Acta Sci. Math., 42 (1980), 95—98.

Remarks on a paper of L. Szabó and Á. Szendrei

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The aim of this note is to give an infinite version of the Theorem of L. SZABÓ and Á. SZENDREI [4]. We shall do this without using I. Rosenberg's Theorem [3] and those parts of [4] which make use of it. We adopt the terminology of [2] and [4].

Theorem. *An at least four element non-trivial algebra with triply transitive automorphism group either has the interpolation property or is equivalent to an affine space over* GF (2).

Most of the proof follows closely that of L. SZABÓ and Á. SZENDREI [4], we shall write out only those parts which are different. We do not need Proposition 1 of [4]. We formulate Proposition 2 in a slightly different way: we consider not necessarily finite algebras and local term functions instead of term functions. The proof is literally the same.

Lemma 1. *Let A be an algebra with at least four elements and with a triply transitive automorphism group. If A does not have the interpolation property but has a three-place non-trivial local term function f, then f is a minority function such that $f(a, b, c) \notin \{a, b, c\}$ whenever the elements $a, b, c \in A$ are all different.*

Proof. The proof that $f(a, b, c) \notin \{a, b, c\}$ if $|\{a, b, c\}|=3$ and that condition $(*)$ of [4] holds, is literally the same as in [4]. This is the beginning of their proof of Lemma 1; thereby we need the infinite version of B. Csákány's Theorem, which is an immediate consequence of the finite one. For, given an (infinite) algebra A with a pattern function $p(x_1, \ldots, x_k)$ which can be interpolated on every finite subset of A^k, and a partial function f on a finite subset $H \subset A^k$, let B denote the subset of A which consists of the elements occurring as coordinates in H or being values of f on H. Then take the polynomial function $\tilde{p}$ which interpolates p on B^k;

Received January 9, in revised form October 24, 1979.

* This work was carried out while the second author was visiting at the Technische Hochschule Darmstadt. He is indebted to Prof. R. WILLE for the invitation.

$(B, \tilde{p})$ is, by Csákány's Theorem, functionally complete, and this gives a representation of f in terms of $\tilde{p}$, hence as a polynomial function on A.

Now it suffices to show that if the local term function f is not a minority function, then A has the interpolation property. For this end we show first that in this case A has the 2-interpolation property. Further, it suffices to consider functions in one variable only: if we take two distinct elements of A^k for some $k \in N$, they differ in at least one component i, and then we consider the i-th projection. Given arbitrary elements $x, y, u, v \in A$, $x \neq y$, we have to show the existence of a unary polynomial function g such that $g(x)=u$, $g(y)=v$. Supposing that A has at least five elements, it is sufficient to prove this if x, y, u, v are all distinct. (In fact, in the other case we can choose two elements e, f both distinct from x, y, u, v, and then send x, y first to e, f and then e, f to u, v.) Since f is not a minority function, at least one of the values $f(x, y, y)$, $f(y, x, y)$, $f(y, y, x)$ is equal to y. Suppose e.g. $f(y, y, x)=y$. By $(*)$ we have elements $c, d \in A$ such that $f(y, x, d)=v, f(x, v, c)=u$. Then we take g to be a (unary) polynomial function which interpolates $f\big(f(\xi, x, d), v, c\big)$ at $\xi=x, y$. (In case A has four elements, by somewhat more, but still elementary, computation one can construct this polynomial function g, thus avoiding the use of Rosenberg's Theorem.)

Now we use induction and prove that if A has the $(n-1)$-interpolation property $(n>2)$ then it has the n-interpolation property, too. Let $g\colon A^k \to A$ and $x_1, \dots, x_n \in A^k$ be different elements and put $a_i=g(x_i)$, $i=1, \dots, n$. Since g has the $(n-1)$-interpolation property, we have polynomial functions $f_1, \dots, f_5$ such that

$$f_1(x_i) = a_i, \quad i = 1, 2, 4, \dots, n; \quad f_2(x_i) = a_i, \quad i = 1, 3, 4, \dots, n;$$

$$f_3(x_i) = \begin{cases} a_i & i = 4, \dots, n, \\ f_1(x_3) & i = 3, \\ f_2(x_2) & i = 2, \end{cases}$$

and for arbitrary elements $d, u \in A$,

$$f_4(x_i) = \begin{cases} a_i & i = 2, 4, \dots, n, \\ d & i = 3; \end{cases}$$

$$f_5(x_i) = \begin{cases} a_i & i = 1, 4, \dots, n, \\ u & i = 3. \end{cases}$$

If $f_1(x_3)=a_3$, then we are done. Suppose therefore $f_1(x_3) \neq a_3$ and by using $(*)$ choose d, u so that $f\big(f_1(x_3), d, u\big)=a_3$. By assumption, f is not a minority function, hence we have, say, $f(y, y, x)=y$. If we have in addition $f(y, x, y)=f(x, y, y)=x$, then we take a polynomial function p which interpolates $f(f_1, f_2, f_3)$ on $\{x_1, \dots, x_n\}$. It is easy to see that $p(x_i)=a_i$; $i=1, \dots, n$. If $f(y, x, y)$ or $f(x, y, y)$, say $f(y, x, y)$, is also y, then we consider a polynomial function q which interpolates $f(f_1, f_4, f_5)$ on $\{x_1, \dots, x_n\}$ and again we obtain that $q(x_i)=a_i$, $i=1, \dots, n$.

Lemma 2. *Let A be an algebra with at least four elements and with a triply transitive automorphism group. Suppose that there exists an at least quaternary (say n-ary) non-trivial local term function f which turns into a projection whenever we identify any two of its variables. Then A has the interpolation property.*

Proof. Again we repeat the beginning of the proof in [4] and obtain property $(**)$. Along the same lines as in Lemma 1 we show first that A has the 2-interpolation property. Take again four different elements $x, y, a, b \in A$. By $(**)$ there exist elements $d_3, \ldots, d_n, d'_3, \ldots, d'_n$ in A such that $f(x, y, d_3, \ldots, d_n) = a$ and $f(y, a, d'_3, \ldots, d'_n) = b$. Consider now a polynomial function g which interpolates $f(f(\xi, y, d_3, \ldots, d_n), a, d'_3, \ldots, d'_n)$ at $\xi = x, y$. This function does the job.

Suppose next that A has the $(m-1)$-interpolation property $(m>2)$. We show that it has the m-interpolation property as well. Consider a function $h\colon A^k \to A$ and put $a_i = h(x_i)$, $i = 1, \ldots, m$. By assumption we have a polynomial function f_1 such that $f_1(x_i) = a_i$, $i = 2, 3, \ldots, m$. If $f_1(x_1) = a_1$ then we are done. Suppose $f_1(x_1) \neq a_1$, then choose an element $b \notin \{a_1, f_1(x_1)\}$, and consider a polynomial function f_2 such that:

$$f_2(x_i) = \begin{cases} b & i = 1 \\ a_i & i = 3, \ldots, m. \end{cases}$$

By $(**)$ there are $t_3, \ldots, t_n$ in A such that $f(f_1(x_1), b, t_3, \ldots, t_n) = a_1$. Next we choose a polynomial function f_3 such that:

$$f_3(x_i) = \begin{cases} t_3 & i = 1 \\ a_i & i = 2, 4, \ldots, m. \end{cases}$$

Finally, we take a polynomial function r which interpolates $f(f_1, f_2, f_3, t_4, \ldots, t_n)$ on $\{x_1, \ldots, x_m\}$, then we have $h(x_i) = r(x_i)$, $i = 1, \ldots, m$.

As a next step, we transfer Lemma 3 of [4], together with its proof, with the obvious modifications to the infinite case.

Lemma 4. *Let A be an algebra with at least four elements and with triply transitive automorphism group. If A does not have the interpolation property, then A admits no essentially quaternary local term function.*

Proof. Suppose h is an essentially quaternary local term function on A, then it has the properties (1)—(7) of Lemma 3. Since h depends on the first variable, one can find elements a, b, c, d in A such that $h(a, b, c, d) := s \neq h(b, b, c, d) = m(b, c, d) := t$, where m is the unique non-trivial ternary local term function on A. A short elementary computation shows that (at least) b, c, d, t must be all different. Let Θ be a congruence of A and $u \Theta v$ with $u \neq v$, and choose an arbitrary $z \notin \{u, v\}$. If $h(a, b, c, d) \neq a$, then just as it is done at the corresponding place in the proof of the Theorem in [4], we see that a, $m(b, c, d)$, $h(a, b, c, d)$ are all different. Now we can find a $\pi\in\mathrm{Aut}\, A$ such that $\pi(a)=v$, $\pi(h(a, b, c, d))=z$, $\pi(m(b, c, d))=u$, and we have $h(v, \pi b, \pi c, \pi d)=z$, $h(u, \pi b, \pi c, \pi d)=u$, which implies $z=h(v, \pi b, \pi c, \pi d)\Theta h(u, \pi b, \pi c, \pi d)=u$, hence $\Theta=A^2$. Suppose now $h(a, b, c, d)=a$, then again we follow the corresponding lines in the proof of the Theorem in [4] and obtain that $a, b, m(b, c, d)$ are all different. Further we choose a $\pi\in\mathrm{Aut}\, A$ with $\pi a=u$, $\pi b=v$, $\pi(m(b, c, d))=z$, and conclude that $u=h(u, v, \pi c, \pi d)\Theta h(v, v, \pi c, \pi d)= =z$, whence $\Theta=A^2$. By this we have that A is simple, and by Lemma 3, A has a unique non-trivial ternary local term function m, which is a minority function. This implies that m remains unchanged if we permute its variables, furthermore $m(m(x, y, z), y, z)=x$ for all $x, y, z\in A$ (cf. (8) in [4]). In particular, since $m(b, c, d)=t$, we get $m(t, c, d)=b$.

On the other hand, A does not have the interpolation property, hence by M. Istinger, H. K. Kaiser and A. F. Pixley [1], Corollary 3.9, we know: If q is a binary local polynomial function and r an element of A such that $q(x, r)=q(r, x)=r$ (for all $x\in A$), then q is the constant function with value r. Consider $q(x, y)= =h(a, m(x, y, t), x, y)$. Then we have $q(x, y)=t$ for all $x, y\in A$, which contradicts $q(c, d)=h(a, m(c, d, t), c, d)=h(a, m(t, c, d), c, d)=h(a, b, c, d)=s\neq t$. This completes the proof of Lemma 4.

Now we continue the proof of the Theorem exactly as it is done in [4].

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