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Integrability of Rees—Stanojević sums

BABU RAM

1. A sequence $\langle a_n \rangle$ of positive numbers is called quasi-monotone if $n^{-\beta} a_n \downarrow 0$ for some β, or equivalently if $a_{n+1} \leqq a_n (1+\alpha/n)$.

We say that a sequence $\langle a_k \rangle$ of numbers satisfies

Condition S^* if $a_k \to 0$ as $k \to \infty$ and there exists a sequence $\langle A_k \rangle$ such that $\langle A_k \rangle$ is quasi-monotone, $\sum_{k=0}^{\infty} A_k < \infty$, and $|\Delta a_k| \leqq A_k$ for all k.

Condition S^* is weaker than Condition S of Sidon introduced in [4].

Recently, REES and STANOJEVIĆ [3] (see also GARRETT and STANOJEVIĆ [2]) introduced the modified cosine sums

$$g_n(x) = \frac{1}{2} \sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta a_j \cos kx \tag{1}$$

and obtained a necessary and sufficient condition for the integrability of the limit of these sums.

The object of this paper is to show that Condition S^* is sufficient for integrability of the limit of (1).

2. We require the following lemmas for the proofs of our results:

Lemma 1. (FOMIN [1]) *If* $|c_k| \leqq 1$, *then*

$$\int_0^{\pi} \left| \sum_{k=0}^{n} c_k \frac{\sin (k+1/2)x}{2 \sin x/2} \right| dx \leqq C(n+1),$$

where C is a positive absolute constant.

Lemma 2. (SZÁSZ [5]) *If* $\langle a_n \rangle$ *is quasi-monotone with* $\sum a_n < \infty$, *then* $n a_n \to 0$ *as* $n \to \infty$.

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3. We prove

Theorem. *Let the sequence $\langle a_k \rangle$ satisfy Condition S^*. Then*

$$g(x) = \lim_{n\to\infty} \sum_{k=1}^{n} \left[\frac{1}{2} \Delta a_k + \sum_{j=k}^{n} \Delta a_j \cos kx \right]$$

exists for $x \in (0, \pi]$ and $g(x) \in L(0, \pi)$.

Proof. We have

$$g_n(x) = \sum_{k=1}^{n} \left[\frac{1}{2} \Delta a_k + \sum_{j=k}^{n} \Delta a_j \cos kx \right] =$$

$$= \sum_{k=1}^{n} \frac{1}{2} \Delta a_k + \sum_{k=1}^{n} a_k \cos kx - a_{n+1} D_n(x) + \frac{1}{2} a_{n+1}.$$

Making use of Abel's transformation, we obtain

$$g_n(x) = \tag{2}$$

$$= \sum_{k=1}^{n} \frac{1}{2} \Delta a_k + \sum_{k=1}^{n-1} \Delta a_k \left(D_k(x) + \frac{1}{2} \right) + a_n \left(D_n(x) + \frac{1}{2} \right) - a_{n+1} D_n(x) - a_1 + \frac{1}{2} a_{n+1} =$$

$$= \sum_{k=1}^{n-1} \Delta a_k D_k(x) + a_n D_n(x) - a_{n+1} D_n(x).$$

The last two terms tend to zero as $n \to \infty$ for $x \neq 0$ and since

$$|D_k(x)| = O(1/x) \quad \text{if} \quad x \neq 0 \quad \text{and} \quad \sum_{k=0}^{\infty} |\Delta a_k| < \infty,$$

the series $\sum_{k=1}^{\infty} \Delta a_k D_k(x)$ converges. Hence $\lim_{n\to\infty} g_n(x)$ exists for $x \neq 0$. Now applications of Abel's transformation and Lemma 1 yield

$$\int_0^{\pi} |g(x)| \, dx = \int_0^{\pi} \left| \sum_{k=1}^{\infty} \Delta a_k D_k(x) \right| dx = \tag{3}$$

$$= \int_0^{\pi} \left| \sum_{k=1}^{\infty} A_k \frac{\Delta a_k}{A_k} D_k(x) \right| dx \leqq \sum_{k=1}^{\infty} |\Delta A_k| \int_0^{\pi} \left| \sum_{j=0}^{k} \frac{\Delta a_j}{A_j} D_j(x) \right| dx \leqq$$

$$\leqq C \sum_{k=1}^{\infty} (k+1) |\Delta A_k| =$$

$$= C \left[\sum_{k=1}^{\infty} (k+1) \left| A_k \left(1 + \frac{\alpha}{k} \right) - \frac{\alpha A_k}{k} - A_{k+1} \right| \right] \leqq$$

$$\leqq C \sum_{k=1}^{\infty} (k+1) \left| A_k \left(1 + \frac{\alpha}{k} \right) - A_{k+1} \right| + C\alpha \sum_{k=1}^{\infty} \frac{k+1}{k} A_k =$$

$$= C \sum_{k=1}^{\infty} (k+1) \Delta A_k + 2C\alpha \sum_{k=1}^{\infty} \frac{k+1}{k} A_k,$$

the last step being the consequence of $A_k(1+\alpha/k)\geqq A_{k+1}$. But

$$\sum_{k=1}^{n} A_k = \sum_{k=1}^{n-1} (k+1)\Delta A_k+(n+1)A_n-A_1.$$

Applications of $\sum_{0}^{\infty} A_k<\infty$ and Lemma 2 yield

$$\sum_{k=1}^{\infty}(k+1)\Delta A_k = \sum_{k=1}^{\infty} A_k+A_1<\infty; \tag{4}$$

(3) and (4) now imply the conclusion of the Theorem.

Corollary. *Let $\langle a_k\rangle$ be a sequence satisfying the condition S^*. Then*

$$\frac{1}{x}\sum_{k=1}^{\infty} \Delta a_k \sin(k+1/2)x = \frac{h(x)}{x}$$

converges for $x\in(0,\pi]$ *and* $\frac{h(x)}{x}\in L(0,\pi)$.

Proof. This follows immediately, namely by (2), $2\sin\frac{x}{2}g(x)=h(x)$.

References

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DEPARTMENT OF MATHEMATICS
MAHARSHI DAYANAND UNIVERSITY
ROHTAK-124001, INDIA