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Best approximation in Banach spaces with unconditional Schauder decompositions

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1. Introduction. The problem of the best approximation in Banach spaces with bases was initiated by NIKOLSKII [4]. SINGER [7, 8] carried out analogous study for the spaces with unconditional bases which has been further continued by RETHERFORD [5] and RETHERFORD and JAMES [6]. It has been pointed out that the results in these two settings are oftenly different. Motivated by this work and keeping in mind that a Banach space does not necessarily possess a basis as encountered by ENFLO [1], we consider Banach spaces with unconditional Schauder decomposition. In section 2, we give the notations and terminology. In section 3, the notions of NT-, NK- and NTK-norms have been defined in terms of the best approximation and a characterisation of each of these norms has been obtained. Also, it has been shown that every NT-norm is an NK-norm whereas the converse is not true is ascertained by giving a counterexample. Finally, it has been shown in section 4, that it is always possible to introduce an equivalent NTK-norm on a Banach space having an unconditional Schauder decomposition.

2. Notations and terminology. Let E be a Banach space, Z a linear subspace of E and x an element of E. An element $z_0 \in Z$ is a best approximation of x from Z provided

$$\|x-z_0\| = \inf\{\|x-z\|:\ z\in Z\}.$$

Thus, to every linear subspace Z of E and an element $x\in E$, there corresponds a bounded, closed and convex (possibly empty) set $B_Z(x)=\{z_0\in Z: z_0$ is a best approximation of $x\}$. We denote by π_Z the mapping of E into Z given by $\pi_Z(x)=z_0$ provided $B_Z(x)=\{z_0\}$.

A sequence (M_i) of nontrivial subspaces of E, is called a decomposition of E provided for each $x\in E$ there exists a unique sequence (x_i) such that $x_i\in M_i$

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and $x=\sum_{i=1}^{\infty} x_i$, the convergence being in the norm topology of E. It is possible to define for each i a projection $P_i\colon E\to M_i$ as $P_i(x)=x_i$. If each P_i is continuous, then (M_i) is called a Schauder decomposition, and we write (M_i, P_i). A decomposition (M_i, P_i) is said to be unconditional Schauder if it is Schauder with the property that $x=\sum_{i=1}^{\infty} x_{p(i)}$, for each permutation p of ω (the positive integers).

Let Σ denote the family of all finite subsets of ω. For $\sigma\in\Sigma$, let

$$L_\sigma = [\bigcup_{i\in\sigma} M_i] \quad\text{and}\quad L^\sigma = [\bigcup_{i\in\omega\setminus\sigma} M_i],$$

where the bracketed expressions denote the closed linear spans of the indicated sets. Also, we put

$$S_\sigma(x) = \sum_{i\in\sigma} P_i(x) \quad\text{and}\quad S^\sigma(x) = x - S_\sigma(x).$$

3. NT- and NK-norms. *Definition.* Let (M_i, P_i) be an unconditional Schauder decomposition of E. Then the norm $\|\ \|$ on E is called an *NT-norm* with respect to (M_i, P_i) if for every $x\in E$ and $\sigma\in\Sigma$, there exists a unique $y_0=\pi_{L_\sigma}(x)\in L_\sigma$, best approximation of x from L_σ, such that $\pi_{L_\sigma}(x)=S_\sigma(x)$; *NK-norm* with respect to (M_i, P_i) if for every $x\in E$ and $\sigma\in\Sigma$, there exists a unique $y_0=\pi_{L^\sigma}(x)\in L^\sigma$, best approximation of x from L^σ, such that $\pi_{L^\sigma}(x)=S^\sigma(x)$; and *NTK-norm* with respect to (M_i, P_i) if it is simultaneously an NT-norm and NK-norm with respect to this decomposition.

Now we characterise these norms as follows:

Theorem 1. *Let E be a Banach space with an unconditional Schauder decomposition (M_i, P_i). Then the norm on E is an*

(a) *NT-norm if and only if*

$$\|\sum_{i\in\omega\setminus\beta} x_i\| < \|\sum_{i\in\omega\setminus\alpha} x_i\|, \tag{3.1}$$

for every pair $\alpha, \beta\in\Sigma$ with $\alpha\subset\beta$ and every sequence $(x_i)_{i\in\omega\setminus\alpha}$ with $x_i\in M_i$ and $\sum_{i\in\beta\setminus\alpha} x_i\neq 0$, for which the series in (3.1) *are convergent*;

(b) *NK-norm if and only if*

$$\|\sum_{i\in\alpha} x_i\| < \|\sum_{i\in\beta} x_i\|, \tag{3.2}$$

for every pair $\alpha, \beta\in\Sigma$ with $\alpha\subset\beta$ and every finite sequence $(x_i)_{i\in\beta}$ with $x_i\in M$ and $\sum_{i\in\beta\setminus\alpha} x_i\neq 0$.

Proof. (a) Assume that the norm on E is an NT-norm. Let $\alpha, \beta\in\Sigma$ with $\alpha\subset\beta$ be arbitrary such that $\sum_{i\in\omega\setminus\alpha} x_i$ is convergent. Then

$$\pi_{L_\beta}(\sum_{i\in\omega\setminus\alpha} x_i) = S_\beta(\sum_{i\in\omega\setminus\alpha} x_i) = \sum_{i\in\beta\setminus\alpha} x_i,$$

and so

$$\|\sum_{i\in\omega\setminus\beta} x_i\| = \|\sum_{i\in\omega\setminus\alpha} x_i - \pi_{L_\beta}(\sum_{i\in\omega\setminus\alpha} x_i)\| < \|\sum_{i\in\omega\setminus\alpha} x_i\|,$$

which verifies the necessary part.

Conversely, for every $x=\sum_{i\in\omega} x_i\in E, \sigma\in\Sigma$ and $y=\sum_{i\in\sigma} y_i\in L_\sigma$ with $y\neq S_\sigma(x)$, we have (by (3.1) with $\beta=\sigma, \alpha=\emptyset$)

$$\|x-S_\sigma(x)\| = \|\sum_{i\in\omega\setminus\sigma} x_i\| < \|\sum_{i\in\omega\setminus\sigma} x_i - \sum_{i\in\sigma}(y_i-x_i)\| = \|x-y\|,$$

and thus the norm on E is an NT-norm.

(b) Assume that the norm on E is an NK-norm. Let $(x_i)_{i\in\beta}$ be a finite sequence and let $\alpha\subset\beta$ be such that $\sum_{i\in\beta\setminus\alpha} x_i\neq 0$. Then

$$\pi_{L^\alpha}(\sum_{i\in\beta} x_i) = \sum_{i\in\beta} x_i - S_\alpha(\sum_{i\in\beta} x_i) = \sum_{i\in\beta\setminus\alpha} x_i.$$

Hence

$$\|\sum_{i\in\alpha} x_i\| = \|\sum_{i\in\beta} x_i - \pi_{L^\alpha}(\sum_{i\in\beta} x_i)\| < \|\sum_{i\in\beta} x_i\|.$$

In order to establish the converse part, let $x=\sum_{i\in\omega} x_i$, $\sigma\in\Sigma$ and $y=\sum_{i\in\omega\setminus\sigma} y_i\in L^\sigma$ with $y\neq S^\sigma(x)$ be arbitrary. Then there exists in $\omega\setminus\sigma$ a smallest index i_0, such that $y_{i_0}\neq x_{i_0}$. Hence, applying (3.2) successively, we obtain

$$\|x-S^\sigma(x)\| = \|\sum_{i\in\sigma} x_i\| = \|\sum_{i\in\sigma} x_i - \sum_{\substack{i\in\omega\setminus\sigma\\ i<i_0}}(y_i-x_i)\| < \|\sum_{i\in\sigma} x_i - \sum_{\substack{i\in\omega\setminus\sigma\\ i\leq i_0}}(y_i-x_i)\| \leq \dots$$

$$\dots \leq \|\sum_{i\in\sigma} x_i - \sum_{i\in\omega\setminus\sigma}(y_i-x_i)\| = \|x-y\|,$$

and thus the norm on E is an NK-norm. This completes the proof of the theorem.

Further, we give the relation between NT- and NK-norms.

Theorem 2. *Let E be a Banach space with an unconditional Schauder decomposition (M_i, P_i). Then every NT-norm with respect to (M_i) is an NK-norm (whence also an NTK-norm) with respect to (M_i).*

Proof. It follows by using (3.1) and (3.2).

The converse of Theorem 2 is not necessarily true. Consider for instance, the Banach space

$$c_0(\chi) = \{\bar{x} = (x_i)\colon x_i\in\chi,\ \operatorname*{Lim}_{i\to\infty} x_i = 0 \text{ in the norm of } \chi\},$$

the norm on $c_0(\chi)$ being given by $|(x_i)|=\sup_i \|x_i\|$, where $(\chi, \|\ \|)$ is any Banach space. On $c_0(\chi)$, define another norm $\|\ \|^*$ as:

$$\|(x_i)\|^* = \sup_{2\leq n<\infty}\ \sup_{p\in\pi_{1,n}} \left(2^{-n}\|x_1\|/n + \sum_{i=2}^{\infty} 2^{-i}\|x_{p(i)}\|\right),$$

where $\pi_{1,n}$ denote the collection of all permutations of the set $\{2, 3, \ldots, n-1, n+1, n+2, \ldots\}$ and $p(n)=n$ for every $p\in\pi_{1,n}$. The norms $\| \ \|$ and $\| \ \|^*$ are equivalent since $\frac{1}{8}\|x\| \leqq \|x\|^* \leqq \frac{5}{8}\|x\|$. We observe that the sequence (N_i) with $N_i = \{\delta_i^{x_i}: x_i\in\chi\}$, where $\delta_i^{x_i}$ we mean the sequence $(0, 0, \ldots, x_i, 0, \ldots)$, i.e. the ith entry in $\delta_i^{x_i}$ is x_i and all others are zero, forms an unconditional Schauder decomposition of $c_0(\chi)$ (see [2], p. 291 and [3], p. 95). Let $\alpha, \beta\in\Sigma, \alpha\subset\beta$ and $(\delta_i^{x_i})_{i\in\beta}$ with $\sum_{i\in\beta\setminus\alpha}\delta_i^{x_i}\neq 0$ be a finite sequence. Then

$$\|\sum_{i\in\alpha}\delta_i^{x_i}\|^* < \|\sum_{i\in\beta}\delta_i^{x_i}\|^*,$$

hence the norm $\| \ \|^*$ on $c_0(\chi)$ is an NK-norm. To show that $\| \ \|^*$ is not an NT-norm, it is enough to establish that

$$\left\|\sum_{m=1}^{\infty}\delta_m^{x_m}\right\|^* = \left\|\sum_{m=2}^{\infty}\delta_m^{x_m}\right\|^*, \tag{3.3}$$

with $x_m = \frac{1}{m}x$ for any $0\neq x\in\chi$.

Obviously

$$\left\|\sum_{m=1}^{\infty}\frac{1}{m}\delta_m^{x}\right\|^* \geqq \left\|\sum_{m=2}^{\infty}\frac{1}{m}\delta_m^{x}\right\|^*. \tag{3.4}$$

Furthermore, let $n\geqq 2$ be fixed. If, for a pair $i, i+j\in\{2, 3, \ldots, n-1, n+1, \ldots\}$ and a $p\in\pi_{1,n}$, we have $\frac{1}{p(i)} < \frac{1}{p(i+j)}$, then for the permutation $p'\in\pi_{1,n}$ defined by

$$p'(i) = p(i+j), \quad p'(i+j) = p(i), \quad p'(k) = p(k) \quad (k\neq i, \ i+j)$$

we have

$$\sum_{\substack{m=2\\ m\neq n}}^{\infty}\frac{\|x\|}{p'(m)2^m} > \sum_{\substack{m=2\\ m\neq n}}^{\infty}\frac{\|x\|}{p(m)2^m},$$

since

$$\frac{a}{2^i}+\frac{b}{2^{i+j}} > \frac{b}{2^i}+\frac{a}{2^{i+j}}, \quad \text{for } a > b \geqq 0.$$

Consequently, for every $n\geqq 2$ and $p\in\pi_{1,n}$, we have

$$\frac{\|x\|}{n2^n}+\sum_{\substack{m=2\\ m\neq n}}^{\infty}\frac{\|x\|}{p(m)2^m} \leqq \frac{\|x\|}{n2^n}+\sum_{\substack{m=2\\ m\neq n}}^{\infty}\frac{\|x\|}{m2^m} = \sum_{m=2}^{\infty}\frac{\|x\|}{m2^m} = \sup_{2\leqq n<\infty}\sum_{\substack{m=2\\ m\neq n}}^{\infty}\frac{\|x\|}{m2^m} \leqq$$

$$\leqq \sup_{2\leqq n<\infty}\ \sup_{\tau\in\pi_{1,n}}\sum_{\substack{m=2\\ m\neq n}}^{\infty}\frac{\|x\|}{\tau(m)2^m} = \left\|\sum_{m=2}^{\infty}\frac{1}{m}\delta_m^{x}\right\|^*,$$

which together with (3.4) implies (3.3).

4. An NTK-norm. If E is a Banach space with an unconditional Schauder decomposition (M_i, P_i) then it is always possible to introduce on E an NTK-norm equivalent to the original norm on E. Consider for instance

$$\|x\|_{\mathrm{NTK}} = \sum_{i\in\omega} \|P_i(x)\| 2^{-i} + \sup_{\sigma\in\omega} \Big\|\sum_{i\in\sigma} P_i(x)\Big\|.$$

This clearly defines a norm on E, and is equivalent to the original norm on E which follows from

$$\|x\| \leqq \|x\|_{\mathrm{NTK}} \leqq \max_{1\leqq i<\infty} \|P_i(x)\| + \sup_{\sigma\in\omega} \Big\|\sum_{i\in\sigma} P_i(x)\Big\| \leqq 3K\|x\|.$$

Finally, let $\alpha, \beta\in\Sigma$ with $\alpha\subset\beta$ and $(x_i)_{i\in\omega\setminus\beta}$ with $\sum_{i\in\beta\setminus\alpha} x_i \neq 0$, $x_i\in M_i$, $i\in\omega$, be such that $\sum_{i\in\omega\setminus\beta} x_i$ converges. Then, we have $\omega\setminus\alpha=(\omega\setminus\beta)\cup(\beta\setminus\alpha)$, hence

$$\Big\|\sum_{i\in\omega\setminus\beta} x_i\Big\|_{\mathrm{NTK}} = \sum_{i\in\omega\setminus\beta} \|x_i\| 2^{-i} + \sup_{\sigma\in\Sigma} \Big\|\sum_{i\in\sigma\cap(\omega\setminus\beta)} x_i\Big\| <$$
$$< \sum_{i\in\omega\setminus\alpha} \|x_i\| 2^{-i} + \sup_{\sigma\in\Sigma} \Big\|\sum_{i\in\sigma\cap(\omega\setminus\alpha)} x_i\Big\| = \Big\|\sum_{i\in\omega\setminus\alpha} x_i\Big\|_{\mathrm{NTK}}.$$

Thus by Theorem 1, $\|\ \|_{\mathrm{NTK}}$ is an NT-norm and hence an NTK-norm by Theorem 2.

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