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On a partial solution of the transitive algebra problem

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Let $B(H)$ denote the Banach algebra of all bounded linear operators on an infinite-dimensional separable complex Hilbert space H. A subalgebra $\mathcal{A}$ of $B(H)$ is called transitive if it is weakly closed, contains the identity operator and its only invariant subspaces are $\{0\}$ and H. $B(H)$ is obviously transitive. Whether there exists any other transitive algebra is a well known open problem, the so-called 'transitive algebra problem'. The problem was first raised by KADISON [5] and it continues to be still unsolved. However, partial solutions of the problem have been obtained by many mathematicians; see, for example, ARVESON [1], BARNES [2], DOUGLAS and PEARCY [3], NORDGREN [8], NORDGREN, RADJAVI and ROSENTHAL [9], and RADJAVI and ROSENTHAL [10], [11]. The first such solution was given by ARVESON [1] who proved that if a transitive algebra $\mathcal{A}$ contains a maximal abelian self-adjoint algebra, then $\mathcal{A}=B(H)$. In the same paper, he also proved that $B(H)$ is the only transitive algebra containing a simple unilateral shift. By using Arveson's techniques, NORDGREN, RADJAVI and ROSENTHAL [9] have shown that a transitive algebra containing a Donoghue operator (backward weighted shift with a monotone decreasing and square-summable weight sequence) equals $B(H)$. The purpose of this note is to go a step further in this direction and show that every transitive algebra containing a certain type of weighted shift, more general than a Donoghue operator, coincides with $B(H)$. Our result assumes significance in the light of the conjecture that every transitive algebra containing a weighted shift is equal to $B(H)$.

We shall denote by $H^{(n)}$ the direct sum of n copies of H, and by $A^{(n)}$ the operator on $H^{(n)}$ which is the direct sum of n copies of A.

Let $\{w_k\}_{k=1}^{\infty}$ be a bounded sequence of non-zero complex numbers and let $\{e_k\}_{k=0}^{\infty}$ be an orthonormal basis of H. The operator T on H defined by the requirement

$$Te_0 = 0 \quad \text{and} \quad Te_k = w_k e_{k-1} \quad (k = 1, 2, \ldots)$$

is called a *weighted unilateral* (backward) *shift* with the weight sequence $\{w_k\}_{k=1}^{\infty}$.

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We may and shall assume, without any loss of generality, that the weights w_k are positive real numbers [4]. In this case, $\{w_k\}_{k=1}^{\infty}$ is said to be of bounded pth-power variation if

$$\sum_{k=1}^{\infty} |w_k - w_{k+1}|^p < \infty.$$

(For $p=1$, we simply say "bounded variation".)

The following theorem is an important tool to obtain our results:

Theorem A. [9, Corollary 1] *If a transitive algebra $\mathscr{A}$ contains an operator A such that*

(i) *every eigenspace of A is one-dimensional, and*
(ii) *for every n, each non-trivial invariant subspace of $A^{(n)}$ contains an eigenvector of $A^{(n)}$,*

then $\mathscr{A}=B(H)$.

In the rest of this paper, $\mathscr{A}$ will denote a transitive algebra containing a weighted unilateral shift T with the weight sequence $\{w_k\}_{k=1}^{\infty}$. Our first result is

Theorem 1. *If $\{w_k\}_{k=1}^{\infty}$ is of bounded variation and*

$$\delta = \delta(n) = \sum_{k=0}^{\infty} \left(\frac{w_{k+2} \dots w_{k+n}}{w_2 \dots w_n}\right)^2 < \infty \tag{1}$$

for all $n \geqq 2$, then $\mathscr{A}=B(H)$.

Proof. We know that there is a disc of eigenvalues for a backward weighted shift, but they are all of multiplicity one. Thus T satisfies condition (i) of Theorem A. Next, let $(x_1, x_2, \dots, x_n)$ be a non-zero element of a non-zero invariant subspace M of $T^{(n)}$ and let

$$x_j = \sum_{i=0}^{\infty} x_{ij} e_i, \quad 1 \leqq j \leqq n.$$

If, for each j, the sequence $\{x_{ij}\}_{i=0}^{\infty}$ has only finitely many non-zero terms, then the invariant subspace of $T^{(n)}$ generated by $(x_1, x_2, \dots, x_n)$ is finite-dimensional and thus contains an eigenvector. We therefore assume, without loss of generality, that for every $m \geqq 0$, there is a number $r=r(m) \geqq m$ and a number $s=s(m)$, $1 \leqq s(m) \leqq n$, such that

$$|x_{r,s}| = \max_{i \geqq m; 1 \leqq j \leqq n} \{|x_{ij}|\} > 0. \tag{2}$$

Now, for a given integer m, we have

$$\frac{(T^{(n)})^r (x_1, x_2, \dots, x_n)}{x_{r,s} w_r \dots w_1} = \left(\frac{x_{r,1}}{x_{r,s}} e_0, \frac{x_{r,2}}{x_{r,s}} e_0, \dots, \frac{x_{r,n}}{x_{r,s}} e_0\right) + (y_{r,1}, y_{r,2}, \dots, y_{r,n}),$$

where

$$y_{r,j} = \sum_{k=r+1}^{\infty} \frac{x_{k,j} w_k \dots w_{k-r+1}}{x_{r,s} w_r \dots w_1} e_{k-r}.$$

Now

$$\|y_{r,j}\|^2 = \sum_{k=r+1}^{\infty} \left(\frac{w_k \dots w_{k-r+1}}{w_r \dots w_1}\right)^2 \left|\frac{x_{k,j}}{x_{r,s}}\right|^2 = \sum_{k=0}^{\infty} \left(\frac{w_{k+2} \dots w_{k+r+1}}{w_1 \dots w_r}\right)^2 \left|\frac{x_{k+r+1,j}}{x_{r,s}}\right|^2 \geqq$$

$$\leqq \sum_{k=0}^{\infty} \left(\frac{w_{k+2} \dots w_{k+r+1}}{w_1 \dots w_r}\right)^2, \quad \text{by (2)},$$

$$= \frac{1}{w_1^2} \sum_{k=0}^{\infty} \left(\frac{w_{k+2} \dots w_{k+r}}{w_2 \dots w_r}\right)^2 w_{k+r+1}^2 = \frac{1}{w_1^2} \sum_{k=0}^{\infty} \sum_{i=0}^{k} \frac{w_{i+2} \dots w_{i+r}}{w_2 \dots w_r}\Big)^2 (w_{k+r+1}^2 - w_{k+r+2}^2)$$

(by Abel's transformation [12])

$$\leqq \frac{\delta}{w_1^2} \sum_{k=0}^{\infty} |w_{k+r+1}^2 - w_{k+r+2}^2|, \quad \text{by (1)},$$

$$= \frac{\delta}{w_1^2} \sum_{k=0}^{\infty} |w_{k+r+1} - w_{k+r+2}| (w_{k+r+1} + w_{k+r+2}) \leqq$$

$$\leqq \frac{2\delta\mu}{w_1^2} \sum_{k>r} |w_k - w_{k+1}|, \quad \text{where} \quad \mu = \sup_k \{w_k\},$$

and hence $y_{rj} \to 0$ as $m \to \infty$.

Also, for each j $(1 \leqq j \leqq n)$, the sequence $\left\{\frac{x_{r,j}}{x_{r,s}}\right\}_{m=1}^{\infty}$ is contained in the unit disc, and hence admits a convergent subsequence converging to a number, say z_j. A routine check reveals that a number j_0 lying between 1 and n will occur infinitely often as a value $s = s(m)$ and corresponding to this j_0, we have $z_{j_0} = 1$. The upshot of the above deliberation is that M contains an eigenvector of $T^{(n)}$, viz. $(z_1 e_0, z_2 e_0, \dots, z_n e_0)$. Thus, T also satisfies condition (ii) of Theorem A and we are done.

Theorem 2. *If $\{w_k\}_{k=1}^{\infty}$ is of bounded* p*th-power variation and*

$$\delta = \delta(n) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \frac{w_{j+2} \dots w_{j+n}}{w_2 \dots w_n}\right)^q < \infty \tag{3}$$

for all $n \geqq 2$, where $1 < p < \infty$ and q is the Hölder conjugate of p, then $\mathscr{A} = B(H)$.

Proof. Proceeding as in the proof of Theorem 1, we have

$$\|y_{r,j}\| = \left(\sum_{k=r+1}^{\infty}\left(\frac{w_k\dots w_{k-r+1}}{w_r\dots w_1}\right)^2\left|\frac{x_{k,j}}{x_{r,s}}\right|^2\right)^{1/2} \leqq \sum_{k=r+1}^{\infty}\left(\frac{w_k\dots w_{k-r+1}}{w_r\dots w_1}\right)\left|\frac{x_{k,j}}{x_{r,s}}\right| \leqq$$

$$\leqq \sum_{k=r+1}^{\infty}\frac{w_k\dots w_{k-r+1}}{w_r\dots w_1}, \quad \text{by (2)}$$

$$= \frac{1}{w_1}\sum_{k=0}^{\infty}\frac{w_{k+2}\dots w_{k+r}}{w_2\dots w_r}w_{k+r+1} \leqq \frac{1}{w_1}\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\frac{w_{j+2}\dots w_{j+r}}{w_2\dots w_r}\right)|w_{k+r+1}-w_{k+r+2}|$$

(by Abel's transformation [12])

$$= \frac{1}{w_1}\left(\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\frac{w_{j+2}\dots w_{j+r}}{w_2\dots w_r}\right)\right)^{1/q}\left(\sum_{k>r}|w_k-w_{k+1}|^p\right)^{1/p} \quad \text{(by Hölder's inequality)}$$

$$= \frac{\delta^{1/q}}{w_1}\left(\sum_{k>r}|w_k-w_{k+1}|^p\right)^{1/p}, \quad \text{by (3);}$$

and hence, $y_{r,j}\to 0$ as $m\to\infty$.

The rest of the proof follows as that for Theorem 1.

Let l^p, $1<p<\infty$, be the Banach space of all complex pth-power summable sequences $x=\{x_0, x_1, x_2, \dots\}$ with the norm

$$\|x\| = \left(\sum_{k=0}^{\infty}|x_k|^p\right)^{1/p}.$$

Then a weighted unilateral (backward) shift T on l^p appears as

$$T\{x_0, x_1, x_2, \dots\} = \{w_1x_1, w_2x_2, \dots\}.$$

We denote by $\mathscr{L}$ a strongly closed subalgebra of $B(l^p)$ containing the identity operator and with no non-trivial invariant subspaces. We have the following analogue of Theorem 1 for l^p spaces, which we state without proof:

Theorem 3. *If $\mathscr{L}$ contains T with*

$$\sum_{k=0}^{\infty}\left(\frac{w_{k+2}\dots w_{k+n}}{w_2\dots w_n}\right)^p<\infty \quad \textit{for all} \quad n\geqq 2,$$

then $\mathscr{L}=B(l^p)$.

Remark. A subalgebra $\mathcal{L}$ of $B(H)$ is called *strictly cyclic* if there exists a vector $x_0 \in H$ such that $\{Ax_0: A \in \mathcal{L}\} = H$, and an operator $A \in B(H)$ is strictly cyclic if the algebra generated by A is strictly cyclic. LAMBERT [7] has shown that every transitive algebra which contains a strictly cyclic algebra equals $B(H)$. It follows, in particular, that every transitive algebra containing a strictly cyclic operator is equal to $B(H)$. Every Donoghue operator is strictly cyclic [6]. Whether the weighted shifts T in our Theorems 1 and 2 are also strictly cyclic, is not known. In case they are, these theorems will follow as corollaries to LAMBERT's theorem [7, Theorem 4.5]. In fact, we strongly feel that the following is true:

Conjecture. Every weighted unilateral shift whose weight sequence is of bounded variation and square-summable is strictly cyclic.

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