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Weighted shifts quasisimilar to quasinilpotent operators

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1. Introduction. The purpose of this note is to resolve certain questions raised in [8] and [9] concerning quasisimilarity and quasinilpotent operators. We prove that a weighted shift is quasisimilar to a quasinilpotent operator if and only if it is a direct sum of quasinilpotents (Theorems 2.7 and 2.8). As an application, we show that there exist operators T such that T and T^* are quasiaffine transforms of quasinilpotent operators but such that T is not quasisimilar to any quasinilpotent operator (Corollary 2.9). In section 3 we relate our results to several open problems concerning quasisimilarity and spectra.

Let $\mathfrak{H}$ denote a separable infinite dimensional complex Hilbert space and let $\mathscr{L}(\mathfrak{H})$ denote the algebra of all bounded linear operators on $\mathfrak{H}$. Let $\mathscr{N}$ and $\mathscr{Q}$ denote, respectively, the subsets of $\mathscr{L}(\mathfrak{H})$ consisting of all nilpotent and quasinilpotent operators. For T in $\mathscr{L}(\mathfrak{H})$, let $\mathfrak{M}(T)=\{x\in\mathfrak{H}: \|T^n x\|^{1/n}\to 0\}$. $\mathfrak{M}(T)$ is a linear manifold whose closure is hyperinvariant for T; moreover, T is quasinilpotent if and only if $\mathfrak{M}(T)=\mathfrak{H}$ [7, Lemma, page 28].

An operator X in $\mathscr{L}(\mathfrak{H})$ is a *quasiaffinity* if X is injective and has dense range. An operator B is a *quasiaffine transform* of an operator A if there exists a quasiaffinity X such that $AX=XB$. Operators A and B are *quasisimilar* if they are quasiaffine transforms of each other [18]. C. Apostol, R. G. Douglas, and C. Foiaş [4, Corollary, page 413] gave necessary and sufficient conditions for two nilpotent operators to be quasisimilar, but analogous results for quasinilpotent operators appear to be unknown. The present note concerns the quasisimilarity orbit of $\mathscr{Q}$. Let $\mathscr{Q}_{af}=\{T\in\mathscr{L}(\mathfrak{H}): T$ is a quasiaffine transform of some quasinilpotent operator$\}$, and let $\mathscr{Q}^*_{af}=\{T\in\mathscr{L}(\mathfrak{H}): T^*$ is in $\mathscr{Q}_{af}\}$. Let $\mathscr{Q}_{qs}$ denote the quasisimilarity orbit of $\mathscr{Q}$, i.e. $\mathscr{Q}_{qs}=\{T\in\mathscr{L}(\mathfrak{H}): T$ is quasisimilar to some quasinilpotent operator$\}$.

In [8] and [9] we obtained the following invariants for membership in $\mathscr{Q}_{qs}$. A compact subset $K\subset \mathbf{C}$ is the spectrum of an operator in $\mathscr{Q}_{qs}$ if and only if K

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is connected and contains 0 [8, Theorem 3.11]. If T is in $\mathcal{Q}_{qs'}$, then T satisfies the following properties:

(I) $\mathfrak{M}(T)$ and $\mathfrak{M}(T^*)$ both contain orthonormal bases for $\mathfrak{H}$; in particular, $\mathfrak{M}(T)$ and $\mathfrak{M}(T^*)$ are dense in $\mathfrak{H}$ [8, Proposition 3.13].

(II) If $\mathfrak{M}\neq\{0\}$ is an invariant subspace for T, then $\sigma(T|\mathfrak{M})$ is connected and contains 0; if additionally, $\mathfrak{M}\neq\mathfrak{H}$, then $\sigma((1-P_{\mathfrak{M}})T|(1-P_{\mathfrak{M}})\mathfrak{H})$ is connected and contains 0 [8, Theorem 3.1]. ($P_{\mathfrak{M}}$ denotes the orthogonal projection of $\mathfrak{H}$ onto $\mathfrak{M}$ and $\sigma(\cdot)$ denotes the spectrum of an operator.) Each operator satisfying (I) also satisfies (II) [8, Proposition 3.15]; several equivalent reformulations of (II) are given in [9, section 3].

Note that $\mathcal{Q}_{qs}\subset\mathcal{Q}_{af}\cap\mathcal{Q}_{af}^*$ and that if T is in $\mathcal{Q}_{af}$, then $\mathfrak{M}(T^*)$ is dense [8, Lemma 3.12]. C. Apostol [3] proved that $\mathfrak{M}(T^*)$ is dense if and only if T is a quasiaffine transform of a compact quasinilpotent operator. Thus an operator T satisfies (I) if and only if T is in $\mathcal{Q}_{af}\cap\mathcal{Q}_{af}^*$.

In [8] we studied whether (I) actually implies membership in $\mathcal{Q}_{qs}$, or equivalently (in view of Apostol's result), whether $\mathcal{Q}_{qs}=\mathcal{Q}_{af}\cap\mathcal{Q}_{af}^*$. In [8] we obtained an affirmative answer to this question for decomposable operators (including normal, spectral, compact, and Riesz operators) and for hyponormal operators. If T is decomposable and $\mathfrak{M}(T^*)$ is dense, then T is quasinilpotent [8, Corollary 3.4]; moreover, the only hyponormal operator satisfying $\mathfrak{M}(T)^-=\mathfrak{H}$ is the zero operator [8, Theorem 3.6]. In section 2 we show that despite these positive results, $\mathcal{Q}_{qs}$ is actually a proper subset of $\mathcal{Q}_{af}\cap\mathcal{Q}_{af}^*$, so that neither (I) nor (II) necessarily implies membership in $\mathcal{Q}_{qs}$.

2. Weighted shifts in $\mathcal{Q}_{qs}$. Let $I=\mathbf{Z}$ or $\mathbf{Z}^+$ and let $\alpha=\{\alpha_n\}_{n\in I}$ denote a bounded sequence of complex numbers. Let $\{e_n\}_{n\in I}$ denote an orthonormal basis for $\mathfrak{H}$. The *weighted shift with weight sequence* α, W_α, is defined by the relations $W_\alpha e_n=\alpha_n e_{n+1}$ $(n\in I)$. If $I=\mathbf{Z}^+$, W_α is a *unilateral* shift, while if $I=\mathbf{Z}$, W_α is a *bilateral* shift. T. B. Hoover [14] exhibited weight sequences α and β, both with infinitely many zero terms, such that W_α and W_β are quasisimilar, W_α is quasinilpotent, and the spectrum of W_β is the closed unit disk. In this section we characterize the weighted shifts in $\mathcal{Q}_{qs}$.

For T in $\mathcal{L}(\mathfrak{H})$ and $n\geqq 0$, let $\mathfrak{M}_n(T)=\ker(T^{n+1})\ominus\ker(T^n)$. Let $\mathfrak{P}(T)=\bigvee_{n=1}^{\infty}\ker(T^n)=\sum_{n=0}^{\infty}\oplus\mathfrak{M}_n(T)$, and let $\mathfrak{M}_\infty(T)=\mathfrak{H}\ominus\mathfrak{P}(T)=\bigcap_{n=1}^{\infty}(\mathfrak{H}\ominus\ker(T^n))$. In the sequel, $\dim\mathfrak{M}$ refers to the orthogonal dimension of a closed subspace $\mathfrak{M}\subset\mathfrak{H}$.

Lemma 2.1. *If A and B are quasisimilar operators in $\mathcal{L}(\mathfrak{H})$, then A and B have the following properties:*

1) $\dim\mathfrak{M}_n(A)=\dim\mathfrak{M}_n(B)$ *for* $0\leqq n\leqq\infty$;

2) $\dim\ker(A^n)=\dim\ker(B^n)$ *for* $n>0$.

Proof. Let X and Y denote quasiaffinities such that $AX=XB$ and $YA=BY$. To prove 1) it suffices to show that $\dim \mathfrak{M}_n(B) \geqq \dim \mathfrak{M}_n(A)$ for $0 \leqq n \leqq \infty$, for then 1) follows by symmetry. Let $0 \leqq n < \infty$; we may assume that $\dim \mathfrak{M}_n(A) > 0$. Let $\{e_k\}_{0 \leqq k < p}$ $(0 < p \leqq \infty)$ denote an orthonormal basis for $\mathfrak{M}_n(A)$. Let $P_0 = 0$ and for $n > 0$, let P_n denote the orthogonal projection onto $\ker(B^n)$; note that $P_{n+1} - P_n$ is the projection onto $\mathfrak{M}_n(B)$.

We show that $\{(1-P_n)Ye_k\}_{0 \leqq k < p}$ is an independent sequence in $\mathfrak{M}_n(B)$. Since $A^{n+1}e_k = 0$, then $B^{n+1}Ye_k = YA^{n+1}e_k = 0$, so $(1-P_n)Ye_k = (P_{n+1}-P_n)Ye_k \in \mathfrak{M}_n(B)$. Suppose $0 \leqq j < p$, $c_0, \ldots, c_j \in \mathbf{C}$ and $\sum_{i=0}^{j} c_i(1-P_n)Ye_i = 0$. Then $\sum c_iYe_i = P_n \sum c_iYe_i \in \ker(B^n)$, and so $YA^n(\sum c_ie_i) = B^n(\sum c_iYe_i) = 0$. Since Y is injective, $\sum c_ie_i \in \ker(A^n)$, and thus $0 = (\sum c_ie_i, e_m) = c_m$ for $0 \leqq m \leqq j$. Therefore $\{(1-P_n)Ye_k\}_{0 \leqq k < p}$ is independent, and it follows (via Gram—Schmidt) that $\dim \mathfrak{M}_n(B) \geqq p = \dim \mathfrak{M}_n(A)$. This completes the proof of 1) for $n < \infty$.

Note that if $y \in \mathfrak{M}_\infty(A)$, then $X^*y \in \mathfrak{M}_\infty(B)$. Indeed, if $z \in \mathfrak{H}$, $n > 0$, and $B^nz = 0$, then $(X^*y, z) = (y, Xz) = 0$ since $Xz \in \ker(A^n)$ and $y \in \mathfrak{M}_\infty(A)$. Since X^* is injective, it follows that $\dim \mathfrak{M}_\infty(B) \geqq \dim \mathfrak{M}_\infty(A)$; the reverse inequality follows by symmetry.

For 2), note that since $\ker(A^{n+1}) = \ker(A^n) \oplus \mathfrak{M}_n(A)$, $\mathfrak{M}_0(A) = \ker(A)$, $\ker(B^{n+1}) = \ker(B^n) \oplus \mathfrak{M}_n(B)$, and $\mathfrak{M}_0(B) = \ker(B)$, the result follows from 1) by induction on n.

Corollary 2.2. *Let A and B be quasisimilar operators in $\mathcal{L}(\mathfrak{H})$. Then there is an operator B' unitarily equivalent to B such that $\mathfrak{M}_n(A) = \mathfrak{M}_n(B')$ for $0 \leqq n \leqq \infty$.*

Proof. For $0 \leqq n \leqq \infty$, let P_n and Q_n denote, respectively, the orthogonal projections onto $\mathfrak{M}_n(A)$ and $\mathfrak{M}_n(B)$. Note that $\sum_{0 \leqq n \leqq \infty} P_n = \sum_{0 \leqq n \leqq \infty} Q_n = 1$ and $P_iP_j = Q_iQ_j = 0$ for $i \neq j$ $(0 \leqq i, j \leqq \infty)$. Lemma 2.1 implies that there exists an isometric operator V_n which maps $\mathfrak{M}_n(A)$ onto $\mathfrak{M}_n(B)$. Let $V = \sum_{0 \leqq n \leqq \infty} V_nP_n$ (strong convergence); then $V^* = \sum V_n^*Q_n$ and V is unitary. If $B' = V^*BV$, it follows that $\mathfrak{M}_n(A) = \mathfrak{M}_n(B')$ for each n.

Remark. An analogue of Corollary 2.2 for $n=0$ is implicit in the proof of [19, Lemma 2].

For T in $\mathcal{L}(\mathfrak{H})$, let $(T)' = \{S \in \mathcal{L}(\mathfrak{H}) : TS = ST\}$ and let $(T)'' = \{R \in \mathcal{L}(\mathfrak{H}) : RS = SR$ for each S in $(T)'\}$. In the sequel $r(T)$ denotes the spectral radius of T.

Lemma 2.3. *Let A, B, X, and Y be operators such that $AX=XB$ and $YA=BY$. If $R \in (B)''$, then $XRY \in (A)'$ and $r(XRY) \leqq r(YX)r(R)$.*

Proof. The hypothesis implies that $XRYA = XRBY = XBRY = AXRY$, so XRY commutes with A. Since $R \in (B)''$ and $YX \in (B)'$, R commutes with YX, and thus $r(XRY) = r(YXR) \leqq r(YX)r(R)$.

Corollary 2.4. *If A is in $\mathcal{Q}_{qs}$, then A commutes with a nonzero quasinilpotent operator.*

Proof. Let $B \in \mathcal{Q}$ be quasisimilar to A and let X and Y denote quasiaffinities such that $AX = XB$ and $YA = BY$. Lemma 2.3 implies that XBY is a quasinilpotent operator commuting with A; moreover, since X is injective and Y has dense range, XBY is nonzero if B is nonzero. If $B=0$, then $A=0$, so the result is clear in this case also.

Lemma 2.5. *Let W be a noninvertible injective weighted shift such that $r(W)>0$. If S commutes with W, then $\sigma(S)$* (the spectrum of S) *has nonempty interior or S is a scalar multiple of the identity.*

Proof. The proof depends on several results from [15] to which we refer the reader for complete details. We consider first the case when W is a unilateral shift. In this case S may be represented as a multiplication operator M_Φ on a space of formal power series $H^2(\beta)$ [15, Theorem 3(b)]. The power series for the multiplier Φ is convergent in $D = \{z \in \mathbf{C}: |z| < r(W)\}$ [15, Theorem 10(iii)], and thus represents an analytic function $\Phi(z)$ in D. Now $\sigma(M_\Phi)$ coincides with the spectrum of Φ in $H^\infty(\beta)$ [15, Proposition 20], and thus $\sigma(M_\Phi)$ contains $\Phi(D)$ [15, page 79]. If M_Φ is not a scalar multiple of the identity, then Φ is non-constant, and it follows that $\Phi(D)$, and thus also $\sigma(M_\Phi)$, has nonempty interior. The proof for the case when W is a non-invertible bilateral shift is analogous; the pertinent results are [15, Theorem 3(a)], [15, Theorem 10′(iii—b)], and the remarks of [15, page 83].

Remark. The conclusion of Lemma 2.5 may fail if W is invertible; consider the unweighted bilateral shift, whose spectrum is the unit circle. Note also that there exist noninjective, non-quasinilpotent weighted shifts which commute with nonzero quasinilpotent operators.

Corollary 2.6. *If W is a noninvertible injective weighted shift and $r(W)>0$, then W commutes with no nonzero quasinilpotent operator.*

Theorem 2.7. *Let $W = W_\alpha$ be a bilateral weighted shift. The following are equivalent.*

1) $W \in \mathcal{Q}_{qs}$;
2) *W is a direct sum of quasinilpotent operators, and if α has at most finitely many zero terms, then W is quasinilpotent.*

Proof. The implication 2)⇒1) follows from [8, Proposition 3.10]. For the converse, we assume that $W \in \mathcal{Q}_{qs}$ and we consider several cases depending on the number and location of the zero terms in the weight sequence α. Note that since $W \in \mathcal{Q}_{qs}$, then W is noninvertible [14], [12].

i) W is injective. Since $W\in\mathcal{Q}_{qs}$, Corollary 2.4 implies that W commutes with a nonzero quasinilpotent; thus Corollary 2.6 implies that W is quasinilpotent.

ii) For each integer N, there exist integers m and n, $n<N<m$, such that $\alpha_m=\alpha_n=0$. It is clear that in this case W is an infinite direct sum of finite dimensional nilpotent operators.

iii) There exist integers n and m, $n\leqq m$, such that $\alpha_n=0$, $\alpha_m=0$, and $\alpha_k\neq 0$ for $k<n$ or $k>m$. We consider only the case $n<m$; the case $n=m$ may be treated similarly. Let $\mathfrak{H}_1=\langle e_n, e_{n-1}, e_{n-2}, \ldots\rangle$, $\mathfrak{H}_2=\langle e_{n+1}, \ldots, e_m\rangle$, and $\mathfrak{H}_3=\langle e_{m+1}, e_{m+2}, \ldots\rangle$. Relative to the decomposition $\mathfrak{H}=\mathfrak{H}_1\oplus\mathfrak{H}_2\oplus\mathfrak{H}_3$, the operator matrix of W is of the form $W=W_\beta^*\oplus N\oplus W_\gamma$, where W_β and W_γ are injective unilateral weighted shifts on $\mathfrak{H}_1$ and $\mathfrak{H}_3$ respectively, and $N^{m-n}=0$.

Suppose that W is quasisimilar to a quasinilpotent operator Q. Let X and Y be quasiaffinities such that $WX=XQ$ and $YW=QY$. Note that $\mathfrak{P}(W)=\mathfrak{H}_1\oplus\mathfrak{H}_2$ and $\mathfrak{M}_\infty(W)=\mathfrak{H}_3$. Corollary 2.2 implies that there is an operator Q' unitarily equivalent to Q such that $\mathfrak{M}_n(W)=\mathfrak{M}_n(Q')$ $(0\leqq n\leqq\infty)$, and thus $\ker(W^n)=\ker(Q'^n)$ for $n\geqq 0$. Let U denote a unitary operator such that $Q'=U^*QU$. Let $X'=XU$ and $Y'=U^*Y$; clearly X' and Y' are quasiaffinities and (*) $W^nX'=X'Q'^n$ and $Y'W^n=Q'^nY'$ for $n>0$. Since $\ker(W^n)=\ker(Q'^n)$, the preceding equations imply that $\mathfrak{M}\equiv\mathfrak{P}(W)=\mathfrak{P}(Q')=\mathfrak{H}_1\oplus\mathfrak{H}_2$ is an invariant subspace for X' and Y'.

Relative to the decomposition $\mathfrak{H}=\mathfrak{M}\oplus\mathfrak{M}^\perp$, the operator matrices of X', Y', Q', and W are of the form

$$\begin{pmatrix} X_{11} & X_{12}\\ 0 & X_{22}\end{pmatrix},\quad \begin{pmatrix} Y_{11} & Y_{12}\\ 0 & Y_{22}\end{pmatrix},\quad \begin{pmatrix} Q_{11} & Q_{12}\\ 0 & Q_{22}\end{pmatrix},\quad\text{and}\quad \begin{pmatrix} Z & 0\\ 0 & W_\gamma\end{pmatrix},$$

where X_{22} and Y_{22} have dense range, and $Q_{22}\in\mathcal{Q}$. The equations (*) imply that $W_\gamma X_{22}=X_{22}Q_{22}$ and $Y_{22}W_\gamma=Q_{22}Y_{22}$. Lemma 2.3 implies that $R=X_{22}Q_{22}Y_{22}$ is a quasinilpotent operator commuting with W_γ, and we assert that R is nonzero. For otherwise, since $Y_{22}X_{22}$ commutes with Q_{22}, it follows that $0=Y_{22}R=Y_{22}X_{22}Q_{22}Y_{22}=Q_{22}Y_{22}X_{22}Y_{22}$. Since X_{22} and Y_{22} have dense range, it follows that $Q_{22}=0$. Now (*) implies that $W_\gamma=0$, which is a contradiction.

Thus R is a nonzero quasinilpotent commuting with W_γ, so Corollary 2.6 implies that W_γ is quasinilpotent. By applying the preceding method to W^*, we conclude that W_β is also quasinilpotent. (Note that $W^*=W_\beta\oplus N^*\oplus W_\gamma^*$, so that $\mathscr{P}(W^*)=\mathfrak{H}_2\oplus\mathfrak{H}_3$.) Since $W_\gamma, W_\beta\in\mathcal{Q}$ and $N\in\mathcal{N}$, it follows that W is quasinilpotent, which (together with case i)) completes the proof of the second part of 2).

iv) There exists a largest integer N such that $\alpha_N=0$, but there exists no smalles, such integer. Let $\mathfrak{M}=\langle e_N, e_{N-1}, \ldots\rangle$. Relative to the decomposition $\mathfrak{H}=\mathfrak{M}\oplus\mathfrak{M}^\perp$, $W=N\oplus W_\gamma$, where N is an infinite direct sum of finite dimensional nilpotentst and W_γ is an injective unilateral weighted shift. Since $\mathfrak{P}(W)=\mathfrak{M}$, the method of case iii) implies that W_γ is quasinilpotent, so W is a direct sum of quasinilpotents.

v) There is a smallest integer N such that $\alpha_N=0$, but there is no largest such integer. The desired conclusion that W is a direct sum of quasinilpotents follows by applying case iv) to W^*, which is a bilateral weighted shift relative to the basis $\{f_n\}_{n=-\infty}^{+\infty}$, where $f_n=e_{-n}$.

Theorem 2.8. *Let* $W=W_\gamma$ *be a unilateral weighted shift. The following are equivalent.*

1) $W\in\mathcal{Q}_{qs}$;
2) $W\in\mathcal{Q}$ *or* γ *has infinitely many zero terms;*
3) W *is a direct sum of quasinilpotent operators.*

Proof. The implication 2)⇒3) is obvious and 3)⇒1) follows from [8, Prop. 3.10]. Suppose that $W_\gamma\in\mathcal{Q}_{qs}$. Let W_β be a quasinilpotent injective unilateral weighted shift and let $W_\alpha=W_\beta^*\oplus W_\gamma$. Thus W_α is a bilateral weighted shift and W_α is in $\mathcal{Q}_{qs}$ [8]. If at most a finite number of the weights of W_α are zero, then Theorem 2.7 implies that $W_\alpha\in\mathcal{Q}$, from which it follows that $W_\gamma\in\mathcal{Q}$. In the remaining case, W_α has infinitely many zero weights, and since W_β is injective, these weights correspond to zero terms in γ.

Corollary 2.9. $\mathcal{Q}_{qs}$ *is a proper subset of* $\mathcal{Q}_{af}\cap\mathcal{Q}_{af}^*$.

Proof. According to [9, Example 3.2], there exists a non-quasinilpotent injective unilateral weighted shift W such that $\mathfrak{M}(W)$ and $\mathfrak{M}(W^*)$ both contain the orthonormal basis $\{e_n\}_{n=0}^{\infty}$. Thus $\mathfrak{M}(W)$ and $\mathfrak{M}(W^*)$ are both dense, and so [3] implies that $W\in\mathcal{Q}_{af}\cap\mathcal{Q}_{af}^*$. However, since W is injective and non-quasinilpotent, Theorem 2.8 implies that W is not in $\mathcal{Q}_{qs}$.

Remark. The shift W in the preceding proof satisfies properties (I) and (II) of section 1. It follows that in general neither property implies membership in $\mathcal{Q}_{qs}$. These results provide negative answers to Question 3.9, Question 3.14, and Question 3.16 of [8]. Theorem 2.7 and Theorem 2.8 answer [8, Question 3.7]. We note also that it is possible to prove Theorem 2.8 directly, without recourse to Theorem 2.7, by employing the same technique used to prove Theorem 2.7.

In [11] C. Foiaş and C. Pearcy proved that if Q is quasinilpotent, then Q and Q^* are quasiaffine transforms of compact operators (which are necessarily quasinilpotent). (This result also follows from [3].) In [11, Proposition 1.5] it is also proved that there exists a quasinilpotent operator that is not quasisimilar to any compact operator. The shift W of Corollary 2.9 is an example of a non-quasinilpotent operator such that W and W^* are quasiaffine transforms of compact operators but such that W is not quasisimilar to any compact operator. The fact that W and

W^* are quasiaffine transforms of compact operators follows from [3]. Now each nonzero operator quasisimilar to a compact operator commutes with a nonzero compact operator [11, Proposition 1.5]; since the spectrum of a compact operator is countable, Lemma 2.5 implies that W is not quasisimilar to any compact operator.

3. Conclusion. In this section we relate our results to a conjecture of [13], discuss some related questions. Let $\{\mathfrak{M}_n\}_{1\leqq n<k}$ $(2<k\leqq\infty)$ denote a sequence of closed subspaces of $\mathfrak{H}$. The sequence $\{\mathfrak{M}_n\}$ is said to be a *basic sequence* for an operator T in $\mathscr{L}(\mathfrak{H})$ if the following properties are satisfied: 1) for each n, $\mathfrak{M}_n$ is invariant for T, i.e. $T\mathfrak{M}_n\subset\mathfrak{M}_n$; 2) For each n, $\mathfrak{M}_n$ and $\bigvee_{m\neq n}\mathfrak{M}_m$ are complementary in $\mathfrak{H}$; 3) If $k=\infty$, then $\bigcap_{n=1}^{\infty}(\bigvee_{m\geqq n}\mathfrak{M}_m)=\{0\}$. The *trivial* basic sequence for any operator is the sequence $\mathfrak{M}_1=\{0\}$, $\mathfrak{M}_2=\mathfrak{H}$. The concept of a basic sequence is due to C. Apostol [1].

D. A. Herrero [13, Conjecture 1] stated the following

Conjecture H. [13] *If an operator T has no non-trivial basic sequence, then each operator S quasisimilar to T satisfies $\sigma(S)=\sigma(T)$.*

Theorem 2.8 can be interpreted as offering some (albeit limited) support to this conjecture. Indeed, an injective unilateral weighted shift T has no nontrivial pair of complementary invariant subspaces [15, Corollary 2, page 63]. Thus T has no nontrivial basic sequence, and Theorem 2.8 shows that if $r(T)>0$, then $r(S)>0$ for each operator S quasisimilar to T. We can show a bit more. Suppose T shifts the basis $\{e_n\}_{n=0}^{\infty}$. Let X and Y be operators with dense range and let S be an operator such that $TX=XS$ and $YT=SY$. Since XY commutes with T, [16, page 780] implies that the matrix of XY relative to $\{e_n\}$ is given by a formal power series $\sum_{n=0}^{\infty}a_nT^n$. Since XY has dense range, $|a_0|>0$. By method quite different than that used in section 2 it can be shown that if $\sum_{n=1}^{\infty}|a_n|\,\|S^n\|<|a_0|$, then $r(S)\geqq r(T)$; note that if $a_n=0$ for each $n\geqq 1$, then XY is invertible, so T and S are similar.

In a different direction, S. Clary [5] has studied subnormal operators quasisimilar to the unweighted unilateral shift U. It follows from [5] that there exists subnormal operators S such that S and U are quasisimilar but not similar; however, quasisimilar subnormal operators do have equal spectra [6, Theorem 2]. The preceding remarks suggest the following question.

Question 3.1. If T is an injective unilateral weighted shift and S is quasisimilar to T, does $\sigma(S)=\sigma(T)$?

C. Apostol [1] proved that an operator T is quasisimilar to a normal operator if and only if T has a basic sequence $\{\mathfrak{M}_n\}$ such that each restriction $T|\mathfrak{M}_n$ is similar to a normal operator. In [9, Theorem 5.5] it is proved that if an operator T has a basic sequence $\{\mathfrak{M}_n\}$ such that each restriction $T|\mathfrak{M}_n$ is a spectral operator, then T is quasisimilar to a spectral operator. The proof of this result also yields the following sufficient condition for membership in $\mathcal{Q}_{qs}$.

Proposition 3.2. *If $\{\mathfrak{M}_n\}$ is a basic sequence for an operator T such that each restriction $T|\mathfrak{M}_n$ is quasinilpotent, then T is in $\mathcal{Q}_{qs}$.*

Question 3.3. Is the converse of Proposition 3.2 true?

The results of [8] show that if T is in $\mathcal{Q}_{qs}$ and T is decomposable or hyponormal, then T is quasinilpotent, so Question 3.3 has an affirmative answer for operators in these classes. More generally, the answer is affirmative for each operator T satisfying property (C) in the sense of [17], since for each such operator, $\mathfrak{M}(T)$ is closed. The answer is also affirmative for weighted shifts; indeed Theorem 2.7 and Theorem 2.8 may be reformulated as follows.

Theorem 3.4. *A weighted shift W is in $\mathcal{Q}_{qs}$ if and only if there exists a basic sequence $\{\mathfrak{M}_n\}$ for W such that each restriction $W|\mathfrak{M}_n$ is quasinilpotent.*

Proof. Theorem 2.7 and Theorem 2.8 imply that if a weighted shift W is in $\mathcal{Q}_{qs}$, then W is a direct sum of quasinilpotents; this direct sum decomposition gives rise to the desired basic sequence. The converse follows from Proposition 3.2.

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