently $\overline{S \cup L}^{\mathscr{F}(A)} \supseteqq \text{l.c.}(S)$ so really only the other containment need be argued. Let $h \in \overline{S \cup L}^{\mathscr{F}(A)}$. Note $h \restriction A \in \text{oc}(S)$ in $(\mathscr{F}(A); S, L)$, since by Lemma 16 we have $m = h \restriction A \in \text{oc}(S)$ in $(A; S, L)$. In fact $h \in \text{oc}(S)$ in $(\mathscr{F}(A); S, L)$. To see that we need only check $h(a_x)$ for $a_x \in \mathscr{F}(A)$. If $h(a) \notin [x]$ we get $h(a_x) = (h(a))_x = (f(a))_x$ for some $f \in S$ and if $h(a) \in [x]$ we get $h(a_x) = ha = fa = f(a_x)$ for some $f \in S$ by use of Lemma 15 and the definition of action by $S$ in $\mathscr{F}(A)$ (see Defn. 6). Now apply Lemma 19 with $(B; S, L) = (\mathscr{F}(A); S, L)$ to get $h \in \overline{S \cup L}^{\mathscr{F}(A)} \cap \text{oc}(S) \Rightarrow h \in \text{l.c.}(S)$ on $(\mathscr{F}(A); S, L)$ as required. □

Lemma 21. *The local closure of any finite monoid $S$ is equal to $S$.*

Proof. Let the monoid $S$ be represented on some set $A$ and assume that $h \in$ local closure $S$ and $h \notin S$. For each $f \in S$ let $a_f \in A$ be such that $h(a_f) \neq f(a_f)$ then $D = \{a_f; f \in S\}$ is finite and clearly $h \restriction D \neq f \restriction D$ for any $f \in S$, contrary to the selection of $h$ in the local closure of $S$. Hence each $h$ in local closure $S$ also belongs to $S$. □

Theorem 2. *For each universal algebra $\mathfrak{A}$ there is a universal algebra $\mathfrak{B}$ satisfying* $\text{End}\,\mathfrak{A} \cong \text{End}\,\mathfrak{B}$ *and* $\text{Su}\,\mathfrak{A} = \text{Su}\,\mathfrak{B}$; *moreover every finitely generated subalgebra of $\mathfrak{B}$ is generated by a single element.*

Proof. Let $\mathfrak{A} = \langle A, F \rangle$, $S = \text{End}\,\mathfrak{A}$ and $L = \text{Su}\,\mathfrak{A}$. For any $C \subseteqq A$ we set $C^* = \bigcup_{n=1}^{\infty} C^n$. (Remark that we do not distinguish between $C$ and $C^1$ and thus $C \subseteqq C^*$.) With any $\varphi \in S$ we associate a transformation $\varphi^*: A^* \to A^*$ defined by $\varphi^*((x_1, \ldots, x_k)) = (\varphi(x_1), \ldots, \varphi(x_k))$, $(x_1, \ldots, x_k) \in A^*$. Let $S^* = \{\varphi^* | \varphi \in S\}$ and $L^* = \{C^* | C \in L\}$. Then $S^* \cong S$ and $L^* \cong L$. We shall construct an algebra $\mathfrak{B} = \langle A^*, G \rangle$ such that $S^* = \text{End}\,\mathfrak{B}$, $L^* = \text{Su}\,\mathfrak{B}$ and every finitely generated subalgebra of $\mathfrak{B}$ is generated by a single element.

Let $g_1, g_2$ be unary operations and $h$ a binary operation on $A^*$ defined by the rules:

$$g_1((x_1, \ldots, x_k)) = x_1, \quad g_2((x_1, \ldots, x_k)) = (x_k, x_1, \ldots, x_{k-1})$$

and

$$h((x_1, \ldots, x_k), (y_1, \ldots, y_l)) = (x_1, \ldots, x_k, y_1, \ldots, y_l)$$

for every $(x_1, \ldots, x_k)$, $(y_1, \ldots, y_l) \in A^*$. Furthermore, with each operation $f \in F$ we associate an operation $f_{\mathfrak{B}}$ on $A^*$ as follows. The arity of $f_{\mathfrak{B}}$ equals the one of $f$ and $f_{\mathfrak{B}}$ is defined by

$$f_{\mathfrak{B}}((x_1^1, \ldots, x_{k_1}^1), \ldots, (x_1^n, \ldots, x_{k_n}^n)) = f(x_1^1, \ldots, x_1^n),\ (x_1^i, \ldots, x_{k_i}^i) \in A^*, \quad i = 1, \ldots, n.$$

Now set $G = \{f_{\mathfrak{B}} | f \in F\} \cup \{g_1, g_2, h\}$.

First consider End $\mathfrak{B}$. It is clear that $S^* \subseteq \mathrm{End}\, \mathfrak{B}$. Let $\Phi \in \mathrm{End}\, \mathfrak{B}$. If $x \in A$ then $\Phi(x) = \Phi(g_1(x)) = g_1(\Phi(x)) \in A$ showing that $\Phi \restriction A = \varphi \in A^A$. Furthermore, if $f \in F$ is $n$-ary and $x_1, \ldots, x_n \in A$, then $\varphi(f(x_1, \ldots, x_n)) = \Phi(f_{\mathfrak{B}}(x_1, \ldots, x_n)) = f_{\mathfrak{B}} \cdot (\Phi(x_1), \ldots, \Phi(x_n)) = f(\varphi(x_1), \ldots, \varphi(x_n))$, i.e. $\Phi \restriction A = \varphi \in \mathrm{End}\, \mathfrak{A} = S$. Now we show by induction on $k$ that (1) $\Phi((x_1, \ldots, x_k)) = (\varphi(x_1), \ldots, \varphi(x_k))$, $(x_1, \ldots, x_k) \in A^*$. If $k = 1$ then (1) holds. Supppose (1) holds for $k-1$. Then $\Phi((x_1, \ldots, x_k)) = \Phi(h((x_1, \ldots, x_{k-1}), x_k)) = h(\Phi((x_1, \ldots, x_{k-1}), \Phi(x_k)) = h((\varphi(x_1), \ldots, \varphi(x_{k-1})), \varphi(x_k)) = (\varphi(x_1), \ldots, \varphi(x_k))$. Hence $\Phi = \varphi^* \in S^*$.

Now consider Su $\mathfrak{B}$. It is clear that $L^* \subseteq \mathrm{Su}\, \mathfrak{B}$. Let $B \in \mathrm{Su}\, \mathfrak{B}$. Taking into account that $g_1$, $g_2$ and $h$ are operations of $\mathfrak{B}$, one can show that $B = (B \cap A)^*$. Furthermore, $B \cap A \in \mathrm{Su}\, \mathfrak{A} = L$. $B = (B \cap A)^* \in L^*$. Finally, if a subalgebra $B$ of $\mathfrak{B}$ is generated by the elements $(x_1^1, \ldots, x_{k_1}^1), \ldots, (x_1^s, \ldots, x_{k_s}^s) \in A^*$ then $B$ is also generated by $(x_1^1, \ldots, x_{k_1}^1, \ldots, x_1^s, \ldots, x_{k_s}^s) \in A^*$ which completes the proof. □

Corollary 4. *If the monoid $N$ and the algebraic lattice $L$ are jointly algebraic and $S$ is a finite submonoid of $N$, then $S$ and $L$ are jointly algebraic.*

Proof. Let $(A; N, L)$ be algebraic, with each compact $t \in L$ singleton generated in that representation. By Theorem 1 $(\mathcal{F}(A); \mathrm{l.c.}(S), L)$ is algebraic. By Lemma 21 $\mathrm{l.c.}(S) = S$ since $S$ is finite, hence $(\mathcal{F}(A); S, L)$ is algebraic and $S$ and $L$ are (abstractly) jointly algebraic. □

Corollary 5. *If $S \subset T$ are two monoids and if $L$ is an algebraic lattice for which the highest element 1 is compact and if $T$ and $L$ are jointly algebraic, then $S$ and $L$ are jointly algebraic.*

Proof. Let $\mathfrak{A} = \langle A; \mathcal{P} \rangle$ be such that $L = \mathrm{Su}\, \mathfrak{A}$ and $T = \mathrm{End}\, \mathfrak{A}$. We may assume each compact $t \in L$ is singleton generated in $\mathfrak{A}$. For the triple $(A; T, L)$ given by $\mathfrak{A}$ we have $(\mathcal{F}(A); \mathrm{l.c.}(S)., L)$ algebraic. In fact by Lemma 17 each compact $t \in L$ is singleton generated in this representation. In particular $1 \in L$ which is compact by hypothesis is singleton generated. It follows that $\mathrm{l.c.}(S) = S$ in that representation, hence $(\mathcal{F}(A); S, L)$ is algebraic and $S, L$ are (abstractly) jointly algebraic. □

Corollary 6. *If the monoid $T$ and the algebraic lattice $L$ are jointly algebraic but not both infinite then every submonoid of $T$ is jointly algebraic with $L$.*

Proof. Follows now immediately from Corollaries 4 and 5. □

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