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A problem of Sz.-Nagy

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1. Introduction

Let $\mathfrak{H}$ be a complex Hilbert space. Relatively simple proofs of the following results are given.

(a) A power bounded operator T on $\mathfrak{H}$ is similar to a unitary operator if and only if T is surjective and if there exists a constant M such that

$$(1-|\lambda|)\|x\| \leqq M\|Tx-\lambda x\|, \quad |\lambda|<1, \quad x\in\mathfrak{H}.$$

(b) Let iA be the generator of a strongly continuous group $\{P_t\colon t\in\mathbf{R}\}$ in $\mathfrak{H}$. Suppose that $\sup\{\|P_{-t}\|\colon t\geqq 0\}$ is finite. Then A is similar to a selfadjoint operator if and only if there is a constant M such that

$$\operatorname{Re}\lambda\|x\| \leqq M\|\lambda x-iAx\|, \quad \operatorname{Re}\lambda>0, \quad x\in D(A).$$

By spectral theory the "only if" parts are obvious. For a contraction T, statement (a) is due to GOHBERG and KREIN [3], who deduced it from a theorem of SZ.-NAGY and FOIAŞ [10]. In the latter theorem the authors provide a sufficient condition for an invertible contraction T to be similar to a unitary operator, in terms of the characteristic operator function $\Theta_T(\lambda)$ of T. This condition is that a constant N exists for which

$$\|x\| \leqq N\|\Theta_T(\lambda)x\|, \quad |\lambda|<1, \quad x\in\mathfrak{H}.$$

For the concept of characteristic operator function and its connection with the theory of unitary dilations we refer to SZ.-NAGY and FOIAŞ [11, Chapitre VI, pp. 228—230, and Chapitre IX, p. 334].

The problem of finding a simpler proof of statement (a), avoiding characteristic functions and dilation theory, was pointed out by SZ.-NAGY in [2]. In the present paper we shall give a solution. We shall even do it for non-contractive, but power

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bounded operators. Indeed, the proof of (a) shall be reduced to the comparatively simpler theorem of Sz.-Nagy [9] which asserts that an invertible operator S is similar to a unitary operator if (and only if) $\sup\{\|S^n\|: n\in\mathbf{Z}\}$ is finite.

Statement (b), the continuous counterpart of (a), is entirely new.

2. Main results

We shall need a few definitions. A linear operator T on $\mathfrak{H}$ is said to be *power bounded* if $\sup\{\|T^n\|: n\in\mathbf{N}\}$ is finite. Let A and B be linear operators with domain and range in $\mathfrak{H}$. Then A is said to be *similar* to B if there exists a bounded linear operator V with bounded everywhere defined inverse such that $AV=VB$.

Theorem 1. *A power bounded operator T on $\mathfrak{H}$ is similar to a unitary operator if and only if it satisfies one of the following conditions (in* (ii)′ *T is supposed to be a contraction):*

(i) *T has power bounded inverse S.*

(ii) *The operators $(T-\lambda I)^{-1}$, $|\lambda|<1$, exist and*

$$\sup\{(1-|\lambda|)\|(T-\lambda I)^{-1}\|: |\lambda|<1\}<\infty.$$

(ii)′ *The operators $\Theta_T(\lambda)^{-1}$, $|\lambda|<1$, exist and*

$$\sup\{\|\Theta_T(\lambda)^{-1}\|: |\lambda|<1\}<\infty.$$

(iii) *T has an inverse S for which the operators $(I-\lambda S)^{-1}$, $|\lambda|<1$, exist and for which*

$$\liminf_{r\uparrow 1}\sup\{(1-r^2)\|(I-\lambda S)^{-1}\|: |\lambda|=r\}<\infty.$$

(iv) *T is surjective and there is a constant M such that*

$$(1-|\lambda|)\|x\|\leqq M\|Tx-\lambda x\|,\quad |\lambda|<1,\quad x\in\mathfrak{H}.$$

Proof. Sz.-Nagy [9] proves the sufficiency of (i) by means of an invariant mean on $\mathbf{Z}$. The necessity of (i) is trivial. The implications (i)⇒(ii), (ii)⇒(iii) and (iii)⇒(iv) are more or less trivial. The implication (iv)⇒(ii) follows from the fact that boundary points of the spectrum of a closed linear operator are approximate eigenvalues; e.g. Halmos [4, Problem 63, p. 39]. In [10] Sz.-Nagy and Foiaş use unitary dilation theory to prove the sufficiency of (ii)′. By establishing certain mutual inequalities between $\|\Theta_T(\lambda)^{-1}\|$ and $\|(T-\lambda I)^{-1}\|$, $|\lambda|<1$, Gohberg and Krein [3] prove the equivalency of (ii) and (ii)′. See also Krein [5, 6] and Sz.-Nagy and Foiaş [11, Chapitre IX, p. 334].

A simple proof of the implication (iii)⇒(i) runs as follows. Since it neither uses unitary dilation theory nor characteristic functions it solves a problem posed by Sz.-Nagy in [2, p. 585].

Fix x in $\mathfrak{H}$ and r in $[0, 1)$. Denote

$$M(r) = \sup \{(1-r^2)\|(I-\lambda S)^{-1}\|: |\lambda| = r\}$$

for $0<r<1$ and put $M_0=\sup \{\|T^n\|: n\in \mathrm{N}\}$. From (iii) it follows that the spectral radius $\varrho(S)$ of S satisfies $\varrho(S)\leqq 1$. Since $\|T^n\|\leqq M_0$, $n\in\mathrm{N}$, it also follows that $\varrho(T)\leqq 1$. Hence, for $|\lambda|<1$, we have norm convergence in both expansions

$$(I-\lambda S)^{-1} = \sum_{n=0}^{\infty} \lambda^n S^n, \quad (I-\bar{\lambda} T)^{-1} = \sum_{n=0}^{\infty} \bar{\lambda}^n T^n.$$

So, since $ST=I$, we have with $\lambda=re^{it}$, $0\leqq r<1$,

$$\sum_{n=-\infty}^{\infty} r^{|n|} e^{int} S^n = \sum_{n=0}^{\infty} r^n e^{int} S^n + \sum_{n=1}^{\infty} r^n e^{-int} T^n =$$

$$= (I-re^{it}S)^{-1} + re^{-it}T(I-re^{-it}T)^{-1} = (1-r^2)(I-re^{it}S)^{-1}(I-re^{-it}T)^{-1}.$$

Thus, by (iii), it follows that

$$\sum_{n=-\infty}^{\infty} r^{2|n|}\|S^n x\|^2 = \frac{1}{2\pi}\int_{-\pi}^{+\pi}\left\|\sum_{n=-\infty}^{\infty} r^{|n|} e^{int} S^n x\right\|^2 dt =$$

$$= \frac{1}{2\pi}\int_{-\pi}^{+\pi} \|(1-r^2)(I-re^{it}S)^{-1}(I-re^{-it}T)^{-1}x\|^2 dt \leqq$$

$$\leqq M(r)^2 \cdot \frac{1}{2\pi}\int_{-\pi}^{+\pi} \|(I-re^{-it}T)^{-1}x\|^2 dt =$$

$$= M(r)^2 \cdot \frac{1}{2\pi}\int_{-\pi}^{+\pi}\left\|\sum_{n=0}^{\infty} r^n e^{-int} T^n x\right\|^2 dt = M(r)^2 \sum_{n=0}^{\infty} r^{2n}\|T^n x\|^2.$$

Consequently,

$$\sum_{n=1}^{\infty} r^{2n}\|S^n x\|^2 \leqq \left(M(r)^2-1\right)\sum_{n=0}^{\infty} r^{2n}\|T^n x\|^2.$$

Next, fix m in N, $m\geqq 1$. Then,

$$r^{2m}\|S^m x\|^2 = (1-r^2)\sum_{n=m}^{\infty} r^{2n}\|T^{n-m}S^n x\|^2 \leqq$$

$$\leqq (1-r^2) - M_0^2 \sum_{n=m}^{\infty} r^{2n}\|S^n x\|^2 \leqq (1-r^2)M_0^2 \sum_{n=1}^{\infty} r^{2n}\|S^n x\|^2$$

and, by what is proved above,

$$r^{2m}\|S^m x\|^2 \leqq (1-r^2)M_0^2\left(M(r)^2-1\right)\sum_{n=0}^{\infty} r^{2n}\|T^n x\|^2 \leqq$$

$$\leqq (1-r^2)M_0^2\left(M(r)^2-1\right)M_0^2(1-r^2)^{-1}\|x\|^2 = M_0^4\left(M(r)^2-1\right)\|x\|^2.$$

Since $0<r<1$ is arbitrary, we conclude that

$$\|S^m\| \leqq \liminf_{r \uparrow 1} M_0^2 (M(r)^2-1)^{1/2}, \quad m \geqq 1.$$

Hence (i) follows.

Remark 1. The operator $(1-r^2)(I-rS)^{-1}(I-rS^{-1})^{-1}$ can be considered as kind of an operator valued Poisson kernel.

Remark 2. In [7] SHIELDS discusses a number of boundedness properties of powers of an operator in relation to the boundedness properties of its resolvent family. See also VAN CASTEREN [12] where similar questions are considered.

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Next we describe the continuous analogue of Theorem 1. For a proof the reader will need Stone's theorem and some other standard facts on strongly continuous semigroups. For all this we refer to YOSIDA [13].

Theorem 2. *Let iA be the generator of a strongly continuous group $\{P_t \colon t \in \mathbf{R}\}$. Assume that* $\sup\{\|P_{-t}\| \colon t \geqq 0\}$ *is finite. Then A is similar to a selfadjoint operator if and only if it satisfies one of the following conditions:*

(i) $\sup\{\|P_s\| \colon s \geqq 0\} < \infty$.

(ii) *The inverses* $(\lambda I - iA)^{-1}$, $\operatorname{Re}\lambda > 0$, *exist and*

$$\sup\{\operatorname{Re}\lambda \|(\lambda I - iA)^{-1}\| \colon \operatorname{Re}\lambda > 0\} < \infty.$$

(iii) *The inverses* $(\lambda I - iA)^{-1}$, $\operatorname{Re}\lambda > 0$, *exist and*

$$\liminf_{\omega \downarrow 0} \sup\{\omega \|\lambda I - iA)^{-1}\| \colon \operatorname{Re}\lambda = \omega\} < \infty.$$

(iv) *There is a constant M such that*

$$\operatorname{Re}\lambda \|x\| \leqq M \|\lambda x - iAx\|, \quad \operatorname{Re}\lambda > 0, \quad x \in D(A).$$

Proof. We only prove the implication (iii)⇒(i). Here we use Plancherel's theorem in $L^2(\mathrm{R}, \mathfrak{H})$; e.g. see EDWARDS and GAUDRY [1, § 3.4, p. 53] or STEIN [8, Chapter II, § 5, pp. 45—47].

Fix x in $\mathfrak{H}$ and $\omega > 0$. Put

$$M(\omega) = \sup\{2\omega \|(\lambda I - iA)^{-1}\| \colon \operatorname{Re}\lambda = \omega\}.$$

From standard semigroup considerations it follows by (iii) that the integral

$$\int_{-\infty}^{\infty} e^{-\omega|s| - i\xi s} P_s x \, ds$$

exists and that

$$\int_{-\infty}^{\infty} e^{-\omega|s|-i\xi s} P_s x\, ds = \int_0^{\infty} e^{-\omega s-i\xi s} P_s x\, ds + \int_0^{\infty} e^{-\omega s+i\xi s} P_{-s} x\, ds =$$

$$= ((\omega-i\xi)I-iA)^{-1}x+((\omega-i\xi)I+iA)^{-1}x = 2\omega((\omega+i\xi)I-iA)^{-1}((\omega-i\xi)I+iA)^{-1}x.$$

So by Plancherel's theorem it follows that

$$\int_{-\infty}^{\infty} e^{-2\omega|s|}\|P_s x\|^2 ds = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big\|\int_{-\infty}^{\infty} e^{-\omega|s|-i\xi s}P_s x\, ds\Big\|^2 d\xi =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\big\|2\omega((\omega+i\xi)I-iA)^{-1}((\omega-i\xi)I+iA)^{-1}x\big\|^2 d\xi \leqq$$

$$\leqq M(\omega)^2\frac{1}{2\pi}\int_{-\infty}^{\infty}\big\|((\omega-i\xi)I+iA)^{-1}x\big\|^2 d\xi =$$

$$= M(\omega)^2\frac{1}{2\pi}\int_{-\infty}^{\infty}\Big\|\int_0^{\infty} e^{-\omega s+i\xi s}P_{-s}x\, ds\Big\|^2 d\xi = M(\omega)^2\cdot\int_0^{\infty} e^{-2\omega s}\|P_{-s}x\|^2 ds.$$

Put $M_0=\sup\{\|P_{-t}\|:\ t\geqq 0\}$ and fix $S>0$. Then

$$e^{-2\omega S}\|P_S x\|^2 = 2\omega\int_S^{\infty} e^{-2\omega s}\|P_{-(s-S)}P_s x\|^2 ds \leqq$$

$$\leqq 2\omega M_0^2\int_S^{\infty} e^{-2\omega s}\|P_s x\|^2 ds \leqq 2\omega M_0^2\int_0^{\infty} e^{-2\omega s}\|P_s x\|^2 ds$$

and by what is proved above,

$$e^{-2\omega S}\|P_S x\|^2 \leqq 2\omega M_0^2\big(M(\omega)^2-1\big)\int_0^{\infty} e^{-2\omega s}\|P_{-s}x\|^2 ds \leqq$$

$$\leqq 2\omega M_0^4\big(M(\omega)^2-1\big)\int_0^{\infty} e^{-2\omega s} ds\cdot\|x\|^2 = M_0^4\big(M(\omega)^2-1\big)\|x\|^2.$$

Consequently, we conclude that

$$\|P_s\| \leqq M_0^2 \liminf_{\omega\downarrow 0}\big(M(\omega)^2-1\big)^{1/2}, \quad s\geqq 0.$$

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